Moduli space of genus g stable maps

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Abstract

We construct the moduli space of genus g stable maps and use it to prove Kontsevich's formula for rational plane curves, we also introduce Gromov-Witten invariants and provide a sketch of recursion for $\mathbb{P}^r$. We use the Hilbert scheme in our construction and give a detailed proof of the existence of the Hilbert scheme, we work mainly with $\mathbb{P}^r$ and only look at GW invariants in the genus 0 case, we also provide some Mathematica code for calculating GW invariants.

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1 Introduction

We want to study curves, in specific, we want to ask questions about the number of curves that satisfy certain geometric conditions, for example, going through a point, of being incident to a higher dimension subspace, etc.
To answer questions like this in a systematic way it is useful to try and find a geometric space whose points represent curves, called a moduli space. We can then hopefully translate the incidence conditions to geometric subspaces and do intersection theory in the moduli space to try and answer the questions.

As a trivial example of this, consider (directed) line segments in $\mathbb{R}^2$, one way we can describe a line segment is by giving its start and end points, we can package that as a vector (s_1, s_2, e_1, e_2) where the line segment starts at (s_1, s_2) and ends at (e_1, e_2). If we allow line segments with no length then we find that $\mathbb{R}^4$ parametrizes line segments. If we want to consider all line segments that start at a point (a, b) we can translate this into the condition of lying in a 2 dimensional plane in $\mathbb{R}^4$, and we have the same condition for all line segments ending at a given point.
If we then want the answer to the trivial question of how many line segments start at a given point and end at another given point we need to find the intersection of 2 linear subspaces of codimension 2 in $\mathbb{R}^4$, this is just a single point.

We also see that if we know the dimension of the moduli space, then for us to expect a finite number of curves we need to intersect subspaces whose codimensions sum to the dimension of the moduli space, if we also know what the codimension of a condition is, then this allows us to calculate how many conditions we need to apply.
In many simple situations it is relatively easy to calculate the "expected" dimension, as there is a straightforward way of parametrizing "nice" objects, and the "nice" objects should be dense in the moduli space, that enables us to find how many conditions we need to impose, even if we do not know what the moduli space actually is.

To do intersection theory we need a complete space and it turns out that the "nice" objects often form a noncomplete moduli space, so we need to compactify the moduli space in such a way that every point still corresponds to a curve, but we might have to allow "bad" curves, it is easy to see that we will need singularities as it is easy to find a family of curves that depend on a parameter t, such that the curve is smooth for all $t \neq 0$, but is not smooth if $t = 0$. $(xy = t)$

It can turn out that the compactification can increase the dimension, but in the situations we will consider that will not happen. We will be interested in maps from marked curves into a fixed space, often $\mathbb{P}^r$, we will have to allow maps from a reducible source with mild singularities, called stable maps, see Section 3 for the details. This compactification is by Kontsevich [9].

We will use the Hilbert scheme in our construction of the moduli space, so in Section 2 we will construct it. Finally we will study some of the enumerative properties of the moduli space in Section 4. We will prove Kontsevich's formula for rational plane curves which gives a recursive formula for the number of degree d rational planar curves. Then we will define Gromov-Witten invariants and prove some basic results about them including a sketch of recursion for $\mathbb{P}^r$.

These moduli spaces are normally talked about as stacks, but we will not use that language, we assume many introductory results from scheme theory, including results about flatness and base change, the main results used will be stated, see Hartshorne [5] or Vakil [14] for details. For an introduction to the moduli space of stable maps in the genus 0 case see [8], this book has a lot of motivation and intuition, for proofs see the notes [3].

2 The Hilbert Scheme

The first (or key) step in showing that there is a moduli space of stable maps is to show the existence of a simpler object, the Hilbert scheme which is informally the collection of all closed subschemes of a given scheme.
This section is based on [13] and [12], (which is also section 5 of the book [2]).

2.1 Representable functors and the Hilbert functor

There are two key ideas for the topic of representable functors.

First, given a specific collection of geometric objects, it would be good if we could give that collection some geometry. This sounds rather vague but for a concrete example, the projective line $\mathbb{P}^1_{\mathbb{C}}$ is the information of all lines in $\mathbb{C}^2$, but $\mathbb{P}^1_{\mathbb{C}}$ is more then just the set of lines in $\mathbb{C}^2$. In general if given some collection of geometric objects, we can give that collection a "natural" scheme structure, then we can use geometric tools to analyse and study the collection, and we can talk about two objects being "close" or "connected", or the tangent space to an object, etc.
The second idea is actually Yoneda's lemma, this tells us that maps from any scheme into a scheme X contains the same information as the original scheme X.

Putting these two ideas together we get the following definition, which is in more generality then just for schemes.

Definition 2.1. Let $\mathfrak{F} : \mathfrak{C} \to Set$ be a contravariant functor, we say that $\mathfrak{F}$ is *representable* if there exists an object X of $\mathfrak{C}$ and $\eta \in \mathfrak{F}(X)$ such that $\alpha_\eta : \mathfrak{Hom}(-, X) \to \mathfrak{F}(-)$ is a natural isomorphism of functors, where $\alpha_\eta(f) = \mathfrak{F}(f)(\eta)$.

We say that X *represents* $\mathfrak{F}$.

This is equivalent to giving X and an isomorphism $\alpha : \mathfrak{Hom}(-, X) \to \mathfrak{F}(-)$. (Yoneda, $\eta = \alpha_X(id_X)$).
As usual we get that X and η are unique up to unique isomorphism.
Many objects can be defined as representable functors, for example projective space, grassmannians, the fibre product and many more.

We also have closely related concepts of fine and coarse moduli spaces.
From now on Sch/S is the category of schemes over S, an object is a scheme X and a map $X \to S$, a morphism is a map $X \to Y$, such that the triangle formed using the maps to S commutes, when talking about objects we will drop the map to S.

Definition 2.2. A *coarse moduli space* for $\mathcal{F} : Sch/S \to Set$ is a pair (M, β) where M is a scheme (over S) and $\beta : \mathcal{F}(-) \to \mathfrak{Hom}(-, M)$ is a natural transformation such that the following properties hold.

1) For any algebraically closed field k we have $\beta_{\operatorname{Spec} k} : \mathfrak{F}(\operatorname{Spec} k) \to \mathfrak{Hom}\,(\operatorname{Spec} k, M)$ is an isomorphism.

2) If we have (M', β') satisfying 1) then we have a unique morphism $f : M \to M'$ such that

$$\begin{array}{ccc} \mathfrak{F}(-) & \xrightarrow{\beta'} & \mathfrak{Hom}\,(-, M') \\ & {\scriptstyle\beta}\searrow \quad \nearrow{\scriptstyle f\circ} & \\ & \mathfrak{Hom}\,(-, M) & \end{array}$$

commutes.

If β is isomorphism (i.e, if there is a $\epsilon \in \mathfrak{F}(M)$ such that $\beta_M(\epsilon) = id_M$ and if $\beta_T(x) = \beta_T(y)$ then $x = y$ for any T) then in fact $\mathfrak{F}$ is representable by M, and we call M (and β) a *fine moduli space.*

Part one of the definition says that the underlying set of points is the correct one. However it is possible that there is no universal family, where the *universal family* is the element $\epsilon \in \mathfrak{F}(M)$ that maps to id_M. It is universal because given any family $\mathcal{E}$ over Z we get a map $Z \to M$, and we can pullback ϵ to Z along this map, this pullback is isomorphic to our original family $\mathcal{E}$.
Part two says that M is universal.

Now we have done the basic setup we can talk about the Hilbert functor.

We want to be able to describe all subschemes of a given scheme X over S, informally we want the collection of all subschemes of X packaged together in a natural way.
For notation given a fibre product

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & S \end{array}$$

where X and Y are schemes over S, we write X_Y for the fibre product $X \times_S Y$, and if we have a sheaf $\mathcal{E}$ on X, we denote its pullback along f' by $\mathcal{E}_Y$.
The most common situation for us will be when X is projective space over S.

Definition 2.3. Let X be an S scheme, an *algebraic family of closed subschemes of X/S parameterized by T* is a closed subscheme $Z \subset X_T$.
The family Z is called *flat* if the induced map $Z \to T$ is flat.

Definition 2.4. Let X be a finite type scheme over S, where S is Noetherian.
Let $\mathfrak{Hilb}_{X/S}\,(T)$ be the set of flat algebraic subschemes of X_T parametrized by T.

We have that $\mathfrak{Hilb}_{X/S}$ is actually a contravariant functor as given $T^{'} \to T$ and $Z \in \mathfrak{Hilb}_{X/S}(T)$ we can pullback Z to $Z \times_T T'$ and get an element of $\mathfrak{Hilb}_{X/S}(T^{'})$.
Looking at the definition of flatness does not explain why we require it, but flatness will be used many times. One way to think about flatness is as an algebraic version of continuity (without it, the moduli spaces are huge and can be more badly behaved), the following result also provides some motivation.

Lemma 2.5. *Let $X \to S$ be projective, then for an algebraic family Z parametrized by T we have a Hilbert polynomial P_{Z_t} associated to the subscheme Z_t of $\mathbb{P}^N_S$ for any geometric point t of T.*
If T is connected then Z is a flat family if and only if P_{Z_t} is independent of t.

From now on assume that $X \to S$ is projective, we want to show that $\mathfrak{Hilb}_{X/S}$ is representable. Using the above lemma we see that if it is representable it will be a collection of open and closed subschemes indexed by Hilbert polynomials. Let $\mathfrak{Hilb}^P_{X/S}$ be the subfunctor of $\mathfrak{Hilb}_{X/S}$ where the flat families have Hilbert polynomial P. ($\mathfrak{F}$ is a *subfunctor* of $\mathfrak{G}$ if $\mathfrak{F}(X) \subset \mathfrak{G}(X)$ for all X, and the inclusion of functors is a natural transformation.)
It is therefore sufficient to show that $\mathfrak{Hilb}^P_{X/S}$ is representable, before we do so we need a few more results.

2.2 m-regularity

The first result we need is a refining of the vanishing of cohomology after twisting enough.

Definition 2.6. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$, we say that it is *m-regular* if $H^i\left(\mathbb{P}^n, \mathcal{F}(m-i)\right) = 0$ for all $i > 0$.

We will sometimes just write $H^i(\mathcal{F})$ if the space is clear enough. It is also known as Castelnuovo-Mumford regularity, as the definition was first given by Mumford, based on ideas of Castelnuovo.

Proposition 2.7. *Let $\mathcal{F}$ be an m-regular sheaf on $\mathbb{P}^n$, then we have*

1) $\mathcal{F}$ is m'-regular for all $m' \geq m$

2) $H^0\left(\mathcal{F}(m')\right) \otimes H^0\left(\mathcal{O}_{\mathbb{P}^n}(1)\right) \to H^0\left(\mathcal{F}(m'+1)\right)$ is surjective for $m' \geq m$

3) $\mathcal{F}(m')$ is generated by global sections for $m' \geq m$

Proof. We will show this by using induction on n.
If $n = 0$ all the 3 statements hold.

Assume $n > 0$, first we note that we have the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0$$

for a generic hyperplane $H \cong \mathbb{P}^{n-1}$. As H is generic we get the following exact sequence

$$0 \to \mathcal{F}(k-1) \to \mathcal{F}(k) \to \mathcal{F}_H(k) \to 0$$

by tensoring with $\mathcal{F}(k)$. Looking at part of the associated long exact sequence we get

$$H^i(\mathcal{F}(m-i)) \to H^i(\mathcal{F}_H(m-i)) \to H^{i+1}(\mathcal{F}(m-i-1))$$

this shows that $\mathcal{F}_H$ is also m-regular, and therefore $1), 2), 3)$ hold for it as $\dim H < n$.
Another part of the long exact sequence gives us

$$H^i(\mathcal{F}(m-i)) \to H^i(\mathcal{F}(m-i+1)) \to H^i(\mathcal{F}_H(m-i+1))$$

again for $i > 0$. The first term is zero by m-regularity and the third term is zero by the induction assumption, therefore $H^i(\mathcal{F}(m-i+1))$ is zero, and so $\mathcal{F}$ is $(m+1)$-regular. By iterating this process we get 1).

