Non-commutative resolutions of quotient singularities

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Abstract

We show the existence of non-commutative (crepant) resolutions of quotient singularities, where the quotient singularity arises as a reductive group acting suitably well-behaved on a vector space. We first go through the simpler case of torus actions, then generalize to connected reductive groups. We use a mixture of algebraic and geometric methods, as well as results from Lie group theory, especially key results about their representation theory.

Contents

1 Introduction

The idea of trying to resolve a singularity by replacing a space X which is not smooth with another space $\tilde{X}$, birational to X and smooth has been very useful in the study of schemes. This is a well known and very commonly used tool in algebraic geometry, if in addition one is working with Calabi-Yau schemes then one may want a resolution that has an extra property, being crepant.
This type of resolution is very geometric, recently the idea of taking a different type of resolution has appeared, using non-commutative rings instead. This is no longer a geometric idea, however it still has connections to geometry and even to classical resolutions of singularities in certain cases. It is more of a categorical resolution, however it can be phrased without using category theory, these resolutions are called non-commutative (crepant) resolutions, we will give the definition first and then provide some of the motivation.

Definition 1.1. Let S be a scheme, such that S is a normal noetherian domain.
A non-commutative resolution of S is an algebra over S which has finite global dimension and is of the form $\Lambda = \mathrm{End}_S(M)$, for M a non-zero, finitely generated, reflexive S-module.
It is called crepant if S is Gorenstein and Λ is a maximal Cohen-Macaulay S-module.
We abbreviate with NC(C)R.

We will be interested in the case where we have an action of G on a vector space V, we will look for a NC(C)R of $k[V]^G$, which is the coordinate ring of the *quotient singularity* V/G. We will mainly consider infinite reductive groups, but we will briefly state what happens if G is finite, all are fields have characteristic 0, and we will work over $\mathbb{C}$ for concreteness.

Some of the motivation for various parts of the definition is;
- finite global dimension of a commutative Noetherian local ring corresponds to being regular, so we take that as part of the definition.
- Λ being CM corresponds to the crepant part, a commutative resolution is crepant if the pull-back of the canonical sheaf is the canonical sheaf, here it has a categorical meaning that is generalized from the geometric meaning.
- Gorenstein says that the singularities can't be too bad and we still have a canonical line bundle.
- being maximal Cohen-Macaulay (CM) gives us a nice category where we have some sort of generalization of Serre duality (a Serre functor), (so having a canonical line bundle is wanted).
- One can show that you only need to consider algebras of the form $\mathrm{End}_{k[X]}(M)$ with M reflexive and finitely generated if you want a crepant resolution.

For more motivation and discussion see the survey paper by Leuschke [8], in there are many of the above motivations including the reasons for requiring the form $\mathrm{End}_S(M)$. In section K of [8] the definition of a NCCR is given, it is different to the one above, but later it is shown to be equivalent to the above definition, there is also many results and other thoughts on NC(C)R's. In this survey paper there is also a brief discussion of categorical desingularizations (which are another type of "resolution") and how they relate to NCCR's.

The first mentions of non-commutative resolutions were by Bondal and Orlov [1] and Van den Bergh [13], not as a main idea, but as something that could be useful to show other results. Van den Bergh only defines an NCCR in the appendix of [13], it had been used as an intermediary step in proving two derived categories were equivalent. Looking at derived categories and equivalences was also the reason that Bondal and Orlov thought of it.

An earlier theorem of Auslander showed the existence of NCCR's for G a finite group acting on $\mathbb{C}[x_1, \cdots, x_n]$ with some conditions (the concept of an NCCR was not around then), see Section 2 for a brief overview.
There is also a classical theorem of Hironaka [4] that says that there is always a resolution of singularities in a variety, but there is not always a crepant resolution, in some cases if we have a NCCR we can build a crepant commutative resolution from the NCCR, see the introduction of [9] for references. However this is not our aim, we are just looking at NCCR's for their own right.

Much of this report is based off the work of Špenko and Van den Bergh [9], all the results on NC(C)R's for reductive groups come from there, we have tried to add details to the proofs for clarity and provide motivation for some of the techniques and methods that they used. For examples of NC(C)R's see sections 5-10 of that paper, they also prove more results then we cover here.

The outline of this report is as follows;
We will very briefly state what happens in the special case that G is a finite group, then we will look at an example of a $\mathbb{C}^*$ action where we can calculate things by hand. Next we will prove a general result about NCCR's for actions by tori. Finally we generalize the methods used to a general connected reductive group.
It will turn out that all the NC(C)R's of X that we consider are of the form $(U \otimes k[X])^G$ called *modules of covariants*, see definition 4.6.

2 Finite group actions

In this section G will be a finite group, unless mentioned otherwise, acting on linearly on $\mathbb{C}^n$. We consider $\mathbb{C}^n/G$, the quotient singularity, and we want to know when we have a NCCR and what it looks like. This is the case that was first understood although not with the language we're using, see for example the McKay correspondence, briefly covered in [8, Section J].
For a general introduction to NCCR's for finite groups with many examples and motivation see [15].

Definition 2.1. Let A be a $\mathbb{C}$-algebra, G a group acting on A.
The skew group ring $A\#G$ is the vector space $A \otimes_{\mathbb{C}} \mathbb{C}G$ with multiplication given by

$$(a_1 \otimes g_1)(a_2 \otimes g_2) = (a_1 \cdot g_1(a_2)) \otimes g_1 g_2.$$

We use the skew group ring because it stores information about G-equivariant modules. Given an A-module M which has a G action as well we have that M is a $A\#G$-module if M is a G-equivariant module. If we have a $A\#G$-module N then we can define a A-module action and a G action on N and N is a G-equivariant module. So we have a correspondence between G-equivariant A-modules and $A\#G$-modules. This also works for G infinite.
We also have

$$\operatorname{Hom}_A(M, N)^G = \operatorname{Hom}_{A\#G}(M, N)$$

for M, N $A\#G$-modules. (G acts on $\operatorname{Hom}_A(M, N)$ by $(gf)(m) = gf(g^{-1}m)$). This gives us;

Theorem 2.2. *Let G be a finite group acting on A, a polynomial algebra, then we have*

$$\text{gl.dim } A\#G = \text{gl.dim } A.$$

Proof. Sketch only;
This follows straightforwardly from the fact that $(-)^G$ is exact which implies

$$\text{Ext}^i_{A\#G}(M,N) = \text{Ext}^i_A(M,N)^G$$

This gives us that gl.dim $A\#G \leq$ gl.dim A. Then take the module $\mathbb{C}$, it's Koszul resolution is G-equivariant, and it has proj.dim $\mathbb{C} =$ gl.dim A. (Note, gl.dim A is the number of variables of A.) We therefore have gl.dim $A\#G \geq$ proj.dim $\mathbb{C} =$ gl.dim A. □

Remark 2.3. If G is a reductive group acting on a smooth connected affine variety X and A is the coordinate ring then we have gl.dim $A\#G \leq$ gl.dim A, with equality if G fixes a point in X. (The residue field of the fixed point plays the role of $\mathbb{C}$ above.)

The following theorem contains a summary of the situation for a linear action of a finite group.