To show 2) consider the diagram.

$$\begin{array}{ccc} H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) & \xrightarrow{i} & H^0(\mathcal{F}_H(k)) \otimes H^0(\mathcal{O}_H(1)) \\ \downarrow \alpha & & \downarrow \beta \\ H^0(\mathcal{F}(k+1)) & \xrightarrow{j} & H^0(\mathcal{F}_H(k+1)) \end{array}$$

By the induction assumption, for $k \geq m$ we have that β is surjective. We also have the following two sequences.

$$H^0(\mathcal{F}(k)) \to H^0(\mathcal{F}_H(k)) \to H^1(\mathcal{F}(k-1))$$
$$H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathcal{O}_H(1)) \to H^1(\mathcal{O}_{\mathbb{P}^n})$$

In the first sequence the final term is zero by 1) which has been shown already. The final term of the second sequence is zero by standard results, therefore both of the maps at the start of the sequences are surjective. As i is the tensor product of these maps it is therefore also surjective.

The diagram commutes and therefore $j \circ \alpha$ is also surjective.
We also have that $\ker j = H^0(\mathcal{F}(k))$ and therefore $\ker j \subseteq \operatorname{Im} \alpha$.

Let $b \in H^0(\mathcal{F}(k+1))$ and let a be any lift of $j(b)$ to $H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, then $b = \alpha(a) + (b - \alpha(a))$, this shows that $H^0(\mathcal{F}(k+1)) = \operatorname{Im} \alpha + \ker j$.

Putting these together we get that $H^0(\mathcal{F}(k+1)) = \operatorname{Im} \alpha$ and therefore 2) is proved.

To prove 3), note that we already have the result for $k >> 0$ by Serre, then use 2) to see that the global sections of $\mathcal{F}(k+1)$ are expressible in terms of the global sections of $\mathcal{F}(k)$. We can continue to reduce k this way until $k = m$, this shows 3). □

Proposition 2.8. *Let P be a Hilbert polynomial, then there exists an integer m depending on P such that any closed subscheme, Z, of $\mathbb{P}^n$ with Hilbert polynomial P has an m-regular ideal sheaf $\mathcal{I}_Z$.*

Proof. We will show this by using induction on n.
Again $n = 0$ is trivial, so assume $n > 0$, let $\mathcal{I}_Z$ be denoted by $\mathcal{I}$, and let H be a generic hyperplane, then as above we get the exact sequence

$$0 \to \mathcal{I}(-1) \to \mathcal{I} \to \mathcal{I}_H \to 0$$

We have that $\mathcal{I}_H$ is an ideal sheaf of $\mathcal{O}_H$ so by the induction assumption there is a m_1 such that $\mathcal{I}_H$ is m_1-regular. We also have that the Hilbert polynomial of H is expressible in terms of the Hilbert polynomial of Z using the long exact sequence. (H has Hilbert polynomial $P(t) - P(t-1)$). So in fact m_1 depends on P.
Looking at the following part of the associated long exact sequence

$$H^{i-1}(\mathcal{I}_H(k+1)) \to H^i(\mathcal{I}(k)) \to H^i(\mathcal{I}(k+1)) \to H^i(\mathcal{I}_H(k+1))$$

we get that $H^i(\mathcal{I}(k)) \cong H^i(\mathcal{I}(k+1))$ for $i > 1$ and all $k \geq m_1 - i$ as $\mathcal{I}_H$ is m_1-regular. Using Serre vanishing we get that in fact these cohomology groups must be zero and therefore $\mathcal{I}$ is m_1-regular apart from the vanishing of $H^1(\mathcal{I}(m_1 - 1))$.

Therefore all we need to do is find an $m \geq m_1$ depending on P such that $H^1(\mathcal{I}(m-1)) = 0$.
To do this consider the exact sequence

$$H^0(\mathcal{I}(m+1)) \xrightarrow{j_m} H^0(\mathcal{I}_H(m+1)) \to H^1(\mathcal{I}(m)) \to H^1(\mathcal{I}(m+1)) \to 0$$

for $m \geq m_1 - 2$. This shows that $h^1(\mathcal{I}(m)) \geq h^1(\mathcal{I}(m+1))$
If this was an equality we would have j_m is surjective and by looking at the proof of Proposition 2.7 part 2) we see that j_m surjective implies that j_{m+1} is also surjective and therefore that $h^1(\mathcal{I}(m+1)) = h^1(\mathcal{I}(m+2)) = \cdots$.
By Serre vanishing these must all be zero, and therefore the $h^1(\mathcal{I}(m))$ form a strictly decreasing sequence to zero.
This implies that $H^1(\mathcal{I}(l)) = 0$ for all $l \geq m_1 - 1 + h^1(\mathcal{I}(m_1 - 1))$.
So all we need to do is find such an l that depends only on P.
We have the exact sequence

$$H^0(\mathcal{O}_Z(k)) \to H^1(\mathcal{I}(k)) \to 0$$

for all $k \geq 0$. We also have

$$H^i(\mathcal{O}_{\mathbb{P}^n}(k)) \to H^i(\mathcal{O}_Z(k)) \to H^{i+1}(\mathcal{I}(k))$$

which shows that $H^i(\mathcal{O}_Z(k)) = 0$ for all $i \geq 1$ and $k \geq m_1 - 2$. (recall that $\mathcal{I} = \mathcal{I}_Z$)

Putting these two facts together gives us that $h^1(\mathcal{I}(m_1 - 1)) \leq P(m_1 - 1)$ (m_1 depends on P, so m depends on P.) Therefore if we let $m = m_1 - 1 + P(m_1 - 1)$ we have the wanted result. ☐

2.3 Flattening stratification

If we have a non-flat sheaf over a base S, we need to be able to divide up the base into pieces, above which our sheaf is flat, before we do this we first recall several results on flatness.

Theorem 2.9 (Base change for flat sheaves). *Let F be coherent on $\mathbb{P}^n_S$, flat over S, then if for some $i \geq 0$ we have that $\dim_{K(s)} H^i(\mathbb{P}^n_s, \mathcal{F}_s) = d$ for all $s \in S$ we then have that $R^i\pi_*\mathcal{F}$ is locally free of rank d and $\left(R^{i-1}\pi_*\mathcal{F}\right)_s \to H^{i-1}(\mathbb{P}^n_s, \mathcal{F}_s)$ is an isomorphism for all $s \in S$.*

This is just one part of a more general theorem about base change for flat sheaves, see for example [5, III, Theorem 12.11]. The main way we will use it is when we twist the sheaf $\mathcal{F}$ enough so that all the higher cohomology is zero and therefore the dimension of the zeroth cohomology is determined by the Hilbert polynomial of $\mathcal{F}$. In this case we get that $\pi_{S*}\mathcal{F}(l)$ is locally free of rank $P(l)$ (P the Hilbert polynomial of F, $l >> 0$) and the base change map is an isomorphism for all i, however only $i = 0$ is non-trivial.

We also have a result on base change for when $\mathcal{F}$ coherent on $\mathbb{P}^n_S$ is not necessarily flat over S.

Lemma 2.10. *Consider the following diagram, and let $\mathcal{F}$ be coherent on $\mathbb{P}^n_S$*

$$\begin{array}{ccc} \mathbb{P}^n_T & \longrightarrow & \mathbb{P}^n_S \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi} \\ T & \longrightarrow & S \end{array}$$

Then there exists m such that for all $m' \geq m$ we have that the base change maps

$$\left(\pi_*\mathcal{F}(m')\right)_T \to \pi_{T*}\mathcal{F}_T(m')$$

are isomorphisms for any $T \to S$.

We also have the maps associated to the higher direct images are isomorphisms, but they are all zero, so the above maps are the only non-trivial ones.

Lemma 2.11 (Generic flatness)**.** *Let $\mathcal{F}$ be coherent on $\mathbb{P}^n_S$ and assume that S is an integral scheme then there exists a non-empty open $U \subset S$ such that $\mathcal{F}_U$ is flat over U*

Lemma 2.12. *Let $\mathcal{F}$ be coherent on $\mathbb{P}^n_S$, then $\mathcal{F}$ is flat over S iff $\pi_{S*}\mathcal{F}(l)$ is locally free for all $l >> 0$.*

Proof. sketch only;
If $\mathcal{F}$ is flat this follows from base change above, if the converse holds let $\bar{\mathcal{F}} = \bigoplus_{i \geq l} \pi_{S*}\mathcal{F}(l)$. This is flat over S and $\mathcal{F} = \tilde{\bar{\mathcal{F}}}$, where $\widetilde{M}$ is the sheaf on $\mathbb{P}^n_S$ associated to M. Going from M to $\widetilde{M}$ preserves flatness so we get that $\mathcal{F}$ is flat. □

Definition 2.13. Let $\mathcal{F}$ be coherent on $\mathbb{P}^n_S$.
A *flattening stratification* for $\mathcal{F}$ over S is a finite disjoint collection, S_i, of locally closed subschemes of S, whose union as a set is S, such that for $g : T \to S$ we have that $\mathcal{F}_T$ is flat if and only if $g^{-1}S_i$ is open and closed in T.

If T is connected, this says that $T \to S$ factors through one of the S_i.
To help us find flattening stratifications we need a definition that is motivated by the ideal generated by the minors of a matrix.

Definition 2.14. Let n be a positive integer, let S be a scheme, and $\mathcal{F}$ a coherent sheaf on S. Let $\mathcal{E}_2 \xrightarrow{f} \mathcal{E}_1$ be a locally free presentation of $\mathcal{F}$, where $\mathcal{E}_i$ is locally free of rank e_i.

We have an induced map

$$\bigwedge^{e_1-n} \mathcal{E}_2 \otimes \bigwedge^{e_1-n} \mathcal{E}_1^* \to \mathcal{O}_S$$

given by $x_1 \wedge \cdots x_{e_1-n} \otimes g_1 \wedge \cdots \wedge g_{e_1-n} \mapsto g_1\left(f(x_1)\right) \cdots g_{e_1-n}\left(f(x_{e_1-n})\right)$.
Let the image of this map be $F_n(\mathcal{F})$, if $n \geq e_1$, then we let $F_n(\mathcal{F}) = \mathcal{O}_S$. This is known as the n^{th} *Fitting ideal.*

One can show that this doesn't depend on the choice of locally free presentation.

Lemma 2.15. *Let $\mathcal{F}$ be coherent on S, and let $n > 0$, then $\mathcal{F}$ is locally free of rank n if and only if $F_{n-1}(\mathcal{F}) = 0$ and $F_n(\mathcal{F}) = \mathcal{O}_S$.*

Proof. Sketch only;
If $\mathcal{F}$ is locally free of rank of rank n then we have the presentation $0 \to \mathcal{F}$ and the result follows.

To show the other implication, reduce to the affine local case and assume that $\mathcal{E}_2 \xrightarrow{f} \mathcal{E}_1$ is a free presentation of $\mathcal{F}$. Then f can be thought of as a matrix and we have that $F_i(\mathcal{F})$ is the ideal generated by the $(e_1 - i) \times (e_1 - i)$ minors of f.
The assumption that $F_n(\mathcal{F}) = \mathcal{O}_S$ (and being a local ring) tells us that f has an invertible $e_1 - n$ minor, we can use this minor to split off part of the presentation and be left with a presentation which has $e_1 = n$. Then we use $F_{n-1}(\mathcal{F}) = 0$ to get that $\mathcal{F}$ is locally free of rank n. (The map f must be 0, this is easy to see using both perspectives, either all the 1×1 minors are 0, or any dual map on the image of f is zero). □

Proposition 2.16. *Let $\mathcal{F}$ be coherent on $\mathbb{P}^n_S$.*
Then there exists a flattening stratification indexed by Hilbert polynomials such that for any $T \to S$, $\mathcal{F}_T$ has Hilbert polynomial P if and only if $T \to S$ factors through S_P.

Proof. We will prove this by first proving it for $n = 0$ and then deducing the result from this case.

If $n = 0$, we have $\mathcal{F}$ coherent on S, and we have the Fitting ideals, $F_i(\mathcal{F})$, let $U_i = V\left(F_{i-1}(\mathcal{F})\right) \backslash V\left(F_i(\mathcal{F})\right)$ locally closed. Lemma 2.15 says that $\mathcal{F}_T$ is locally free of rank i if and only if $T \to S$ factors through U_i.

Now let $n > 0$, then generic flatness says that there is an open, non-empty subset $V \subset S_{red}$ such that $\mathcal{F}_U$ is flat over U. (To apply generic flatness we need S to be integral and it might not be, so to do this restrict to one irreducible component and then take the reduced subscheme structure.) As S is Noetherian, we find finitely many U_i. (Once we have found U_1, consider $S_{red} \backslash U_1$ and repeat).
This gives us finitely many subschemes U_i of S, whose union as a set is S such that $\mathcal{F}_{U_i}$ is flat over U_i. This gives us that only finitely many Hilbert polynomials appear as the Hilbert polynomials of the fibres, re-index the U_i by the Hilbert polynomials P_i that appear.

Now we can find $m >> 0$ such that all the fibres are m-regular (for us what we need is $H^i\left(\mathcal{F}_s(l)\right) = 0$. for all $l \geq m$.)

This is done by first finding m_{P_i} such that $\mathcal{F}_{U_{P_i}}$ is m-regular. This can be done by a very similar arguement to Proposition 2.8, or by simply using Serre vanishing and base change.
Using Lemma 2.10 we find that for each U_{P_j} there exists a m'_{P_j} such that for

$$\begin{array}{ccc} \mathbb{P}^n_{U_{P_j}} & \longrightarrow & \mathbb{P}^n_S \\ \downarrow{\scriptstyle \pi_{U_{P_j}}} & & \downarrow{\scriptstyle \pi_S} \\ U_{P_j} & \longrightarrow & S \end{array}$$

we have an isomorphism

$$(\pi_{S*}\mathcal{F}(l))_{U_{P_j}} \cong \pi_{U_{P_j}*}\mathcal{F}_{U_{P_j}}(l)$$

for all $l \geq m'_{P_j}$ and the higher pushforwards are all zero (and therefore also isomorphisms).
Now let $m \geq \max_{i,j}\{m_{P_i}, m'_{P_i}\}$.

Then we have

$$\begin{aligned} H^0\left(\mathbb{P}^n_s, \mathcal{F}_s(l)\right) &\cong \left(\pi_{U_{P_i}*}\mathcal{F}_{U_{P_i}}\right)_s && \text{(by base change)} \\ &\cong \left((\pi_{S*}\mathcal{F}(l))_{U_{P_j}}\right)_s && \text{(by above isomorphism)} \\ &= (\pi_{S*}\mathcal{F}(l))_s \end{aligned}$$

for all $l \geq m$, and we also have that all the higher cohomology on the fibres is zero.
Therefore for $\mathcal{F}_s$, it's Hilbert polynomial for any $s \in S$ is determined by $h^0\left(\mathcal{F}_s(i)\right)$ for $i = m, m+1, \ldots, m+n$.
Then for any $l \geq m+n$ let

$$\bar{\mathcal{F}}_l = \bigoplus_{i=m}^{l} \pi_{S*}\mathcal{F}(i)$$

This is a coherent sheaf on S so it has a flattening stratification by Fitting ideals. Let this stratification be $\{S^l_P\}$.
By construction the Hilbert polynomial of $\mathcal{F}$ is constant in S^l_P, and it is P.
We have that $\left(\bar{\mathcal{F}}_l\right)_{S^l_P}$ is locally free by properties of the stratification, and therefore when $l \geq i \geq m$ we have that $(\pi_{S*}\mathcal{F}(i))_{S^l_p}$ is also locally free. This implies that we have

$$\cdots \supseteq S^{l-1}_P \supseteq S^l_P \supseteq S^{l+1}_P \supseteq \cdots$$

and therefore by Noetherianess this stabilizers (each S^l_P has the same underlying subset) and we get locally closed subschemes S_P of S whose union is S such that $(\pi_{S*}\mathcal{F}(i))_{S_P}$ is locally free (of rank $P(i)$) for all $i \geq m$.
Therefore by Lemma 2.12 we have that in fact $\mathcal{F}_{S_P}$ is flat over S_P with Hilbert polynomial P on each fibre. □

2.4 The Hilbert functor is representable

Now that we have the above collection of results we can go ahead and show that the Hilbert functor is representable.

Theorem 2.17. *The functor $\mathfrak{Hilb}^P_{X/S}$ is representable for $X \to S$ projective, S Noetherian, by a projective scheme $Hilb^P_{X/S}$ over S.*

We will prove this result in three steps, first we will find a natural transformation from the Hilbert functor to the Grassmannian functor in a special case. Second we will show this natural transformation corresponds to an embedding. Third and finally we will reduce to the special case.

The first step is to prove the following

Proposition 2.18. *There exists an injective morphism of functors*

$$\alpha : \mathfrak{Hilb}^P_{\mathbb{P}^n} \to \mathfrak{Grass}^{P(m)}\left(\pi_* \mathcal{O}_{\mathbb{P}^n}(m)\right)$$

Where P is a Hilbert polynomial and m is a large enough integer.

Proof. Let m be at least as large as the integer we find in Proposition 2.8, we have the exact sequence.

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_{\mathbb{P}^n_T} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

Twist it by m and apply π_{T*} this gives us

$$\pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(m) \longrightarrow \pi_{T*}\mathcal{O}_Z(m) \longrightarrow 0$$

This is as $R^1\pi_{T*}\mathcal{I}_Z(m) = 0$ (by Proposition 2.7 and base change for flat sheaves).
We have the fibre product diagram

$$\begin{array}{ccc} \mathbb{P}^n_T & \longrightarrow & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi} \\ T & \longrightarrow & \mathbb{Z} \end{array}$$

and we have $\pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(m) \cong (\pi_*\mathcal{O}_{\mathbb{P}^n}(m))_T$ therefore we have a surjective map $(\pi_*\mathcal{O}_{\mathbb{P}^n}(m))_T \to \pi_{T*}\mathcal{O}_Z(m)$.
We also have that $\pi_{T*}\mathcal{O}_Z(m)$ is locally free of rank $P(m)$ (again by base change for flat sheaves) therefore we have an element of $\mathfrak{Grass}^{P(m)}\left(\pi_*\mathcal{O}_{\mathbb{P}^n}(m)\right)(T)$.
Base change also gives us that this is a natural transformation of functors, therefore we have the map α.

We want to show that α is injective, so let Z give us an element of $\mathfrak{Hilb}^P_{\mathbb{P}^n}(T)$, apply α to get

$$0 \longrightarrow \pi_{T*}\mathcal{I}_Z(m) \longrightarrow \pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(m) \longrightarrow \pi_{T*}\mathcal{O}_Z(m) \longrightarrow 0$$

Pull this back to $\mathbb{P}^n_T$ to get

$$\pi_T{}^*\pi_{T*}\mathcal{I}_Z(m) \longrightarrow \pi_T{}^*\pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(m)$$

We also have the natural map

$$\pi_T{}^*\pi_{T*}\mathcal{O}_{\mathbb{P}^n_T}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^n_T}(m)$$

and composing these two map we get

$$\pi_T{}^*\pi_{T*}\mathcal{I}_Z(m) \longrightarrow \mathcal{O}_{\mathbb{P}^n_T}(m)$$

The image is the sheaf generated by the global sections of $\mathcal{I}_Z(m)$, so by Proposition 2.7, part 3) the image of this map is just $\mathcal{I}_Z(m)$. This shows that we can recover Z from it's image under α, therefore α is injective. □

The next step is to prove that the image is a *locally closed subfunctor*, in this situation it means that the image looks like $\mathfrak{Hom}(-, W)$ where W is locally closed in the Grassmannian scheme. There is a general notation of what it means to be a closed/open/locally closed subfunctor, but the above is enough for our needs, it implies that as the Grassmannian is representable, the Hilbert functor is represented by a locally closed subscheme.

Proposition 2.19. *Let α be as above, then the image of α is equal to the subfunctor $\mathfrak{Hom}(-, G_P)$ where G_P will be defined in the proof.*

Proof. We have that the Grassmannian functor is representable by a scheme $G = Grass^{P(m)}(\pi_*\mathcal{O}_{\mathbb{P}^n}(m))$. (i.e. $\mathfrak{Hom}(-, G) \cong \mathfrak{Grass}^{P(m)}(\pi_*\mathcal{O}_{\mathbb{P}^n}(m))(-)$.)
There is also universal quotient $(\pi_*\mathcal{O}_{\mathbb{P}^n}(m))_G \to \mathcal{Q}$ which gives us

$$0 \longrightarrow \mathcal{K} \longrightarrow (\pi_*\mathcal{O}_{\mathbb{P}^n}(m))_G \longrightarrow \mathcal{Q} \longrightarrow 0$$

Pull the inclusion back to $\mathbb{P}^n_G$ and compose with the natural map as in the previous proof to get

$$\pi_G{}^*\mathcal{K} \longrightarrow \mathcal{O}_{\mathbb{P}^n_G}(m)$$

The image is $\mathcal{I}_Y$ for some closed subscheme Y of $\mathbb{P}^n_G$.
By Proposition 2.16 the sheaf $\mathcal{O}_Y$ has a flattening stratification, let G_P be the part indexed by P. Then given an element Z of $\mathfrak{Hilb}^P_{\mathbb{P}^n}(T)$ we get an element of $\mathfrak{Grass}^{P(m)}(\pi_*\mathcal{O}_{\mathbb{P}^n}(m))(T)$ and therefore a map $j : T \to G$, by the universality of $\mathcal{Q}$ we have that the pullback of Y along j is Z. Now Z is flat over T with Hilbert polynomial P, therefore by the definition of the flattening stratification j factors through G_P, and we have that the image of α lies in $\mathfrak{Hom}(-, G_P)$.

Given a map $T \to G_P$ we have that Y_{G_P} is flat over G_P, so the pullback to T is also flat, and therefore gives an element of $\mathfrak{Hilb}^P_{\mathbb{P}^n}(T)$.
This gives us a map $\beta : \mathfrak{Hom}(-, G_P) \to \mathfrak{Hilb}^P_{\mathbb{P}^n}(-)$ and by construction it is an inverse to α. □

The two results above show that $\mathfrak{Hilb}^P_{\mathbb{P}^n}$ is representable by a quasi-projective scheme over $\mathbb{Z}$. If we have a DVR and Z is flat over the complement of the closed point, then it extends uniquely to a flat subscheme over the DVR by [5, III, Proposition 9.8]. This tells us that in fact it is also proper by the valuation criterion for properness.
We can now prove Theorem 2.17.