Theorem 2.4. *Let $V = \mathbb{C}^n$, $G \subset GL(V)$ finite.*
Then if G contains no non-trivial pseudo-reflections[1] *we have an isomorphism*

$$\mathbb{C}[V]\#G \cong End_{\mathbb{C}[V]^G}(\mathbb{C}[V])$$

(Consider $\mathbb{C}[V]$ as a $\mathbb{C}[V]^G$-module)
and $\mathbb{C}[V]\#G$ is a NCR.
If in addition $G \subset SL(V)$ then $\mathbb{C}[V]\#G$ is a NCCR.

Proof. The isomorphism was first proved by Auslander, (for power series rings) for a proof of the isomorphism as stated above see [5, Theorem 3.2]. Using Theorem 2.2 we get that $\Lambda = \mathbb{C}[V]\#G$ is a NCR. Watanabe [14] showed that $\mathbb{C}[V]^G$ is Gorenstein if $G \subset SL(V)$, one can also show that $\mathbb{C}[V]$ is a maximal CM $\mathbb{C}[V]^G$-module;
putting these together we get the result. □

We can write $\mathbb{C}[V] \cong (\bigoplus_{W\ irred.iso.class} W \otimes \mathbb{C}[V])^G$ and can also show that writing $P = \bigoplus_{W\ irred.iso.class} W$ we have $\Lambda \cong (\text{End}_{\mathbb{C}}(P) \otimes \mathbb{C}[V])^G$, see Corollary 4.7, we see the appearance of the module of covariants here. We see that Λ can be calculated on the quotient or on the original space.

For finite groups we can therefore explain the situation quite nicely, assuming G acts well enough. In general, if G acts on X we will look for a NC(C)R that is a module of covariants. In some situations, one can actually create a commutative crepant resolution using quiver GIT from the non-commutative resolution, see [15, Section 3], going the other way is also possible using tilting bundles.

[1] an element is a pseudo-reflection if it has finite order and fixes a hyperplane pointwise. If G is a group generated by pseudo-reflections then the Chevalley-Shephard-Todd theorem [3] says that $\mathbb{C}^n/G$ is smooth, so there is no singularity.

3 Example for torus action

We want to try and copy the ideas from the finite case and apply it when G is an infinite reductive group, straight away we reach a problem.
In the finite case we have a finite number of irreducible representations, so we just take all of them, and it turns out that that works. In the general reductive case we have an infinite number of irreducible representations, so we need to pick a subset of them. When we do this we no longer have a guarantee of finite global dimension.

We will first do an example of a $\mathbb{C}^*$ action, and then in the next section we will generalize the ideas to $T = (\mathbb{C}^*)^n$ acting on an affine variety (with some conditions on the action). Not everything will be proved in our example, we will use some results from the next section.

Let $\mathbb{C}^*$ act on $\mathbb{C}^3$ with weights $1, 1, -2$, it acts on the coordinate ring $R = \mathbb{C}[x, y, z]$ with weights $-1, -1, 2$.
$\mathbb{C}^*$ has a irreducible representation $V(i)$ for each integer i. Let $R(i) = V(i) \otimes R$, these are projective $R\#\mathbb{C}^*$-modules. Let $T = R \oplus R(1)$. The claim is that $\mathrm{End}_{R\#\mathbb{C}^*}(T)$ has finite global dimension.

We have a functor

$$\mathrm{Hom}_{R\#\mathbb{C}^*}(T, -) : R\#\mathbb{C}^*\text{--}mod \to \mathrm{End}_{R\#\mathbb{C}^*}(T)\text{--}mod$$

and we get $\mathrm{End}_{R\#\mathbb{C}^*}(T)$-modules $M_i = \mathrm{Hom}_{R\#\mathbb{C}^*}(T, R(i))$ for each $i \in \mathbb{Z}$, in particular, M_0 and M_1 are projective. By Proposition 4.10 it is sufficient to check that $M_i = \mathrm{Hom}_{R\#\mathbb{C}^*}(T, R(i))$ has finite projective dimension for each $i \in \mathbb{Z}$.

We will show this by considering the Koszul resolutions of R/z and $R/x, y$. First note that

$$\mathrm{Hom}_{R\#\mathbb{C}^*}(T, (R/z)(k)) = 0$$

if $k < 0$. This is as

$$\mathrm{Hom}_{R\#\mathbb{C}^*}(R(i), (R/z)(k))) = ((R/z)(k - i))^{\mathbb{C}^*}$$

and this is zero if there are no elements with weight 0, this holds for $i = 0, 1$ if $k < 0$.
We have the resolution,

$$0 \to R(2) \xrightarrow{z} R \to R/z \to 0.$$

Tensor with $V(k)$ and then apply $\mathrm{Hom}_{R\#\mathbb{C}^*}(T, -)$. For $k < 0$ we get,

$$0 \to M_{2+k} \to M_k \to 0.$$

Therefore we have,

$$M_1 \cong M_{-1} \cong M_{-3} \cong \cdots \cong M_{-2i-1} \qquad i \geq 0$$
$$M_0 \cong M_{-2} \cong M_{-4} \cong \cdots \cong M_{-2i} \qquad i \geq 0$$

Therefore as both M_0, M_1 are projective $\mathrm{End}_{R\#\mathbb{C}^*}(T)$-modules we get the result for M_i when $i < 0$.

In a similar way $\mathrm{Hom}_{R\#\mathbb{C}^*}(T, (R/x, y)(k)) = 0$ if $k > 1$.
We have the resolution,

$$0 \to R(-2) \xrightarrow{\begin{pmatrix} y & -x \end{pmatrix}} R(-1)^2 \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R \to R/x, y \to 0.$$

So after tensoring with $V(k)$ for $k > 1$ and applying $\mathrm{Hom}_{R\#\mathbb{C}^*}(T, -)$ we get,

$$0 \to M_{k-2} \to M_{k-1}^2 \to M_k \to 0.$$

For $k = 2$ we get that M_2 has a finite resolution using projective $\mathrm{End}_{R\#\mathbb{C}^*}(T)$-modules.
For $k = 3$ we have,

$$0 \to M_1 \to M_2^2 \to M_3 \to 0.$$

As M_1 is projective and M_2 has finite projective dimension we get that M_3 also does using the Ext long exact sequence. In the same way, if $k = 4$ we have

$$0 \to M_2 \to M_3^2 \to M_4 \to 0.$$

and again the Ext long exact sequence gives us finite projective dimension for M_4.
It is clear that this process extends and that we have proj.dim $M_i < \infty$ for all i.
Using Proposition 4.10 we get that this is in fact a NCR, using Proposition 4.9 and Corollary 4.7 we get that it is actually a NCCR.

For later use we will want a straightforward generalization of the Ext long exact sequence trick to give finite projective dimension.

Proposition 3.1. *Given an exact sequence such that every term but one has finite projective dimension then in fact every term has finite projective dimension.*

Proof. We prove this by induction, if we have a 3 term sequence, then the long exact sequence for Ext gives us the result.
Given a sequence of length k

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_{k-1} \longrightarrow M_k \longrightarrow 0$$

either the first 2 terms, or the last two terms have finite projective dimension, without loss of generality assume it is the last two. We then have the sequences

$$0 \to K \to M_{k-1} \to M_k \to 0$$

and

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_{k-2} \longrightarrow K \longrightarrow 0$$

By the base case, K has finite projective dimension, by the induction assumption, so does every term in the second sequence. $\square$

4 Torus actions

The aim of this section is to prove that if we have a torus $T = (\mathbb{C}^*)^n$ acting generically on $\mathrm{Spec}\, SW$ for W a representation of T, $SW = \mathrm{Sym}^\bullet W$, the symmetric algebra on W, then a NC(C)R exists, and we can describe the NC(C)R explicitly.