Proof of Theorem 2.17. Let X be projective over S and fix a closed embedding $X \subset \mathbb{P}^n_S$. First we want to show that it is sufficient to consider $\mathbb{P}^n_S$. We assume that $\mathfrak{Hilb}^P_{\mathbb{P}^n_S}$ is representable by H, with

Z being the universal subscheme of $\mathbb{P}^n_H$. We also have maps

$$(\pi_* \mathcal{I}_X(m))_H \to \left(\pi_* \mathcal{O}_{\mathbb{P}^n_S}(m)\right)_H$$

$$\pi_{H*}\mathcal{O}_{\mathbb{P}^n_H}(m) \to \pi_{H*}\mathcal{O}_Z(m)$$

coming from the normal short exact sequences, here m is large enough for the second map to be surjective, and such that we have the isomorphisms from Lemma 2.10.
Base change gives us that $\left(\pi_* \mathcal{O}_{\mathbb{P}^n_S}(m)\right) \cong \pi_{H*}\mathcal{O}_{\mathbb{P}^n_H}(m)$ so we can compose the maps, call it γ.
Now let $t : T \to H$, we can pullback γ along t. Using base change we find that $((\pi_* \mathcal{I}_X(m))_H)_T \cong \pi_{T*}\mathcal{I}_{X_T}(m)$ and that $(\pi_{H*}\mathcal{O}_Z(m))_T \cong \pi_{T*}\mathcal{O}_{Z_T}(m)$, using both these isomorphism we have that

$$t^*(\gamma) : \pi_{T*}\mathcal{I}_{X_T}(m) \to \pi_{T*}\mathcal{O}_{Z_T}(m)$$

By the construction of γ we have that $t^*(\gamma) = 0$ iff $\pi_{T*}\mathcal{I}_{X_T} \subset \pi_{T*}\mathcal{I}_{Z_T}$ iff $\mathcal{I}_{X_T} \subset \mathcal{I}_{Z_T}$ iff $Z_T \subset X_T$.
This show that the zero locus of γ represents $\mathfrak{Hilb}^P_{X/S}$.
Therefore it is sufficient to consider $X = \mathbb{P}^n_S$.

The final step is showing that in fact we can assume $S = \mathbb{Z}$, we can do this as the Hilbert functor interacts well with base change.
To see this, consider $\mathfrak{Hilb}^P_{\mathbb{P}^n}(T) \times \mathfrak{Hom}(T, S)$, an element is a closed subscheme Z of $\mathbb{P}^n_T$ and a map $T \to S$. We have the diagram

$$\begin{array}{ccccc} \mathbb{P}^n_T & \longrightarrow & \mathbb{P}^n_S & \longrightarrow & \mathbb{P}^n \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & S & \longrightarrow & \mathbb{Z} \end{array}$$

and $(\mathbb{P}^n_S)_T = \mathbb{P}^n_T$ so we have an element of $\mathfrak{Hilb}^P_{\mathbb{P}^n_S}(T)$.
Similarly an element of $\mathfrak{Hilb}^P_{\mathbb{P}^n_S}(T)$ is a closed subscheme of $(\mathbb{P}^n_S)_T$, this is the information of a map $T \to S$ and a closed subscheme of $(\mathbb{P}^n)_T$, therefore we get an element of $\mathfrak{Hilb}^P_{\mathbb{P}^n}(T) \times \mathfrak{Hom}(T, S)$.
These procedures are inverse to each other so we have that

$$\mathfrak{Hilb}^P_{\mathbb{P}^n}(-) \times \mathfrak{Hom}(-, S) \cong \mathfrak{Hilb}^P_{\mathbb{P}^n_S}(-)$$

If $\mathfrak{Hilb}^P_{\mathbb{P}^n}$ is representable by H we also have

$$\mathfrak{Hilb}^P_{\mathbb{P}^n}(-) \times \mathfrak{Hom}(-, S) \cong \mathfrak{Hom}(-, H) \times \mathfrak{Hom}(-, S) \cong \mathfrak{Hom}(-, H \times S)$$

This shows that $H \times S$ represents $\mathfrak{Hilb}^P_{\mathbb{P}^n_S}$.
Therefore it is sufficient to prove that $\mathfrak{Hilb}^P_{\mathbb{P}^n}$ is representable, but we have already shown this in Proposition 2.19. □

The above proof can be generalized to deal with quasi-projective schemes and to show that the quotient functor is representable, however we do not need this level of generality, see [12, Section 5.6] for more details.

3 Moduli space of genus g stable maps

This section is mainly based on [3], everything is over $\mathbb{C}$ for simplicity, for a more conceptual construction, without proofs see [8].
Informally what we want to do is describe all maps from curves into a given space, that is a rather vague idea so we will seek to make it a bit more concrete.
First of all there is the related idea of classifying all curves of a given genus, if one restricts to smooth curves, this can be done and we get a coarse moduli space M_g. We can also allow marked points on the curve, again one can find a coarse moduli space $M_{g,n}$ of smooth genus g curves with n marked distinct points, both spaces are smooth. (In fact $M_{g,n}$ is a fine moduli space if n is large enough)
There is an issue with this space, it is not complete, one can not always take limits, and one can not do intersection theory. We want a way to compactify this space in a natural geometric way, it turns out that by allowing stable n-pointed genus g curves we can get a complete moduli space $\overline{M}_{g,n}$, one can also compactify M_g. For more details on M_g and $\overline{M_g}$ see [4], see [1] for Deligne and Mumford's original paper on the irreducibility of $\overline{M_g}$, see Knudsen [7] for $\overline{M}_{g,n}$.

Definition 3.1. A *stable n-pointed genus g curve* is a connected curve C with arithmetic genus g that has only nodes as singularities, each rational component has at least 3 special points and each component with genus 1 has at least one special point, where a *special point* is either a marked point or a node (an intersection point between components).

The conditions for stability are equivalent to C having finitely many automorphisms that fix the marked points.
It turns out that $\overline{M}_{g,n}$ is projective, but not smooth, unless $g = 0$, in this case it is also a fine moduli space.

3.1 Stable maps

We don't want to just consider curves, we also want to look at their images in a given space, for us we will restrict to looking at maps into $\mathbb{P}^n$, however we will give the definition in more generality

Definition 3.2. An *n-pointed, genus g, quasi-stable curve* $(C, p_1, \dots, p_n,)$ is a projective, connected, reduced curve with arithmetic genus g, which has at worse nodal singularities, and such that the n marked points p_i are distinct and non-singular
A *quasi-stable family* of n-pointed, genus g, quasi-stable curves over S is a flat, projective map $\pi : \mathcal{C} \to S$, with n sections p_i, such that each geometric fibre is a n-pointed, genus g, quasi-stable curve.
Let X be a scheme, a *quasi-stable family of maps* over S from n-pointed, genus g, quasi-stable curves to X is the data $(\pi : \mathcal{C} \to S, p_1, \dots, p_n, \mu : \mathcal{C} \to X)$ where $(\pi : \mathcal{C} \to S, p_1, \dots, p_n,)$ is a family of n-pointed, genus g, quasi-stable curves over S, and $\mu : \mathcal{C} \to X$ is a morphism.

We say two such families of maps

$$(\pi : \mathcal{C} \to S, p_1, \dots, p_n, \mu : \mathcal{C} \to X), (\pi' : \mathcal{C}' \to S, p_1, \dots, p_n, \mu' : \mathcal{C}' \to X)$$

are isomorphic if there exists an isomorphism of schemes $\alpha : \mathcal{C} \to \mathcal{C}'$ such that

- $\pi = \pi' \circ \alpha$
- $p_i' = \alpha \circ p_i$
- $\mu = \mu' \circ \alpha$

If $(C, p_1, \dots, p_n, \mu)$ is a quasi-stable map to X, then the *special points* of an irreducible component $E \subset C$ are the marked points in E and the intersection points between E and other components of C.
We say the map $(C, p_1, \dots, p_n, \mu)$ is *stable* if in addition we have that every irreducible component of genus 0 that is mapped to a point by μ has at least 3 special points, and every irreducible component of genus 1 that is mapped to a point by μ has at least 1 special point.
A family of maps $(\pi : \mathcal{C} \to S, p_1, \dots, p_n, \mu)$ is *stable* if each geometric fibre is stable.

Lemma 3.3. *A quasi-stable map $(C, p_1, \dots, p_n, \mu)$ is stable if and only if $(C, p_1, \dots, p_n, \mu)$ has finitely many automorphisms*

Proof. As the automorphism needs to preserve the map μ, we only have to consider components that are sent to a single point by μ.

If the genus of the component is at least 2, we have finitely many automorphisms.
If the genus is 1 we have that the automorphisms are translations semidirect producted with a finite group, so having to fix a point removes the translations, so the automorphism group is finite iff we have to fix a point.
If the genus is 0, then there exists a unique automorphism of $\mathbb{P}^1$ sending 3 distinct points to $0, 1, \infty$, so we have no automorphisms iff we have to fix at least 3 points. □

This gives some motivation for the definition of stability.
We also have another equivalent condition for stability if $X = \mathbb{P}^r$

Lemma 3.4. *Let $(C, p_1, \dots, p_n, \mu)$ be a quasi-stable map to $\mathbb{P}^r$, let ω_C be the dualizing sheaf, then $(C, p_1, \dots, p_n, \mu)$ is stable if and only if $\omega_C(p_1, \dots, p_n) \otimes \mu^*(\mathcal{O}_{\mathbb{P}^r}(3))$ is ample.*

Proof. Let $\mathcal{L} = \omega_C(p_1, \dots, p_n) \otimes \mu^*(\mathcal{O}_{\mathbb{P}^r}(3))$
This result is shown by checking that the degree of $\mathcal{L}$ is positive on each irreducible component $E \subset C$.
If E has genus at least 2, or if μ doesn't contract E, then the degree is at least 1.
If E has genus 1 and μ contracts E, then the degree of ω_C on E is 0 and each special point increases the degree by 1, so $\mathcal{L}$ has positive degree if and only if there is at least 1 special point.
If E has genus 0 and is contracted by μ we have that that degree of ω_C on E is -2, therefore $\mathcal{L}$ has positive degree on E if and only if we have at least 3 special points. □

Now let X be a scheme over $\mathbb{C}$ and let $\beta \in H_2(X, \mathbb{Z})$, then we say that $\mu : C \to X$ *represents* β if $\mu_*([C]) = \beta$.

Definition 3.5. Fix X and β as above, and define the contravariant functor $\overline{\mathscr{M}_{g,n}}(X, \beta)$ by setting $\overline{\mathscr{M}_{g,n}}(X, \beta)(S)$ to be the set of isomorphism classes of stable families of maps over S of n-pointed,

genus g curves to X that represent β.
If $X = \mathbb{P}^r$, write d for β corresponding to curves of degree d.

Theorem 3.6. *The functor* $\overline{\mathcal{M}_{g,n}}(\mathbb{P}^r, d)$ *has a coarse moduli space*

3.2 The moduli space of genus g stable maps exists

Before we can show that the moduli space exists we need a few more results.

Lemma 3.7. *Let* $(C, p_1, \ldots, p_n, \mu)$ *be a stable map, then* $\mathcal{L} = \omega_C(p_1, \ldots, p_n) \otimes \mu^*(\mathcal{O}_{\mathbb{P}^r}(3))$ *is ample, and there exists* t *depending on* g, n, r, d*, but not on* C *such that we have*

- $h^1(C, \mathcal{L}^t) = 0$
- $\mathcal{L}^t$ *is very ample.*

Proof. By Lemma 3.4 the degree of $\mathcal{L}$ is at least 1 on each component and the degree of ω_C is at most $2g-2$, therefore if $t \geq 2g$ we have

$$\deg \mathcal{L}^{-t} \otimes \omega_C = \deg \omega_C - t \deg \mathcal{L} \leq 2g - 2 - 2g < 0$$

So as we have $h^1(C, \mathcal{L}^t) = h^0(C, \mathcal{L}^{-t} \otimes \omega_C)$ we get the first condition for $t \geq 2g$.

To show the second condition, it is sufficient to show that $H^1(C, \mathcal{L}^t \otimes \mathcal{I}_p \mathcal{I}_q) = 0$ for all $p, q \in C$. (By a long exact sequence we get that $\mathcal{L}^t$ separates points and tangents)
So equivalently we need to show that $H^0(C, \omega_C \otimes \mathcal{L}^{-t} \otimes (I_p I_q)^*) = 0$.
This is a straightforward adaptation of [6, Lemma 3.9].

Let $\gamma : \tilde{C} \to C$ be the normalization at p, q, then we have the divisor $D = \sum p_i + \sum q_i$ where p_i, q_i map to p, q. As C has only node singularities, the degree of D is at most 4.
We have $\gamma_* \mathcal{O}_{\tilde{C}}(-D) \subset I_p I_q$ and injections

$$\mathcal{H}om(I_p I_q, \mathcal{O}_C) \hookrightarrow \mathcal{H}om(\gamma_* \mathcal{O}_{\tilde{C}}(-D), \mathcal{O}_C) \hookrightarrow \gamma_* \mathcal{H}om(\mathcal{O}_{\tilde{C}}(-D), \mathcal{O}_{\tilde{C}})$$

for details on the inclusion and injections see [6, Lemma 3.9].
This gives us

$$(I_p I_q)^* \hookrightarrow \gamma_* \mathcal{O}_{\tilde{C}}(D)$$

We tensor with $\omega_C \otimes \mathcal{L}^{-t}$ to see that it is sufficient to prove that $H^0(C, \omega_C \otimes \mathcal{L}^{-t} \otimes \gamma_* \mathcal{O}_{\tilde{C}}(D)) = 0$.
To do this we check that $\omega_C \otimes \mathcal{L}^{-t} \otimes \gamma_* \mathcal{O}_{\tilde{C}}(D)$ has negative degree for each irreducible component of C. We know that $\mathcal{L}^t$ has degree at least t on each component, and that the degree of $\gamma_* \mathcal{O}_{\tilde{C}}(D)$ is at most 4, we also have that the degree of ω_C is at most $2g-2$ putting these all together we need to have $t > 2g - 2 + 4$ for the line bundle to have negative degree.

Therefore for $t \geq 2g + 3$ we have that $\mathcal{L}^t$ is very ample, and we have the vanishing of the first cohomology group as well. □

Lemma 3.8. *Let* $\pi : \mathcal{C} \to S$ *be a flat family of quasi-stable curves, then we have*

- $\mathcal{O}_S \cong \pi_* \mathcal{O}_{\mathcal{C}}$
- *Let* $\mathcal{N}$ *be a line bundle on* S*, then* $\mathcal{N} \cong \pi_* \pi^* \mathcal{N}$

Proof. Sketch only;
The first result follows from base change and the fact that the geometric fibres of π are connected and reduced.
The second result follows from the first result and the projection formula. $(\pi_* \mathcal{O}_{\mathcal{C}} \otimes \mathcal{N} \cong \pi_* (\mathcal{O}_{\mathcal{C}} \otimes \pi^* \mathcal{N}))$ □

Corollary 3.9. *Let* $\mathcal{L}, \mathcal{M}$ *be line bundles on* $\mathcal{C}$*, then there exists a line bundle* $\mathcal{N}$ *on* S *such that* $\mathcal{L} \otimes \mathcal{M}^{-1} \cong \pi^* \mathcal{N}$ *if and only if the following both hold*

- $\pi_* \left(\mathcal{L} \otimes \mathcal{M}^{-1}\right)$ *is locally free*
- $\pi^* \pi_* \left(\mathcal{L} \otimes \mathcal{M}^{-1}\right) \to \mathcal{L} \otimes \mathcal{M}^{-1}$ *is an isomorphism.*

Proposition 3.10. *Let* $\pi : \mathcal{C} \to S$ *be a flat family of quasi-stable curves.*
Let $\mathcal{L}, \mathcal{M}$ *be line bundles on* $\mathcal{C}$ *such that* $\mathcal{L}$ *and* $\mathcal{M}$ *have the same degree on any geometric fibre when restricted to any irreducible component of that geometric fibre.*
Then there exists a unique closed subscheme $T \to S$ *such that*

- *There exists* $\mathcal{N}$ *on* T *such that* $\mathcal{L}_T \otimes \mathcal{M}_T^{-1} \cong \pi^* \mathcal{N}$
- *If* $(W \to S, \mathcal{N}')$ *is another pair such that* $\mathcal{L}_R \otimes \mathcal{M}_R^{-1} \cong \pi^* \mathcal{N}'$ *then* $W \to S$ *factors through* T*.*

Proof. If we were in the special case that S is reduced and that $\mathcal{L}$ and $\mathcal{M}$ had the same degree on any fibre, i.e. that $\mathcal{L}_s \cong \mathcal{M}_s$ for any $s \in S$, then the scheme S itself satisfies the conditions, as base change gives us that $\pi_* \left(\mathcal{L} \otimes \mathcal{M}^{-1}\right)$ is locally free of rank 1, so it is enough to show that the natural map

$$\pi^* \pi_* \left(\mathcal{L} \otimes \mathcal{M}^{-1}\right) \to \mathcal{L} \otimes \mathcal{M}^{-1}$$

is an isomorphism, but this follows from the assumptions as we can check fibrewise.
If we are in a more general situation more has to be done, the idea is that it is sufficient to check for $S = \operatorname{Spec} Y$ affine. (To reduce to this case we use uniqueness to glue together the result on an affine open cover.) In this case it is possible to describe the objects more explicitly and get the wanted closed subscheme. (It is closed as we have that $\mathcal{L}_s \cong \mathcal{M}_s$ iff we have that $\dim H^0 \left(\mathcal{C}_s, \mathcal{L}_s \otimes \mathcal{M}_s^{-1}\right) = 1$ which is a closed condition.) For the full details see [11, part 10]. □

We now have enough results to prove that the functor $\overline{\mathscr{M}_{g,n}} (\mathbb{P}^n, d)$ has a coarse moduli space.

Proof of Theorem 3.6. Let $(C, p_1, \ldots, p_n, \mu)$ be a stable map from an n-pointed, genus g curve to $\mathbb{P}^r$, which represents d.
Let $\mathcal{L} = \omega_C (p_1, \ldots, p_n) \otimes \mu^* (\mathcal{O}_{\mathbb{P}^r}(3))$ by Lemma 3.4 $\mathcal{L}$ is ample and by Lemma 3.7 there exists a t such that $\mathcal{L}^t$ is very ample for all C.

Let $e = \deg \mathcal{L}^t$, this depends on g, n, d and t only not on C, we then have that $h^0 \left(C, \mathcal{L}^t\right) = e - g + 1$

by Riemann-Roch. ($h^1\left(C,\mathcal{L}^t\right)=0$ by construction of t).
Pick an isomorphism $h^0\left(C,\mathcal{L}^t\right)\cong\mathbb{C}^{e-g+1}=W$. We then get an embedding

$$i:C\hookrightarrow\mathbb{P}\left(W\right)$$

and a map $\alpha:C\to\mathbb{P}\left(W\right)\times\mathbb{P}^r$, given by (i,μ). We also get n points in $\mathbb{P}\left(W\right)\times\mathbb{P}^r$ which are the images of the sections p_i.

Now let H_g be the Hilbert scheme of genus g curves in $\mathbb{P}\left(W\right)\times\mathbb{P}^r$ with multidegree (e,d) and let P_i be the Hilbert scheme of a point in $\mathbb{P}\left(W\right)\times\mathbb{P}^r$.

We have that the stable map $(C,p_1,\ldots,p_n,\mu)$ is associated to a point in $H_g\times P_1\times\cdots P_n=N_{g,n}$. Not all points in $N_{g,n}$ correspond to stable curves, let $I\subset N_{g,n}$ be the closed subscheme containing elements $(C,x_1,\ldots,x_n)$ such that $x_i\in C$ for each i.
There is an open subset U of I such that

- C is quasi-stable
- $C\to\mathbb{P}(W)$ is a nondegenerate embedding
- The points x_i are in the non-singular locus of C inside $\mathbb{P}(W)$

and such that the two line bundles $\mathcal{O}_{\mathbb{P}(W)}(1)\otimes\mathcal{O}_{\mathbb{P}^r}(1)|_C$ and $\omega_C^t\left(tp_1+\cdots+tp_n\right)\otimes\mathcal{O}_{\mathbb{P}^r}(3t+1)|_C$ have the same degree on each irreducible component.
Using Proposition 3.10 we find J a closed subscheme such that the line bundles above are equal.
(We have a family $\pi:\mathcal{C}\to U$ where the fibre of π at $(C,x_1,\ldots,x_n)$ is $(C,p_1,\ldots,p_n,\mu)$ where the p_i come from the x_i and the map μ comes from α, this is well defined by the conditions above. We have two line bundles on $\mathcal{C}$ which on the fibres are the two line bundles above, we get J from Proposition 3.10 and above J, the two line bundles differ by a pullback of some bundle on J, when we look at these line bundles on the fibres $(C,p_1,\ldots,p_n,\mu)$ they become isomorphic as they differ by a trivial bundle.)
By construction J is the locus of stable maps.

We have a $PGL(W)$ action on $\mathbb{P}(W)$ and it induces an action on $\mathbb{P}(W)\times\mathbb{P}^r$ and then on H_g and on P_i, it preserves the conditions above, so we get a $PGL(W)$ action on J.
When we look at two stable maps in the same orbit, i.e $g\cdot(C,x_1,\ldots,x_n)=(C',x_1',\ldots,x_n')$ the action of g induces an isomorphism $C\to C'$ that is in fact an isomorphism of stable maps, we also have that action has finite stabilizers by Lemma 3.3.
Therefore $J/PGL(W)=\overline{M}_{g,n}\left(\mathbb{P}^r,d\right)$ is the moduli space of stable maps of n-pointed, genus g curves to $\mathbb{P}^r$ that represent d. □

If X is a projective variety then $\overline{\mathscr{M}_{g,n}}\left(X,\beta\right)$ has a coarse moduli space that is in fact a closed subscheme of $\overline{M}_{g,n}\left(\mathbb{P}^r,d\right)$, if X is a convex scheme and the genus is 0, then more can be said, see [3] for the details. This is also more naturally phrased using stacks.