4.1 Set-up and Cohen-Macaulayness

First we give the definitions, notation and basic results that we will use, many of the results and definitions will be in greater generality then is needed for this section, the extra generality will be used later.

Definition 4.1. G a reductive group acts *generically* on a smooth affine variety X if
i) X contains a point with a closed orbit and trivial stabilizer
ii) The locus of points not satisfying i) has codimension at least 2.
We say W is *generic* if G acts generically on $\operatorname{Spec} SW$.

From now on all our actions will be generic. The irreducible representation of T are indexed by lists of n integers, we will represent a representation by an integer lattice point in $\mathbb{R}^n$, called a weight. Given a representation W, we let α_i for $i = 1$ to dim W be the weights of W.

Definition 4.2. We call W *quasi-symmetric* if we have,

$$\sum_{\alpha_i \in l} \alpha_i = 0,$$

for all lines l through the origin.

So for a representation of $\mathbb{C}^*$ to be quasi-symmetric we require that the sum of the negative weights is equal to the sum of the positive weights, equivalently that $G \hookrightarrow SL(W)$.
Note that the example from Section 3 is both generic and quasi-symmetric.
Quasi-symmetry is a very strong condition on the action and we will not assume that all our actions are quasi-symmetric, however to get results about NCCR's we will need quasi-symmetry.

Knop showed that SW^G is Gorenstein if W is generic and quasi-symmetric [7]. We also have a category equivalence for reflexive sheaves if G acts generically.

Remark 4.3. A reflexive sheaf is a sheaf that is isomorphic to its double dual, this makes them torsion free, they are almost vector bundles, but can have mild singularities (they aren't affected by codimension 2 phenomena).

Lemma 4.4. *Let R be the coordinate ring of a smooth connected affine variety X.*
Let ref($R\#G$) be the category of G-equivariant R-modules which are reflexive as R-modules, and ref(R^G) be the category of reflexive R^G-modules.
If G acts generically X then,

$$\operatorname{ref}(R\#G) \to \operatorname{ref}(R^G) : M \mapsto M^G$$

$$\operatorname{ref}(R^G) \to \operatorname{ref}(R\#G) : N \mapsto (R \otimes_{R^G} N)^{**}$$

are inverse category equivalences.

Remark 4.5. The first functor makes sense as G acting generically implies that $X \to X//G$ contracts no divisor and therefore [2, Proposition 1.3], gives us that $(-)^G$ preserves reflexive modules.

Definition 4.6. Let U be a finite representation of G. Then $(U \otimes R)^G$ is an R^G-module, called the *module of covariants* associated to U.

If G acts generically [2, Proposition 1.3] tells us that the module of covariants is reflexive, from now on we will not mention reflexive again, but it is there behind the scenes.

Corollary 4.7. *Let G,X,R and U be as above, then*

$$(End_{\mathbb{C}}(U) \otimes R)^G \cong End_{R^G}\left((U \otimes R)^G\right)$$

Remark 4.8. The left hand side is also $\mathrm{End}_{R\#G}(U \otimes R)$, i.e. the equivariant endomorphisms of the equivariant bundle associated to U, the right hand side is endomorphisms of the same bundle pushed down to R^G.

If we replace U by $\mathrm{End}(U)$ we get an algebra, called the *algebra of covariants*, we will be looking for NCCR's that are algebras of covariants, as in the finite group case.

Now we go back to the torus case, given T, a torus, acting on W with weights α_i, let

$$r\Sigma = \left\{ \sum_i a_i \alpha_i \mid a_i \in (-r, 0] \right\} \subset \mathbb{R}^n.$$

We will let $\Sigma = 1\Sigma$, and $r\overline{\Sigma} = \{\sum_i a_i\alpha_i \mid a_i \in [-r, 0]\}$.
Given a weight χ let $V(\chi)$ be the corresponding representation, let $P_\chi = V(\chi) \otimes_{\mathbb{C}} SW$, these form a family of projective generators for the category of finitely generated G-equivariant SW-modules, or $SW\#G$-modules.
Given a finite collection $\mathcal{L}$ of weights, let $P_{\mathcal{L}} = \bigoplus_{\chi\in\mathcal{L}} P_\chi$, and let $\Lambda_{\mathcal{L}} = \mathrm{Hom}_{SW\#G}(P_{\mathcal{L}}, P_{\mathcal{L}})$ and $\tilde{P}_{\mathcal{L},\chi} = \mathrm{Hom}_{SW\#G}(P_{\mathcal{L}}, P_\chi)$.
We want to find conditions on $\mathcal{L}$ so that $\Lambda_{\mathcal{L}}$ has finite global dimension and also for when $\Lambda_{\mathcal{L}}$ is a CM-module.

We will first find the condition for $\Lambda_{\mathcal{L}}$ to be CM, the work was done by Van den Bergh [12], adapted for this case in [9, Section 4.4]. We find it is sufficient to find $\mathcal{L}$ such that if $\chi_1, \chi_2 \in \mathcal{L}$, then $\chi_1 - \chi_2 \in \Sigma$.[2]
To help show this we make a definition, given a subset B of $\mathbb{R}^n$ and $\epsilon \in \mathbb{R}^n$, let $B_\epsilon = \cup_{r>0} B \cap (r\epsilon + B)$. Informally, think of B_ϵ as B with some of the boundary removed, the part that is moved "inwards" by ϵ.

Proposition 4.9. *Let W be generic and quasi-symmetric for T.*
Let $\mathcal{L} = r\overline{\Sigma}_\epsilon \cap \mathbb{Z}^n$, such that ϵ isn't parallel to any face of $\overline{\Sigma}$.
Then $\Lambda_{\mathcal{L}}$ as defined above is Cohen-Macaulay for $r \leq \frac{1}{2}$.

Proof. By the above we only need to check that if we take 2 weights, χ_1, χ_2 in $\mathcal{L}$ and take the difference we end up in Σ.
The quasi-symmetry assumption implies that $\Sigma = \overline{\Sigma} - \partial\overline{\Sigma}$. If $\chi_1 - \chi_2 \notin \Sigma$, then it must be on a face of $\overline{\Sigma}$ and therefore χ_1 and $-\chi_2$ are both on the same face B, but by construction, ϵ must move one of $B, -B$ inwards (it isn't parallel to any face), which implies that one of χ_1, χ_2 isn't in $r\overline{\Sigma}_\epsilon$. $\square$

[2]See Theorem 5.13 and the paragraph above for the full result

4.2 Finite global dimension

The idea is to work by contradiction, pick a "minimal" module with infinite projective dimension, and then show that in fact we have a finite resolution using modules that are smaller, i.e. have finite projective dimension, then using Proposition 3.1, we have that in fact our module has finite projective dimension, contradicting our assumption that we could find one with infinite global dimension. First we show that we only need to consider modules of the form $\tilde{P}_{\mathcal{L},\chi}$.