4 Properties and Gromov-Witten invariants

Now that we have constructed the moduli space of genus g curves we will look at a few properties that it has, we will mainly consider the genus 0 case, as the higher genus moduli spaces are not as well behaved. First we will consider a few maps from the space, and the dimension in the genus 0 case. Then we will define and study Gromov-Witten invariants, again only in genus 0. These are numbers that are associated with the moduli space and have enumerative significance.
There are a number of useful maps from $\overline{M}_{g,n}(\mathbb{P}^n, d)$.
First of all we have natural transformations

$$\alpha_i : \overline{\mathscr{M}_{g,n}}(\mathbb{P}^r, d)(-) \to \mathfrak{Hom}(-, X)$$

which are defined as, let $\mathcal{C}_\mu = (\pi : \mathcal{C} \to S, p_1, \ldots, p_n, \mu)$ be an element of $\overline{\mathscr{M}_{g,n}}(\mathbb{P}^r, d)(S)$, then let $\alpha_{iS}(\mathcal{C}_\mu) = \mu \circ p_i$.
It is a natural transformation as given $f : T \to S$, and $\mathcal{C}_\mu$, we have that $\mathcal{C}_\mu$ pullsback to $\left(\pi_T : \mathcal{C}_T \to T, p_1, \ldots, p_{\bar{p}_i}, \mu \circ \bar{f}\right)$, where $\bar{f}$ is the lift of f and $\bar{p}_i$ is the unique map $T \to \mathcal{C}_T$ we get such that $p_i \circ f = \bar{f} \circ \bar{p}_i$. ($\bar{p}_i$ comes from the universal property of the fibre product.)
Then this element is sent to $\mu \circ \bar{f} \circ \bar{p}_i = \mu \circ p_i \circ f$.
As $\overline{\mathscr{M}_{g,n}}(\mathbb{P}^r, d)$ has a coarse moduli space, $\overline{M}_{g,n}(\mathbb{P}^r, d)$, we get unique evaluation maps

$$\rho_i : \overline{M}_{g,n}(\mathbb{P}^r, d) \to X$$

We also have forgetful maps

$$\overline{M}_{g,n}(\mathbb{P}^r, d) \to \overline{M}_{g,n}$$

that arise from the universal property of $\overline{M_{g,n}}$, and there are forgetful maps.

$$\overline{M}_{g,n}(\mathbb{P}^r, d) \to \overline{M}_{g,n-1}(\mathbb{P}^r, d)$$

We will give a slightly informal arguement to find the dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$.
If we just consider maps $\mathbb{P}^1 \to \mathbb{P}^r$, we can describe them by giving $r+1$ homogeneous polynomials of degree d, this space has dimension $(r+1)(d+1)-1$. (We subtract 1 because rescaling does nothing in projective space.) We still need to deal with automorphisms of $\mathbb{P}^1$, these have dimension 3. Some of these maps have non-trivial automorphisms (as maps), but there is an open set that is automorphism free, if we restrict to the automorphism free ones, we can take the quotient of these maps by the automorphisms of $\mathbb{P}^1$ and get a space of dimension $(r+1)(d+1)-4$. In fact this is a fine moduli space for automorphism-free maps $\mathbb{P}^1 \to \mathbb{P}^r$ of degree d.
As this space is an open dense subset of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ we get that the dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is $(r+1)(d+1)-4+n$. (Each mark adds one dimension.)

Remark 4.1. For higher genus, this type of dimension count does not work as the subset of automorphism free maps is no longer dense, the compactification can add components with higher dimension.
For example consider $\overline{M}_{1,0}\left(\mathbb{P}^2, 3\right)$, the automorphism free curves from $\mathbb{P}^1$ have dimension 9 (dimension of smooth conics in $\mathbb{P}^2$), but the boundary has a component that consists of an elliptic curve contracted to a point, and a rational curve with a degree 3 map, glued at a point. This is $\overline{M}_{1,1}$ and $\overline{M}_{0,1}\left(\mathbb{P}^2, 3\right)$. These spaces have dimension 1 and 9 respectively which shows us that this component has dimension 10. (There is a zero dimension space of genus 1 curves.)

4.1 The boundary of $\overline{M}_{0,n}(\mathbb{P}^r, d)$

We want to describe the extra maps that we have added in the compactification, we will only consider the genus 0 case, these extra maps form the *boundary*.
This section will have few if any proofs, for the details see [3] or [8].
The extra maps in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ are made up of maps from reducible curves, and in fact we can describe them naturally using boundary divisors.

Definition 4.2. Fix n, r and d, then let $A \cup B$ be a partition of $\{1, \ldots, n\}$ and let $d_A + d_B = d$, where $d_A, d_B \geq 0$.
Let $D(A, B; d_A, d_B)$ be the locus of stable maps $\mu : C \to \mathbb{P}^r$ such that

- C is the union of two curves C_A, C_B that meet at a point
- The marks associated to A lie on C_A, and similarly for B.
- $\mu|_A$ represents d_A, and $\mu|_B$ represents d_B

we call $D(A, B; d_A, d_B)$ a *boundary divisor.*

The union of all the $D(A, B; d_A, d_B)$ is a divisor called the boundary.
We also have a map

$$\overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A) \times_{\mathbb{P}^r} \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B) \to D(A, B; d_A, d_B)$$

where the fibre product is over the evaluation maps associated to x.

Lemma 4.3. *The above map is an isomorphism if A and B are not empty, if both A and B are empty and $d_A = d_B = d/2$ then it is two to one, in all other cases it is birational.*

Using this lemma we can show that $D(A, B; d_A, d_B)$ is codimension 1. We have that

$$\begin{aligned}
\dim D(A, B; d_A, d_B) =& \dim \overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A) \times_{\mathbb{P}^r} \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B) \\
=& \dim \overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A) + \dim \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B) - r \\
=& (r+1)(d_A+1) - 4 + n_A + 1 + (r+1)(d_B+1) - 4 + n_B + 1 - r \\
=& (r+1)(d+2) - 6 + n - r \\
=& \dim \overline{M}_{0,n}(\mathbb{P}^r, d) - 1
\end{aligned}$$

This lemma also gives us a recursive structure on the boundary, which will turn out to be very useful.
There are also special boundary divisors that have nice properties.

Definition 4.4. Let $1 \leq i, j, k, l \leq n$ be distinct integers, then let

$$D(i, j|k, l) = \sum_{\substack{i,j\in A \\ k,l\in B}} D(A, B; d_A, d_B)$$

Lemma 4.5. *We have*

$$D(i,j|k,l) \sim D(i,k|j,l)$$

Proof. Sketch:
We have the forgetful map $\overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}$ and there is also another forgetful map $\overline{M}_{0,n} \to \overline{M}_{0,4}$. Let the 4 marks that are not forgotten be i, j, k, l then $D(i,j|k,l)$ is the inverse image of a point in $\overline{M}_{0,4} \cong \mathbb{P}^1$, as any two points in $\mathbb{P}^1$ are equivalent we get the result. □

4.2 Example of plane rational curves of degree d

We consider the problem of finding the number of curves of degree d through a given number of points in $\mathbb{P}^2$.
First of all we have that there exists 1 line going through any two distinct points, and it is a classical result that there exists a unique conic passing through 5 general points.

If we restrict to smooth curves of degree d then the question becomes relatively straightforward. The space of curves of degree d in $\mathbb{P}^2$ is the dimension of $k[x,y,z]_d$, which is the number of monomials of degree d in 3 variables, but we need to subtract one, because scaling by a constant does nothing. Therefore the space of curves of degree d has dimension $\frac{d(d+3)}{2}$. Smooth curves form an open subspace, and the condition of going through a fixed point is a hyperplane, as we have assumed the points are generic, we get that the intersection of $\frac{d(d+3)}{2}$ generic hyperplanes is a point. Therefore we have 1 smooth degree d curve through $\frac{d(d+3)}{2}$ generic points. (If we increase the number of points we would expect 0 solutions, if we decrease we would expect an infinite number of points.)

For a more interesting question we consider rational curves, the degree genus formula for smooth curves tell us that $g = \frac{(d-1)(d-2)}{2}$ and if we allow nodes, each one drops the genus by 1. A node is a codimension 1 condition, so the space of rational curves of degree d has codimension $\frac{(d-1)(d-2)}{2}$. This gives us a space of dimension

$$\frac{d(d+3)}{2} - \frac{(d-1)(d-2)}{2} = 3d - 1$$

So if we look at the number of rational curves of degree d going through $3d-1$ general points we expect the answer to be finite. Let N_d be the answer.
Another way to see this is using the fact that the dimension of $\overline{M}_{0,0}(\mathbb{P}^2, d)$ is $3(d+1) - 4 = 3d - 1$, so informally, we would expect to need $3d-1$ conditions to get a finite subset of curves.

We are going to consider the space $\overline{M}_{0,3d}(\mathbb{P}^2, d)$, for $d \geq 2$, where we denote the marks by $l_1, l_2, p_1, p_2, m_1, \ldots, m_{3d-4}$. We have $3d$ evaluation maps and let L_1, L_2 be two lines in $\mathbb{P}^2$, and $P_1, P_2, M_1, \ldots, M_{3d-4}$ be $3d-2$ points in $\mathbb{P}^r$, the lines and points are generic. (This means that the two lines meet in one point, the other $3d-2$ points are all distinct and do not meet either line, and no $3e$ points lie on a curve of degree e, where we allow the point $L_1 \cap L_2$ as one of the possible $3e$ points.)

Consider the subspace

$$\Gamma = \rho_{l_1}^{-1}(L_1) \cap \rho_{l_2}^{-1}(L_2) \cap \rho_{p_1}^{-1}(P_1) \cap \rho_{p_2}^{-1}(P_2) \cap \rho_{m_1}^{-1}(m_1) \cap \cdots \cap \rho_{m_{3d-4}}^{-1}(M_{3d-4})$$

One can show that in this case the ρ_i are flat, so the codimension is preserved, and by genericness the intersections have the right codimension as well. We have the codimension of Γ is $1+1+2(3d-2)=6d-2$.
We also have that the dimension of $\overline{M}_{0,3d}(\mathbb{P}^2,d)$ is $3(d+1)-4+3d=6d-1$, therefore Γ is a curve, it can also be shown that Γ lies in the locus of automorphism free curves and intersects the boundary divisors transversally.

We want to calculate

$$\Gamma\cap D(l_1,l_2|p_1,p_2)$$

$$\Gamma\cap D(l_1,p_1|l_2,p_2)$$

using Lemma 4.5 we see that both intersections will give us the same number.

First we consider $\Gamma\cap D(l_1,l_2|p_1,p_2)$, this consists of curves C_A,C_B of genus 0 and maps μ_A,μ_B of degree d_A,d_B such that $d_A+d_B=d$, $\mu_A(C_A)$ and $\mu_B(C_B)$ meet in a given point and $\mu_A(l_i)(L_i)\in L_i,\mu_B(p_i)=P_i$ and $\mu(m_i)=M_i$, where mu_A,μ_B are the restriction of μ, and the m_i are distributed across C_A and C_B.
If $d_A=0$ then all the m_i have to lie on C_B as else we would have an $M_i\in L_1\cap L_2$. This gives us $3d-2$ marked points on C_B and we also have the point $\mu(C_A)=L_1\cap L_2$, therefore this divisor intersects Γ at N_d points.