Looking at the definition of $\Lambda_{\mathcal{L}}$ we recall that it is of the form $\mathrm{End}\left(\bigoplus M_i\right)$, we can therefore describe this as a quiver algebra with relations, where each vertex corresponds to a M_i and the arrows are maps $M_i \to M_j$, this gives $\Lambda_{\mathcal{L}}$ a grading.
We have simple modules $S_\chi = V(\chi) \otimes SW/SW_{>0}$, these are twists of the skyscraper sheaf at the origin, and they give us $\Lambda_{\mathcal{L}}$-modules $\tilde{S}_\chi = \mathrm{Hom}_{SW\#G}\left(P_{\mathcal{L}}, S_\chi\right)$. These are simple graded modules called vertex simples as $\mathrm{Hom}_{SW\#G}\left(P_{\chi_i}, S_{\chi_j}\right) = \mathbb{C} \cdot \delta_{ij}$, so $\tilde{S}_\chi$ only "appears" at the vertex labelled by P_χ.

Proposition 4.10. *Let $\Lambda_{\mathcal{L}}$ be as above.*
Then gl.dim $\Lambda_{\mathcal{L}} < \infty$ *if and only if we have* proj.dim $\tilde{P}_{\mathcal{L},\chi} < \infty$ *as a $\Lambda_{\mathcal{L}}$-module, for all χ.*

Proof. One direction is trivial, so assume that all the $\tilde{P}_{\mathcal{L},\chi}$ have finite projective dimension, but that we can find an M with infinite projective dimension. This gives us an infinite minimal free resolution which is constructed as follows;

Pick a minimal generating set m_i of M (use graded Nakayama, lift a basis of $M/\left(\Lambda_{\mathcal{L}}\right)_{>0} M$ to M) then define a map $\Lambda_{\mathcal{L}} \to M$ by sending $1 \mapsto m_i$. Put these maps together to give a surjection onto M, it becomes an isomorphism after quotienting by $\left(\Lambda_{\mathcal{L}}\right)_{>0}$ which implies that we have $\mathrm{Ker} f \subset \left(\Lambda_{\mathcal{L}}\right)_{>0}$, therefore when we extend the resolution with a map g (constructed similarly) we have $\mathrm{Im}\; g \subset \left(\Lambda_{\mathcal{L}}\right)_{>0}$.
Therefore when we tensor the resolution with $S = \Lambda_{\mathcal{L}}/\left(\Lambda_{\mathcal{L}}\right)_{>0}$ which kills all the terms with degree at least 1, the map g becomes 0 as it's image only has terms with degree at least 1. This implies that we have

$$\mathrm{Tor}_i^{\Lambda_{\mathcal{L}}}(M, S) \neq 0$$

for all i. This gives us that S also has infinite projective dimension and as $S = \bigoplus_{\chi\in\mathcal{L}} \tilde{S}_\chi$ we get that at least one of the simple modules $\tilde{S}_\chi$ has infinite projective dimension.

However we also have the Koszul complex of S_χ which gives a finite resolution of S_χ where all the terms involved are finite sums of P_Ψ for some weights Ψ.
Applying $\mathrm{Hom}(P_{\mathcal{L}}, -)$ gives us a finite exact resolution of $\tilde{S}_{\mathcal{L},\chi}$ involving only $\tilde{P}_{\mathcal{L},\Psi}$ and then we use Proposition 3.1 to find that each $\tilde{S}_\chi$ has finite projective dimension and therefore so does $\Lambda_{\mathcal{L}}$. □

We want to generalize the idea from the example in Section 3 to work for T, W arbitrary. To do this let λ be a one parameter subgroup and let K_λ be the subspace of W spanned by the weights α_i such that $\langle\lambda, \alpha_i\rangle > 0$. Geometrically it is the points that flow to 0 under λ.
Then we can take the Koszul resolution of $S(W/K_\lambda)$. We have

$$0 \longrightarrow \wedge^d K_\lambda \otimes SW \longrightarrow \wedge^{d-1} K_\lambda \otimes SW \longrightarrow \cdots \longrightarrow K_\lambda \otimes SW \longrightarrow SW \longrightarrow S(W/K_\lambda) \longrightarrow 0,$$

where $d = \dim_{\mathbb{C}} K_\lambda$. Call this (exact) complex C_λ. Tensoring the sequence with χ preserves exactness, denote it by $C_{\lambda,\chi}$.
Looking more closely at $C_{\lambda,\chi}$, we see that the every term apart from the rightmost one consists of modules of the form P_Ψ where $\Psi = \chi + \alpha_{i_1} + \cdots + \alpha_{i_k}$ where $k \leq d$ and $\langle \lambda, \alpha_{i_j} \rangle > 0$. Note there are only finitely many such modules appearing.

We don't want to pick any χ, it should be related to λ in the following way.

Definition 4.11. Fix $\mathcal{L}, \lambda$, we say χ is *separated* from $\mathcal{L}$ by λ if we have,

$$\langle \lambda, \chi \rangle < \langle \lambda, \mu \rangle$$

for all $\mu \in \mathcal{L}$.

Proposition 4.12. *Assume χ is separated from $\mathcal{L}$ by λ. Let $C_{\mathcal{L},\lambda,\chi}$ be the complex you get by applying* $\mathrm{Hom}(P_{\mathcal{L}}, -)$ *to $C_{\lambda,\chi}$.*
Then $C_{\mathcal{L},\lambda,\chi}$ is an exact complex resolving $\tilde{P}_{\mathcal{L},\chi}$.

Proof. It is clear the the second last term on the right is $\tilde{P}_{\mathcal{L},\chi}$, so it is sufficient to prove that $\mathrm{Hom}(P_{\mathcal{L}}, \chi \otimes S(W/K_\lambda)) = 0$.

For $\mathrm{Hom}(P_\mu, \chi \otimes S(W/K_\lambda)) \neq 0$ we need $S(W/K_\lambda)$ to have weights γ_i such that $-\mu + \chi + \sum \gamma_i = 0$. Rearranging and applying $\langle \lambda, - \rangle$ to both sides we get that we require,

$$\langle \lambda, \mu \rangle = \langle \lambda, \chi \rangle + \langle \lambda, \sum \gamma_i \rangle.$$

Now the weights of $S(W/K_\lambda)$ are of the form $\sum \alpha_{i_j}$ where $\langle \lambda, \alpha_{i_j} \rangle \leq 0$. So we have,

$$\langle \lambda, \mu \rangle \leq \langle \lambda, \chi \rangle$$

But by assumption, we actually have the opposite inequality if $\mu \in \mathcal{L}$, so we have the wanted result. $\square$

Now we can show that we have finite global dimension.

Proposition 4.13. *Let W be generic.*
Let $\mathcal{L} = r\Sigma \cap \mathbb{Z}^n$ for any $r > 1$.
Then we have $\mathrm{gl.dim}\, \Lambda_{\mathcal{L}} < \infty$
If in addition the action is quasi-symmetric then we can instead assume that $r > \frac{1}{2}$.

Proof. Assume not, then by Proposition 4.10 we can find a χ such that $\mathrm{proj.dim}\, \tilde{P}_{\mathcal{L},\chi} = \infty$.
Pick χ such that $\chi \in p_\chi \overline{\Sigma}$ and p_χ is minimal, then such that q_χ which is the number of a_i which are equal to $-p_\chi$ for any way of writing $\chi = \sum a_i \alpha_i$ with $a_i \in [-p_\chi, 0]$ is also minimal.

Note that $\chi \notin p_\chi \Sigma$ by minimality assumptions, and $p_\chi \geq r$, as else $\mathrm{proj.dim}\, \tilde{P}_{\mathcal{L},\chi} = 0$. Then pick λ such that χ is separated from $\mathcal{L}$ by λ. This can always be done, just pick $-\lambda$ to be orthogonal, facing outwards from the face of $p_\chi \overline{\Sigma}$ that χ lies in, if χ is at the intersection of multiple faces, pick any one.

Then we have the complex $C_{\mathcal{L},\lambda,\chi}$ which resolves $\tilde{P}_{\mathcal{L},\chi}$ by Proposition 4.12. This resolution contains a finite number of terms involving a finite number of $P_{\mathcal{L},\Psi}$ for $\Psi = \chi + \alpha_{i_1} + \cdots + \alpha_{i_k}$ where $1 \leq k \leq d$ and $\langle \lambda, \alpha_{i_j} \rangle > 0$.
Note that if $\langle \lambda, \alpha_{i_j} \rangle > 0$ then we must have $a_i = -p_\chi$. (Else we would have $\chi - \epsilon\alpha_i \in \mathcal{L}$ for small ϵ, but $\langle \lambda, \chi - \epsilon\alpha_i \rangle = \langle \lambda, \chi \rangle - \epsilon\langle \lambda, \alpha_i \rangle < \langle \lambda, \chi \rangle$ which can't happen because of the separated assumption.)

As we are adding at least one α_i to χ to get Ψ, we must have p_Ψ or q_Ψ is smaller then p_χ or q_χ.

So by minimality assumptions, every piece of the resolution of $\tilde{P}_{\mathcal{L},\chi}$ has finite projective dimensions, using Proposition 3.1 we find that actually $\tilde{P}_{\mathcal{L},\chi}$ must also have finite projective dimension. This contradicts our original assumption and gives us the result.
If in addition we have a quasi-symmetric action then if we assume that $r > 1/2$, we could have $p_\chi < 1$, then we would have Ψ written as a sum of the α_i with at least one coefficient greater then 0, which isn't allowed, however in this situation, quasi-symmetry comes in, and we use the fact that on the line containing α_i there must be weights on the other side of the origin, and we can use those weights instead. This is what restricts us to $r > 1/2$ as we have $|r - 1| < r$ in this case. $\square$

This proof has a straightforward geometric picture.

We start with a convex shape Σ and we expand it until we first have a weight in the boundary that corresponds to an infinite projective dimensional module (p_χ is minimal). If we have multiple weights in the boundary with this property we pick one in the highest dimensional face possible (q_χ is minimal). Think of this as picking the least "singular" one where in the 3 dimensional case we have a vertex is the most singular, then an edge, followed by a face, and the interior of the shape isn't singular.
Then we show that every term in the resolution we have is less "singular" which means it has finite projective dimension, see Figure 1 for a pictorially example of where the weights of the resolution could lie.

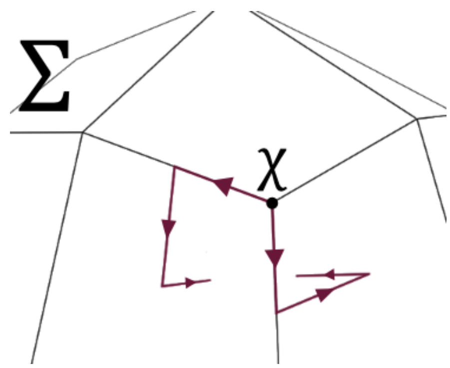

Figure 1: The weights on the red lines are less "singular" then χ.

Putting the above results together we can now prove the existence of NCCR's.

Theorem 4.14. *Let W be generic and quasi-symmetric for $G = (\mathbb{C}^*)^n$. Let $\mathcal{L} = \frac{1}{2}\overline{\Sigma}_\epsilon \cap \mathbb{Z}^n$, such that ϵ isn't parallel to any face of $\overline{\Sigma}$.*
Then $\Lambda_\mathcal{L}$ is a NCCR for SW^G.

Proof. We can adapt the proof of Proposition 4.13 to allow for a shift to v, i.e. we have $\Lambda_\mathcal{L}$ has finite projective dimension for $\mathcal{L} = (r\Sigma + v) \cap \mathbb{Z}^n$, with v any vector.
For r' such that $r' - \frac{1}{2} << 1$ we have $\frac{1}{2}\overline{\Sigma}_\epsilon$ has the same weights as $r'\Sigma + \delta\epsilon$ for small values of δ.

Therefore by Proposition 4.13 we have finite global dimension, and by Proposition 4.9 we have $\Lambda_\mathcal{L}$ is CM, putting these results together we get that $\Lambda_\mathcal{L}$ is a NCCR. $\square$

Note that for $V_{\mathcal{L}} = \bigoplus_{\chi \in \mathcal{L}} V(\chi)$ we have,

$$\begin{aligned}\Lambda_{\mathcal{L}} &\cong \mathrm{Hom}_{SW\#G}\left(P_{\mathcal{L}}, P_{\mathcal{L}}\right)\\ &\cong \mathrm{Hom}_{SW}\left(P_{\mathcal{L}}, P_{\mathcal{L}}\right)^G\\ &\cong \left(\mathrm{Hom}_{\mathbb{C}}\left(V_{\mathcal{L}}, V_{\mathcal{L}}\right) \otimes SW\right)^G\\ &\cong \mathrm{End}_{SW^G}\left(\left(V_{\mathcal{L}} \otimes SW\right)^G\right) && \text{(Cor. 4.7)}\end{aligned}$$

So $\Lambda_{\mathcal{L}}$ is of the right form and is actually an NCCR, even though it is defined in different way.

5 General reductive groups

Having done the torus case and stated the result in the finite case, can we extend to allow more types of groups?
It turns out that we can generalize this to any connected reductive group with a quasi-symmetric and generic action. We will show some of the ideas first, often using the $SL_2(\mathbb{C})$ case as motivation or an example, and then give a proof of the result in full generality.

5.1 Set-up

The set-up for us is, let G be a connected reductive group, W a representation of G, T be a maximal torus inside G, let α_i be the T weights of W, $SW = \mathrm{Sym}^{\bullet} W$ and $X = \mathrm{Spec}\, SW$.

To find NC(C)R's of SW^G we need finite global dimension and Cohen-Macaulayness. We only discuss finite global dimension, the module being CM will follow from [12]. We will however state the result we are using from that paper of Van den Bergh.