If $d_B=0$, then we would have $P_1=P_2$, so $d_B\geq 1$.
If $d_A,d_B\geq 1$ and sum to d we have to give $3d_A-1$ points to C_A and $3d_B-3$ points to C_B as else we would have without loss of generality, $\mu(C_A)$ a curve of degree d_A going through at least $3d_A$ general points.
There are $\binom{3d-4}{3d_A-1}$ ways of picking the $3d_A-1$ marks for C_A, then the rest go to C_B.
We also have N_{d_A} choices for the image of C_A and N_{d_B} choices for the image of C_B. We also need to pick l_1,l_2, the lines L_i meet the image of C_A in d_A places so there are d_A choices for each l_i, we also need to pick the intersection point of $\mu(C_A)$ and $\mu(C_B)$, there are d_Ad_B choices for this point, therefore we have

$$\Gamma\cap D(l_1,l_2|p_1,p_2)=N_d+\sum_{\substack{d_A+d_B=d\\ d_A,d_B\geq 1}}\binom{3d-4}{3d_A-1}d_A^3d_BN_{d_A}N_{d_B}$$

Next we look at $\Gamma\cap D(l_1,p_1|l_2,p_2)$.
If $d_A=0$ then we would have $P_1\in L_1$ which can not happen, similarly we need $d_B\geq 1$ as well.
For $d_A,d_B\geq 1$ we need to give $3d_A-2$ marks to C_A and $3d_B-2$ marks to C_B, for the same reasons as earlier.
We have $\binom{3d-4}{3d_A-2}$ choices for the marks on C_A, and we have N_{d_A} choices for the image of C_A and N_{d_B} choices for the image of C_B. We have to pick l_1 and l_2, there are d_A,d_B choices respectively, finally we have to pick the intersection point, again there are d_Ad_B choices.
Altogether this gives us

$$\Gamma\cap D(l_1,l_2|p_1,p_2)=\sum_{\substack{d_A+d_B=d\\ d_A,d_B\geq 1}}\binom{3d-4}{3d_A-2}d_A^2d_B^2N_{d_A}N_{d_B}$$

Theorem 4.6. *We have a recursive formula to calculate N_d.*

$$N_d = \sum_{\substack{d_A+d_B=d \\ d_A,d_B\geq 1}} \binom{3d-4}{3d_A-2} d_A^2 d_B^2 N_{d_A} N_{d_B} - \binom{3d-4}{3d_A-1} d_A^3 d_B N_{d_A} N_{d_B}$$

Proof. This follows from the formulas above and Lemma 4.5. □

Here are the first few values

d	1	2	3	4	5	6	7	8	9
N_d	1	1	12	620	87,304	26,312,976	14,616,808,192	13,525,751,027,392	19,385,778,269,260,800

Before the discovery of this formula, N_d was know up to N_5, N_4 was found in 1873 by Zeuthen.

4.3 Gromov-Witten invariants

One can take the ideas above and create a more general theory, which we will now briefly do.
This section is based off [3, Section 7] and [8, Chapter 4].
Let $r \geq 2$, $g = 0$ and let $\gamma_1, \dots \gamma_n$ be elements of the cohomology ring, $H^*(\mathbb{P}^r, \mathbb{Z})$ of $\mathbb{P}^r$. We have the evaluation maps $\rho_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \to X$, and we use them to get cohomology class $\rho_1^*(\gamma_1), \dots, \rho_n^*(\gamma_n)$ on $\overline{M}_{0,n}(\mathbb{P}^r, d)$.

Definition 4.7. Let $d \geq 0$, let γ_1, γ_n be cohomology class in $\mathbb{P}^r$, then the *Gromov-Witten invariant* of degree d associated with the classes γ_1, γ_n is

$$I_d(\gamma_1 \cdots \gamma_n) = \int_{\overline{M}_{0,n}(\mathbb{P}^r,d)} \rho_1^*(\gamma_1) \smile \cdots \smile \rho_n^*(\gamma_n)$$

The notation means that we evaluate the homogeneous component of highest codimension on the fundamental class, if the γ_i are homogeneous then it will be zero unless the sum of the codimensions of the γ_i is equal to the dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. These numbers have enumerative significance.

Proposition 4.8. *Let γ_i be as above, and let Γ_i be subvarieties of $\mathbb{P}^r$ that correspond to γ_i by Poincare duality.*
If the codimensions of the Γ_i add up to the dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d) = e$ and the Γ_i are generic then we have

$$I_d(\gamma_1 \cdots \gamma_n) = \#\rho^{-1}(\Gamma_1) \cap \cdots \rho_n^{-1}(\Gamma_n)$$

Proof. Sketch, we assume the γ_i are Chern classes (for example, Γ_i linear subspaces).
Let $\gamma_i = Z(s_i)$, where s_i is a regular section of E_i, a vector bundle of rank e_i, then we have that $\gamma_i = c_{e_i}(E_i)$.
Consider

$$\cap \rho_i^{-1}(\Gamma_i) = \cap \rho_i^{-1}(Z(s_i)) = \cap Z(\rho_i^* s_i)$$

By assumption this scheme has the correct codimension, $e = \sum e_i$ and therefore it corresponds to the chern class $c_e\left(\bigoplus \rho_i^* E_i\right)$. We have

$$c_e\left(\bigoplus \rho_i^* E_i\right) = \smile c_{e_i}\left(\rho_i^* E_i\right) = \smile \rho_i^*\left(c_{e_i}(E_i)\right) = \rho_1^*(\gamma_1) \smile \cdots \smile \rho_n^*(\gamma_n)$$

so we get the wanted result □

For a full proof see [3, Lemma 14], or [8, Lemma 4.1.3].

This shows that $I_d(\gamma_1 \cdots \gamma_n)$ is the number of pointed maps $\mathbb{P}^1 \to \mathbb{P}^r$ that represent d and such that $\rho_i(p_i) \in \Gamma_i$. (One can show that the intersection lies in the locus of smooth curves with no automorphisms.) In particular we have that any rational curve of degree d incident to the Γ_i appears, in fact one can show that if all the classes have codimension at least 2 then the curve intersects each Γ_i once and the inverse image of those points are the marks. Therefore in this case, $I_d(\gamma_1 \cdots \gamma_n)$ is the number of rational curves incident to the Γ_i.
For an example, in $\mathbb{P}^2$ we have that $I_d(h^2 \cdots h^2) = N_d$, where h corresponds to a hyperplane, and there are $3d-1$ classes appearing.

Note that $I_d(\gamma_1 \cdots \gamma_n)$ is invariant under reordering the γ_i, and is linear.
We can also say something about simple GW invariants.

Lemma 4.9. *For $d = 0$, the GW invariants are non-zero only if $n = 3$ and the codimensions sum to r.*

Proof. We have that $\overline{M}_{0,n}\left(\mathbb{P}^r, 0\right) = \overline{M}_{0,n} \times \mathbb{P}^r$ and that each ρ_i is the projection π_2 onto the second factor. We then have

$$\begin{aligned} I_d(\gamma_1 \cdots \gamma_n) &= \int_{\overline{M}_{0,n} \times \mathbb{P}^r} \pi_2^*\left(\gamma_1 \smile \cdots \smile \gamma_n\right) \\ &= \int_{\mathbb{P}^r} \gamma_1 \smile \cdots \smile \gamma_n \pi_{2*}\left(\overline{M_{0,n}} \times \mathbb{P}^r\right) \end{aligned}$$

If $n < 3$ then $\overline{M}_{0,n}\left(\mathbb{P}^r, 0\right) = 0$, and if $n > 3$ then $\overline{M}_{0,n}$ has positive dimension, so the pushforward is zero, therefore we need $n = 3$, and we are left with $\int_{\mathbb{P}^r} \gamma_1 \smile \cdots \smile \gamma_n$, which is the classical intersection number. □

Lemma 4.10. *If one of the classes is the fundamental class, then the GW invariant is non-zero only if $n = 3$ and $d = 0$.*

Proof. This is as if γ_n is the fundamental class, then $\rho_1^*(\gamma_1) \smile \cdots \smile \rho_n^*(\gamma_n)$ is the pullback of a class from $\overline{M}_{0,n-1}\left(\mathbb{P}^r, d\right)$ and the fibres have positive dimensions, therefore for the same reasons as above, $I_d(\gamma_1 \cdots \gamma_n) = 0$. If $d = 0$ and $n = 3$ then there is no forgetful map. □

Lemma 4.11. *The only non-zero GW invariant with less then 3 marks is*

$$I_1(h^r \cdot h^r) = 1$$

i.e. a unique line through 2 distinct points.

Proof. Follows by dimension counting, we have $\dim \overline{M}_{0,n}(\mathbb{P}^r, d) = rd + r + d + n - 3$. This is at least $2r + n - 2$ as $d > 0$, now the max possible codimension is $2r$, and this max is the only way we can reach the needed codimension. □

Lemma 4.12. *Let $d > 0$, if one of the classes is the hyperplane class h, then we have*

$$I_d(\gamma_1 \cdots \gamma_n \cdot h) = dI_d(\gamma_1 \cdots \gamma_n)$$

Proof. We know that $I_d(\gamma_1 \cdots \gamma_n \cdot h)$ is the number of $n+1$-pointed curves representing d, and incident to Γ_i and H, a generic hyperplane. The image will meet H in d places, therefore the result follows. □

This can be proved more formally using the fact that the map which forgets the last mark is generically d to 1 when restricted to the preimage of the hyperplane.
These above results show that for $\mathbb{P}^2$, all the GW invariants follow from the knowledge of $I_d(h^2 \cdots h^2)$. We know from earlier that we must have $3d - 1$ copies of h^2 and therefore the knowledge of $I_d(h^2 \cdots h^2) = N_d$ is all we need to calculate any GW invariant. By Theorem 4.6, we only need the value of $N_1 = I_1(h^2 \cdot h^2) = 1$.

It turns out that this result generalizes to $\mathbb{P}^r$.

Theorem 4.13. *All the Gromov-Witten invariants for $\mathbb{P}^r$ can be found recursively, knowing only $I_1(h^r \cdot h^r) = 1$ as an initial value.*

We will provide a brief sketch of how this works.

We want to generalize the argument used to count rational curves, now one of the key steps was that we could consider the curves C_A and C_B as effectively separate and work in moduli spaces with smaller degree and less marks. To try and generalize we need a result about the cohomology class of the diagonal in $\mathbb{P}^r$.

$$\Delta = \sum_{e+f=r} h^e \times h^f$$

Where Δ is actually the class of the diagonal, this is called the Künneth decomposition of the diagonal. Using this and that the evaluation maps and restriction maps behave well with respect to each other one can show that we have

$$\int_{D(A,B;d_A,d_B)} \rho_1^*(\gamma_1) \smile \cdots \smile \rho_n^*(\gamma_n) = \sum_{e+f=r} I_{d_A}\left(\prod_{a\in A} \gamma_a \cdot h^e\right) I_{d_B}\left(\prod_{b\in B} \gamma_b \cdot h^f\right)$$

where γ_a are the classes that are associated with the marks in A, the h^e, h^f appear because of the glueing marks and the structure of the diagonal, for a proof see [8, Corollary 4.3.3]

Now given any GW invariant $I_d(\gamma_1 \cdots \gamma_n)$ we need to reduce it down to $I_1(h^r \cdot h^r)$, so we will assume that we have know how to deal with all GW invariants which have a smaller degree or less marks.
By the above results we can assume that the codimension of each γ_i is at least 2.