The idea of finding a finite resolution $C_{\mathcal{L},\lambda,\chi}$ of $\tilde{P}_{\mathcal{L},\chi}$, and using that to show that all the projective modules, and therefore the algebra $\Lambda_{\mathcal{L}}$ has finite global dimension, was a useful idea and we want to generalize that.
To do this we need to find a way of constructing a complex that consists of P_{μ}'s, then we can tensor with χ and apply $\mathrm{Hom}_{SW\#G}(P_{\mathcal{L}}, -)$ and hope we get an exact resolution of $\tilde{P}_{\mathcal{L},\chi}$.
In the torus case this was relatively straightforward, we picked a one-parameter subgroup of T and used the Koszul resolution of the part of X that was stable as we took the limit of the parameter.
Try this idea, let λ be a one parameter subgroup of T, let K_{λ} be the subspace spanned by weight vectors v such that $\langle \lambda, v\rangle > 0$. Let Z_{λ} be the linear subspace of X that is preserved under the limit of the one parameter action (as $t \to 0$, $\lambda(t)z$ exists for all $z \in Z_{\lambda}$), it is cut out by K_{λ}.
We have a problem though, Z_{λ} isn't preserved by G, and GZ_{λ} is singular. For an example let $G = SL_2(\mathbb{C})$ acting on maps $\mathbb{C}^n \to \mathbb{C}^2$, by matrix multiplication on the left, let $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$, then Z_{λ} is the matrices with empty bottom row, and GZ_{λ} is matrices of rank at most 1, which has a singularity, the zero matrix, we can however take a "natural" resolution, consider the subspace of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^2) \times \mathbb{P}^1$, containing elements (f, l) such that $f(\mathbb{C}^n) \subset l$. This subspace resolves GZ_{λ}. It is also a sub-bundle of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^2) \times \mathbb{P}^1$ considered as a bundle over $\mathbb{P}^1$.

This is potentially helpful as we have bundles over $\mathbb{P}^1$ which we understand quite well, however was this just a coincidence, or is there something more general going on?

5.2 Borel subgroups

Definition 5.1. Let G be a reductive group, a *Borel* subgroup $B \subset G$ is a maximal closed connected solvable subgroup of G.

Borel subgroups are useful for many reasons in the study of reductive groups, but it is the following result that we want.

Theorem 5.2. *G a connected reductive group, then all Borel subgroups are conjugate and G/B is a projective variety*

Remark 5.3. One can also define Borel subgroups as minimal subgroups P such that G/P is complete, if G/P is complete, P is called parabolic.
Also note that any maximal torus is contained in some Borel subgroup.

Proposition 5.4. *Let $B \subset G$ be a Borel subgroup, we have a correspondence between representations of B and G-equivariant bundles on G/B.*

Proof. First, given a G-equivariant vector bundle $\pi : V \to G/B$, we have $\pi(b \cdot v) = b \cdot \pi(v) = \pi(v)$, therefore B acts on the fibres. Pick the fibre $V_{[e]}$, it is a B-representation.
Second, let W be a B-representation, let $\widetilde{W} = G \times W/\sim$ where $(g, v) \sim (gb^{-1}, bv)$. Define $\pi : \widetilde{W} \to G/B,\ (g, v) \mapsto [g]$. It is easy to check that this is well defined and turns $\widetilde{W}$ into a G-equivariant vector bundle (with the natural G-action on G). The fibre above $[e]$ is isomorphic to W as a B-representation. □

Note that if $W \subset Y$ such that Y is a G-representation then we also have a map $p : \widetilde{W} \to GW, (g, w) \mapsto gw$. This generalizes the resolution that we saw above where $G = SL_2(\mathbb{C})$.
For $SL_2(\mathbb{C})$ we have that $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}$ is a Borel subgroup, now $SL_2(\mathbb{C})$ acts on $\mathbb{P}^1$ transitively and B is the stabilizer of $[1:0]$ so $SL_2(\mathbb{C})/B \cong \mathbb{P}^1$.
Also note that B preserves Z_λ in this case.

In general choose B such that $T \subset B$, and λ such that $\langle \lambda, \rho \rangle \geq 0$ for all roots ρ of B, then Z_λ is preserved by B [9, Section 11.2] so is a B-representation, therefore we get $\widetilde{Z_\lambda}$ and a resolution $\widetilde{Z_\lambda} \to GZ_\lambda$. We also think of X as a B-representation and get $\widetilde{X}$, it is a trivial bundle as we have a global trivialization $X \times G/B \to \widetilde{X}$ given by $(x, [g]) \mapsto (g, g^{-1}x)$, this is well defined as $(x, [gb]) \mapsto (gb, b^{-1}g^{-1}x) \sim (g, g^{-1}x)$. As $Z_\lambda \hookrightarrow X$ as a B-representation we get an inclusion of vector bundles, putting these together we have the following diagram.

$$\begin{tikzcd} \widetilde{Z_\lambda} \arrow[r, hook] \arrow[d] & \widetilde{X} \arrow[r, "\pi"] \arrow[d] & G/B \\ GZ_\lambda \arrow[r, hook] & X & \end{tikzcd}$$

The inclusion of vector bundles induces $SW \otimes \mathcal{O}_{G/B} \to \mathrm{Sym}^{\bullet}_{G/B}\left(\widetilde{(W/K_\lambda)}\right)$

Using the correspondence between B-representations and G-equivariant bundles on G/B we can start creating resolutions and get B-representations appearing.

The resolution we take is the Koszul representation of $\mathcal{O}_{\widetilde{Z_\lambda}}$ as a G/B sheaf, we get

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\bigwedge^{d_\lambda} K_\lambda} \otimes SW \arrow[r] & \cdots \arrow[r] & \widetilde{\bigwedge^2 K_\lambda} \otimes SW \arrow[dll] \\ & \widetilde{\bigwedge K_\lambda} \otimes SW \arrow[r] & SW \otimes \mathcal{O}_{G/B} \arrow[r] & \mathrm{Sym}^{\bullet}_{G/B}\left(\widetilde{(W/K_\lambda)}\right) \arrow[r] & 0 \end{tikzcd}$$

In the torus case we had $G = B$ and the correspondence is effectively an equality, so we were done as the resolution was a resolution of G-representations, however we have to work a bit harder now as we only have B-representations.

To show how this works we will first do a a few calculations on $\mathbb{P}^1$, and then state the general result we will use.
We have representations of B on $\mathbb{C}$ indexed by $\mathbb{Z}$, they are $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot v = a^k v$. (Basically representations of $T \cong \mathbb{C}^*$), they correspond to the line bundles $\mathcal{O}_{\mathbb{P}^1}(k)$. By standard results [10, Tag 01XS] they have cohomology,

$$H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)\right) = \mathbb{C}[x, y]_k$$

$$H^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)\right) = \left(\frac{1}{xy}\mathbb{C}\left[\frac{1}{x}, \frac{1}{y}\right]\right)_k$$

For k positive we get that the cohomology is homogeneous polynomials of degree k, we also get an action of G on the cohomology, and with this action we get the standard irreducible representation of $SL_2(\mathbb{C})$ with highest weight k. If k is negative, we get the representation with highest weight $-k-2$. We can write this weight as $-(k+1)-1$.

The general result is called the Borel-Weil-Bott theorem, see the next section for the definitions of dominant and what $V(\chi)$ is.

Theorem 5.5 (Borel-Weil-Bott). *Let $T \subset B \subset G$ be a maximal torus and Borel subgroup of a reductive group G (over $\mathbb{C}$).*
Given a weight χ of T we have a representation of B and therefore a line bundle L_χ on G/B.
Let W be the Weyl group, define $w \star \lambda = w(\lambda + \rho) - \rho$ where ρ is half the sum of the positive roots of G. Then if there exists $w \in W$ such that $w \star \chi$ is dominant, we have

$$H^{l(w)}\left(G/B, L_\chi\right) = V(w \star \chi).$$

$$H^i\left(G/B, L_\chi\right) = 0 \qquad i \neq l(w)$$

where $l(w)$ is the length[3].

[3] the length of w is $\min\{k | w \text{ can be written as a product of } k \text{ reflections in the simple roots}\}$

For a proof and more details on how to get from a weight of T to a representation of B see [11, Section 16].
In the $\mathbb{P}^1$ case we have $\rho = 1$ for $SL_2(\mathbb{C})$, and the calculations showed a special case of the theorem.

5.3 Proofs in the reductive case

We now have all the results we need to prove the results of Section 4 for a reductive group acting generically on a representation (and similarly, stronger results if the action is quasi-symmetric).
Let $T \subset B \subset G$ be as above. Let $\mathcal{W}$ be the Weyl group, let $X(T)$ be the weights of T, and let $X(T)_{\mathbb{R}}$ be the $\mathbb{R}$ vector space obtained from the lattice $X(T)$, let $\Phi \subset X(T)$ be the roots of G. The roots of B are the negative weights denoted Φ^-, the positive weights are $\Phi^+ = -\Phi^-$. Let $\bar{\rho}$ be half the sum of the positive weights.

Definition 5.6. An element $\chi \in X(T)_{\mathbb{R}}$ is dominant if $\langle \chi, \rho \rangle \geq 0$ for all $\rho \in \Phi^+$.

Let $X(T)^+_{\mathbb{R}}$ be the cone of dominant elements and let $X(T)^+$ be the set of dominant weights. The theorem of the highest weight [6, Section 5] says that each dominant weight corresponds to a irreducible G-representation $V(\chi)$. For χ a dominant weight let $P_\chi = V(\chi) \otimes SW$.
Let W be a representation of G, SW as above, let α_i be the T-weights of W, $i = 1, \dots, \dim W$.

Pick λ such that $\langle \lambda, \rho \rangle \leq 0$ for all $\rho \in \Phi^+$, let K_λ be as above, then we have the resolution

$$
\begin{tikzcd}
0 \arrow[r] & \widetilde{\bigwedge^{d_\lambda} K_\lambda} \otimes SW \arrow[r] & \cdots \arrow[r] & \widetilde{\bigwedge^2 K_\lambda} \otimes SW \arrow[dll, out=-10, in=170] & & \\
& \widetilde{\bigwedge K_\lambda} \otimes SW \arrow[r] & SW \otimes \mathcal{O}_{G/B} \arrow[r] & \mathrm{Sym}^\bullet_{G/B}\left(\widetilde{(W/K_\lambda)}\right) \arrow[r] & 0
\end{tikzcd}
$$

We can tensor it with $\widetilde{\chi}$, for χ dominant, and it stays exact, using a result about homological algebra from [9, Appendix A] we get the following G-equivariant isomorphism in the derived category, the left hand side is a chain complex so it has a differential that we have suppressed from the notation, we do not need to know what it is.

$$\bigoplus_{i \geq 0, j \leq 0} H^i\left(G/B, \left(\widetilde{\chi \otimes \bigwedge^{-j} K_\lambda}\right)\right) \otimes SW[-i-j] \cong \mathcal{R}\Gamma\left(G/B, \widetilde{\chi} \otimes \mathrm{Sym}^\bullet_{G/B}\left(\widetilde{(W/K_\lambda)}\right)\right) \qquad (1)$$

Denote the left hand side by $C_{\lambda,\chi}$.

Let $\mathcal{L}$ be a finite subset of $X(T)^+$ and set $P_{\mathcal{L}} = \bigoplus_{\chi \in \mathcal{L}} P_\chi$, let $\tilde{P}_{\mathcal{L},\chi} = \mathrm{Hom}_{SW\#G}(P_{\mathcal{L}}, P_\chi)$. Finally, given χ a weight, if there exists $w \in W$ such that $w \star \chi = \mu$ is dominant, let $\chi^+ = \mu$.

Theorem 5.7. *Let $C_{\mathcal{L},\lambda,\chi} = \mathrm{Hom}_{SW\#G}(P_{\mathcal{L}}, C_{\lambda,\chi})$.*
If χ is separated from $\mathcal{L}$ by λ, then $C_{\mathcal{L},\lambda,\chi}$ is an exact complex resolving $\tilde{P}_{\mathcal{L},\chi}$. It contains terms of the form $\tilde{P}_{\mathcal{L},\mu}$ where $\mu = (\chi + \alpha_{i_1} + \cdots + \alpha_{i_{-j}})^+$ where $\langle \lambda, \alpha_{i_{-j}} \rangle > 0$.

Proof. Note that this is the general version of Proposition 4.12.

To show it is exact after applying $\mathrm{Hom}_{SW\#G}(\mathcal{L},-)$ it is sufficient to show that the right hand side of (1) has only zero terms, (then it has no cohomology and therefore so does the LHS which is $C_{\mathcal{L},\lambda,\chi}$).
To do this we note that

$$\widetilde{\chi}\otimes \mathrm{Sym}^{\bullet}_{G/B}\left(\widetilde{(W/K_\lambda)}\right)$$

has T weights $\chi+\alpha_{i_1}+\cdots+\alpha_{i_k}$ where $k\in\mathbb{Z}_{\geq 0}$ and $\langle\lambda,\alpha_{i_k}\rangle\leq 0$. (Repetitions are allowed)
Therefore by the Borel-Weil-Bott theorem we get that the RHS has terms of the form $V\left((\chi+\alpha_{i_1}+\cdots+\alpha_{i_k})^+\right)$.
We have $\mathrm{Hom}_{SW\#G}(P_{\chi_1},V(\chi_2))\cong \mathrm{Hom}_G(V(\chi_1),V(\chi_2))$[4] which is non-zero if and only if $\chi_1=\chi_2$.
So it is sufficient to prove that $(\chi+\alpha_{i_1}+\cdots+\alpha_{i_k})^+$ doesn't appear in $\mathcal{L}$.

By a fact about weights and Weyl groups we have $\langle\lambda,\chi^+\rangle\leq\langle\lambda,\chi\rangle$ (see [9, Appendix D]), so using also that χ is separated and that $\langle\lambda,\alpha_{i_k}\rangle\leq 0$ we have

$$\begin{aligned}\langle\lambda,(\chi+\alpha_{i_1}+\cdots+\alpha_{i_k})^+\rangle &\leq \langle\lambda,\chi+\alpha_{i_1}+\cdots+\alpha_{i_k}\rangle\\ &\leq\langle\lambda,\chi\rangle\\ &<\langle\lambda,\mu\rangle\end{aligned}$$

for all $\mu\in\mathcal{L}$. Therefore we have an acyclic resolution.

Using the Borel-Weil-Bott theorem on the left hand side of (1) we see that the final term is $\tilde{P}_{\mathcal{L},\chi}$ ($\chi^+=\chi$ as χ is dominant), and that we get exactly the stated terms in the complex as K_λ has weights such that $\langle\lambda,\alpha_{i_j}\rangle>0$. Also note that there are finitely many terms appearing in the resolution. □

Now we have the resolution $C_{\mathcal{L},\lambda,\chi}$ we can prove a general version of Proposition 4.13. Σ is as in Section 4.

Theorem 5.8. *Let G be a connected reductive group, let W be a generic G-representation.*
Let $\mathcal{L}=(-\bar{\rho}+r\Sigma)\cap X(T)^+$
Then for $\Lambda_{\mathcal{L}}=\mathrm{Hom}_{SW\#G}(P_{\mathcal{L}},P_{\mathcal{L}})$ we have that $\mathrm{gl.dim}\,\Lambda_{\mathcal{L}}<\infty$ if $r>1$.
If in addition, W is quasi-symmetric then we can allow $r>\frac{1}{2}$.

Proof. The proof is very similar to Proposition 4.13, again assume not, then we can find χ dominant, such that $\mathrm{proj.dim}\,\tilde{P}_{\mathcal{L},\chi}=\infty$. Pick such a $\chi\in\left(-\bar{\rho}+p_\chi\overline{\Sigma}\right)$ that is minimal with respect to p_χ and then q_χ as before. Note that p_χ,q_χ do not change under the $\star$ action.

As $r\Sigma$ is convex by construction, and χ does not lie in $\mathcal{L}$ we can find λ' such that

$$\langle\lambda',\chi\rangle>\langle\lambda',\mu\rangle$$

for all $\mu\in-\bar{\rho}+r\Sigma$. We also get $\langle\lambda',\chi+\bar{\rho}\rangle>\langle\lambda',\mu+\bar{\rho}\rangle$. Pick w such that $w\lambda'$ is dominant, then as $r\Sigma$ is preserved by $\mathcal{W}$ we get

$$\langle w\lambda',w(\chi+\bar{\rho})\rangle>\langle w\lambda',\mu+\bar{\rho}\rangle$$

[4] In general we have $\mathrm{Hom}_{SW\#G}(V(\chi)\otimes SW,M)\cong\mathrm{Hom}_G(V(\chi),M)$.

We also have $\langle w\lambda', \chi + \bar{\rho}\rangle \geq \langle w\lambda', w(\chi + \bar{\rho})\rangle$. (again [9, Appendix D]).
Putting these together we get that

$$\langle w\lambda', \chi + \bar{\rho}\rangle > \langle w\lambda', \mu + \bar{\rho}\rangle$$

Now let $\lambda = -w\lambda'$ we have that $\langle \lambda, \chi\rangle < \langle \lambda, \mu\rangle$ for all $\mu \in -\bar{\rho} + r\Sigma$, therefore χ is separated from $\mathcal{L}$ by λ, and also λ is such that $\langle \lambda, \rho\rangle \leq 0$ for all $\rho \in \Phi^+$.

Proposition 5.7 gives us an exact resolution of $\tilde{P}_{\mathcal{L},\chi}$ and in exactly the same way as in the torus case we find that all the terms have finite projective dimension (use that p_χ, q_χ are preserved under $\star$), therefore so does $\tilde{P}_{\mathcal{L},\chi}$. Proposition 4.10 holds for any reductive group, only change required is we only need to check for $\chi \in X(T)^+$, therefore we get the wanted result.
If the action is quasi-symmetric then exactly the same arguement as in the torus case, Proposition 4.13 allows us to assume $r > \frac{1}{2}$. □

The geometric picture of this proof is the same as in the torus case, we just had to work a bit harder to show that we can find a λ with the wanted properties.

Example 5.9. Let $G = SL_2(\mathbb{C})$ and let G act on $\mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}^n, \mathbb{C}^2\right)$ as earlier.
The weights are n copies of 1 and n copies of -1, therefore this action is quasi-symmetric, it is also generic, therefore letting $\mathcal{L} = \left\{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$ we have that $\Lambda_{\mathcal{L}}$ has finite global dimension.
We will see later that if n is odd this is actually a NCCR, the reason for this is that $(1/2)\overline{\Sigma}$ has no weights in the boundary in this case.

Proposition 4.9 also has a generalization for G reductive, here we state the full result of Van den Bergh [12] that we are using to show Cohen-Macaulayness.

Definition 5.10. The elements of $(-2\bar{\rho} + \Sigma) \cap X(T)^+$ are called *strongly critical* weights for G with respect to W.

Theorem 5.11. *[12, Theorem 1.3] Let U be an irreducible representation of G such that U^* has weight χ. Then $(U \otimes SW)^G$ is Cohen-Macaulay if χ is strongly critical.*

Corollary 5.12. *Let G be a connected reductive group, let W be a generic G-representation.*
Let $\mathcal{L}$ be a finite subset of $X(T)^+$.
Then $\Lambda_{\mathcal{L}} = \mathrm{Hom}_{SW\#G}(P_{\mathcal{L}}, P_{\mathcal{L}})$ is Cohen-Macaulay if for all $\chi_1, \chi_2 \in \mathcal{L}$ we have $\chi_1 - w_0\chi_2$ is strongly critical, where w_0 is the longest length element in $\mathcal{W}$.

The arguement to get to the corollary is in [9, Section 4.4].

We can now state the general result for NCCR's.

Theorem 5.13. *Let G be a connected reductive group, W a representation that is generic and quasi-symmetric, if there exists an ϵ that isn't parallel to any face of Σ and is $\mathcal{W}$-invariant then let,*
$\mathcal{L} = \left(-\bar{\rho} + (1/2)\overline{\Sigma}_\epsilon\right) \cap X(T)^+$
Then if $\mathcal{L} \neq \emptyset$, let $P_{\mathcal{L}} = \bigoplus_{\chi \in \mathcal{L}} P_\chi$.
Then we have that $\Lambda_{\mathcal{L}} = \mathrm{Hom}_{SW\#G}(P_{\mathcal{L}}, P_{\mathcal{L}})$ is an NCCR for SW^G.

Proof. As in Theorem 4.14 we can adapt the proof of Theorem 5.8 to allow for a shift by a $\mathcal{W}$-invariant element, so we only need to check the condition in Corollary 5.12. Again this is very

similar to the proof of Proposition 4.9.
We have $-w_0\chi_2 \in -\bar{\rho}+(1/2)\left(-\overline{\Sigma}\right)$ and by quasi-symmetry $-\overline{\Sigma} = \overline{\Sigma}$, therefore $\chi_1 - w_0\chi_2 \in -2\bar{\rho}+\overline{\Sigma}$. If we assume that $\chi_1 - w_0\chi_2 \notin -2\bar{\rho} + \Sigma$ then we would have $\bar{\rho} + \chi_1, \bar{\rho} - w_0\chi_2$ both lie in the same face of $(1/2)\overline{\Sigma}$, but this can not happen as this face would be a face of both $(1/2)\Sigma_\epsilon$ and $(1/2)\Sigma_{-\epsilon}$. Then using Corollary 4.7 and the isomorphisms after Theorem 4.14 we get that $\Lambda_{\mathcal{L}}$ is of the right form to be a NCCR. □

Remark 5.14. Requiring quasi-symmetry is a very strong restriction, the existence of a $\mathcal{W}$-invariant ϵ that is non-zero is also a strong condition, if G is semi-simple then such an ϵ does not exist. Looking closer at the proof we see that if the boundary of $(1/2)\overline{\Sigma}$ contains no weights then the proof works without changing $\overline{\Sigma}$ for $\overline{\Sigma}_\epsilon$. This is something that can be checked in a specific example, see the $SL_2(\mathbb{C})$ example 5.9 for such a situation.

For more examples of NC(C)R's see [9, Section 5-10], they also define something they call a twisted NCCR which is more general and allows for non-connected reductive groups.

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