Rearrange the classes so that γ_n has the least codimension and write $\gamma_n = \alpha_1 \smile \alpha_2$, where α_i has strictly smaller codimension then γ_n.
Like in the rational curve case we look at $\overline{M}_{0,n+1}(\mathbb{P}^r, d)$, with marks $a_1, a_2, p_1, \ldots, p_{n-1}$. Consider the class

$$\rho^*_{a_1}(\alpha_1) \smile \rho^*_{a_2}(\alpha_2) \smile \rho^*_{p_1}(\gamma_1) \smile \cdots \smile \rho^*_{p_{n-1}}(\gamma_{n-1})$$

As earlier, this is the class of a curve and we then integrate this class against $D(a_1, a_2|p_1, p_2)$.
Using the above equality we get that the integral is equal to a sum of products of GW invariants

$$\sum \sum_{e+f=r} I_{d_A}\left(\alpha_a \cdot \alpha_2 \cdot \prod_{a\in A} \gamma_a \cdot h^e\right) I_{d_B}\left(\prod_b \in B\gamma_b \cdot h^f\right)$$

where the first sum is over all boundary divisors with $a_1, a_2 \in A, p_1, p_2 \in B$.
If $d_A, d_B > 0$ then we are done, so we only need to deal with the cases where $d_A = 0$ or $d_B = 0$. If $d_A = 0$, then by earlier results, all the other marks must be associated with B and we get

$$I_0(\alpha_1 \cdot \alpha_2 \cdot h^{r-l_1-l_2}) I_d(\gamma_1 \cdots \gamma_{n-1} \cdot h^{l_1+l_2})$$

Where l_i is the codimension of α_i. The I_0 term is a known integer as it is an intersection in $\mathbb{P}^r$ and $I_d(\gamma_1 \cdots \gamma_{n-1} \cdot h^{l_1+l_2})$ is some known multiple of $I_d(\gamma_1 \cdots \gamma_n)$ as γ_n has codimension $l_1 + l_2$ by construction, therefore this is the GW invariant we are looking for. We also have the term

$$I_d(\gamma_3 \cdots \gamma_{n-1} \cdot h^{m_1+m_2} \cdot \alpha_1 \cdot \alpha_2) I_0(\gamma_1 \cdot \gamma_2 \cdot h^{r-m_1-m_2})$$

where the codimension of γ_i is m_i, again the I_0 term is known and the I_d term is another GW invariant with n marks, however by construction it has a term with lower codimension then the original one, so we can iterate the above process, and each time we do this we get GW invariants that we already know, and GW invariants with smaller minimum codimension, as soon as this codimension reaches 1, we can remove that term by earlier results to get a GW invariant with a lower number of terms, which we already know by assumption.

However this does not give us useful information yet, like in the rational curve case we also integrate against $D(a_1, p_1|a_2, p_2)$, this will give the same answer.
Again we get an expression like above, but this time the only terms we need to consider are the following ones.

$$I_0(\alpha_1 \cdot \gamma_1 \cdot h^{r-l_1-m_1}) I_d(\gamma_2 \cdots \gamma_{n-1} \cdot \alpha_2 \cdot h^{l_1+m_1})$$
$$I_d(\gamma_1 \cdot \gamma_3 \cdots \gamma_{n-1} \cdot h^{l_1+m_1} \cdot \alpha_1) I_0(\gamma_2 \cdot \alpha_2 \cdot h^{r-m_2-l_2})$$

In both cases we know the I_0 invariant and the I_d invariant has a smaller minimum codimension so iterating the process as above gives us an expression that involves only known GW invariants. We then rearrange the equality given by the divisors so that we have $I_d(\gamma_1 \cdots \gamma_n)$ is equal to some expression of GW invariants with lower degree or number of marks, therefore by assumption, we can express $I_d(\gamma_1 \cdots \gamma_n)$ in terms of $I_0(h^r \cdot h^r) = 1$ and the recursion is proved.

As an example, $I_1(h^2 \cdot h^2 \cdot h^2 \cdot h^2) = 2$. (The dimension of $\overline{M}_{0,4}(\mathbb{P}^3, 1)$ is 8 which is the sum of the codimensions)
Applying the above algorithm we get consider the classes h^2, h^2, h^2, h, h with the marks p_1, p_2, p_3, l_1, l_2.
If we consider the intersection with $D(l_1, l_2|p_1, p_2)$ we get a sum of products

$$I_{d_A}\left(\prod \gamma_a \cdot h^e\right) I_{d_B}\left(\prod \gamma_b \cdot h^f\right)$$

where $d_A + d_B = 1$, $e + f = 3$. We must have either d_A or d_B is 0.
If d_A is zero, we have the classes h, h, h^e and the codimensions must sum to 3 so we have $e = 1$, the other term is $I_1(h^2 \cdot h^2 \cdot h^2 \cdot h^2)$ which is the value we are looking for.
If $d_B = 0$ then we have $I_0\left(h^2 \cdot h^2 \cdot h^f\right)$ which has total codimension $4 + e$, and it needs to be 3, therefore this term is zero.
Putting these together we get that the intersection with $D(l_1, l_2|p_1, p_2)$ is $I_1(h^2 \cdot h^2 \cdot h^2 \cdot h^2)$.

Next we intersect with $D(l_1, p_1|p_2, l_2)$, again we have to only consider cases where $d_A = 0$ or $d_B = 0$.
If $d_A = 0$ then we have $I_0\left(h^2 \cdot h \cdot h^e\right)$ and for the codimensions to work we must have $e = 0$, this term is therefore 1, it is multiplied by the GW invariant $I_1\left(h^2 \cdot h^2 \cdot h^1 \cdot h^3\right)$, we can remove the h^1 class in exchange for multiplying by $d = 1$.
If $d_B = 0$ we get exactly the same thing as this is a symmetric situation. Therefore we get that

$$I_1\left(h^2 \cdot h^2 \cdot h^2 \cdot h^2\right) = 2I_1\left(h^2 \cdot h^2 \cdot h^3\right)$$

So we now have to calculate $I_1\left(h^2 \cdot h^2 \cdot h^3\right)$. This time we get the classes h^3, h^2, h, h, intersect with $D(l_1, l_2|p_1, p_2)$.
If $d_A = 0$ then we have $I_0(h \cdot h \cdot h^e)I_1(h^3 \cdot h^2 \cdot h^{3-e})$. This is non-zero only when $e = 1$ and this product is then $I_1\left(h^2 \cdot h^2 \cdot h^3\right)$.
If $d_B = 0$ then as above the codimensions do not work and we get 0.

Intersecting with the other divisor $D(l_1, p_2|l_2, p_2)$ we get
If $d_A = 0$, we have $I_0(h^3 \cdot h \cdot h^e)I_1(h^2 \cdot h^1 \cdot h^{3-e})$ which is zero for all values of e.
If $d_B = 0$ we have $I_1(h^3 \cdot h \cdot h^e)I_0(h^2 \cdot h^1 \cdot h^{3-e})$ which is non-zero if $e = 3$. The I_0 term is 1, and the other term is $I_1(h^3 \cdot h \cdot h^3) = I_1\left(h^3 \cdot h^3\right)$ which is our known initial value, 1.
Putting this altogether we get that

$$I_1(h^2 \cdot h^2 \cdot h^2 \cdot h^2) = 2$$

We see that while the recursion can be done, it is a long computation for even simple GW invariants.

There is some Mathematica code in Appendix A that can calculate any GW invariant of the form $I_d\left(h^{a_1} \cdots h^{a_n}\right)$. (Which is all GW invariants up to linearity).
Some examples are

- On $\mathbb{P}^4$ - $I_2\left(h^3 \cdot h^3 \cdot h^3 \cdot h^3 \cdot h^4\right) = 2$
- On $\mathbb{P}^3$ - $I_3\left(h^2 \cdot h^2 \cdot h^2 \cdot h^2 \cdot h^3 \cdot h^3 \cdot h^3 \cdot h^3\right) = 30$
- On $\mathbb{P}^6$ - $I_5\left(h^4 \cdot h^4 \cdot h^5 \cdot h^5 \cdot h^5 \cdot h^5 \cdot h^5 \cdot h^5 \cdot h^5 \cdot h^5\right) = 15,279$
- On $\mathbb{P}^5$ - $I_4\left(h^2 \cdots h^2\right) = 3,430,726,351,753,800,000$, there are 26 classes of codimension 2.

There is a more general theory of GW invariants for higher genus, but that requires the virtual fundamental class, as $\overline{M}_{g,n}\left(\mathbb{P}^r, d\right)$ has components with too high dimension. One can also talk about axioms for GW invariants, these take some of the results that we proved above and turn them into axioms, for example we want to have permutation invariance, to be able to "pull out" a codimension one class, to be able to split into GW invariants on "smaller" moduli spaces, etc. There are 9 axioms in total, see [10] for more details.

A Appendix A

Here is the Mathematica code that calculates Gromov-Witten invariants using an algorithm that is based on the recursion theorem. The notation is $GW\left[r, d, \{a_1, a_2, \ldots, a_n\}\right]$ is the GW invariant $I_d\left(h^{a_1} \cdots h^{a_n}\right)$ associated to $\mathbb{P}^r$. This code can copied straight into a Mathematica notebook, the explanation of the code follows afterwards.

```
In[1]:= GW[r_,d_,x_] := GW[r,d,x]= If[Total[x] == (r+1)*(d+1) - 4 + Length[x],GW[r,d,x,False,False,False],0]
GW[r_,0,x_, False,False,False] := If[Length[x] == 3 && x[[1]] + x[[2]] + x[[3]] == r, 1,0]
GW[r_,1,{i_,j_}, False,False,False]:=If[i==j==r,1,0]
GW[r_,d_,x_, False, False, False] :=If[MemberQ[x,0],If[Length[x]==3 && x[[1]] + x[[2]]+x[[3]]==r && d==0,1,0],GW[r,d,x, True, False, False]]
GW[r_,d_,x_, True, False, False] := GW[r,d,Sort[x], True, True, False]
GW[r_,d_,x_,True, True,False] := If[x[[1]] ==1 && d> 0, d*GW[r,d,x[[2;;Length[x]]]],GW[r,d,x,True,True,True]]
GW[r_,d_,x_,True,True,True] :=If[Length[x] ≥ 4,

Sum[

GW[r,a,Join[{1},{x[[2]]},x[[y]],{b}]]*

GW[r,d-a,

Join[{x[[1]]-1},{x[[3]]},{r-b}, x[[Complement[Range[4,Length[x]],y]]]]

]

,{b,0,r}, {y,Subsets[Range[4,Length[x]]]}, {a,0,d}] -

Sum[ GW[r,a,Join[{1},{x[[1]]-1},x[[y]],{b}]]*

GW[r,d-a,

Join[{x[[2]]},{x[[3]]},{r-b}, x[[Complement[Range[4,Length[x]],y]]]]

]

,{b,0,r}, {y,Subsets[Range[4,Length[x]]]}, {a,1,d}]

,

Sum[

GW[r,a,Join[{1},{x[[2]]},{b}]]*GW[r,d-a, Join[{x[[1]]-1},{x[[3]]},{r-b}]]

,{b,0,r},{a,0,d}] - Sum[

GW[r,a,Join[{1},{x[[1]]-1},{b}]]*GW[r,d-a, Join[{x[[2]]},{x[[3]]},{r-b}]]

,{b,0,r},{a,1,d}]


]
```

The explanation of what the code does is as follows.

```
In[2]:= GW[r_,d_,x_] := GW[r,d,x]= If[Total[x] == (r+1)*(d+1) - 4 + Length[x],GW[r,d,x,False,False,False],0]
```

This line checks that the codimensions of the classes add up to the dimension of the moduli space, if they do not, it outputs 0, else it adds 3 dummy inputs which will be used later, all with the value False. The ":= GW[r,d,x] = If[..." means that it will save the value of GW[r,d,x] when it calculates it, this makes the program run quicker, at the cost of needing to store more values, however this is not a large cost to pay as it does not take up much space.

```
In[3]:= GW[r_,0,x_, False,False,False] := If[Length[x] == 3 && x[[1]] + x[[2]] + x[[3]] == r, 1,0]

GW[r_,1,{i_,j_}, False,False,False]:=If[i==j==r,1,0]
```

These two lines deal with the two special cases that we already know the value of, when d is 0 and when we only have 2 classes.

```
In[4]:= GW[r_,d_,x_, False, False, False] :=If[MemberQ[x,0],If[Length[x]==3 && x[[1]] + x[[2]]+x[[3]]==r && d==0,1,0],GW[r,d,x, True, False, False]]
```

This line checks if one of the classes is the fundamental class, if it is, it outputs 0 unless the degree is 0, the codimensions sum to r and there are only 3 classes. If the fundamental class does not appear it sets the first dummy variable to True.

```
In[5]:= GW[r_,d_,x_, True, False, False] := GW[r,d,Sort[x], True, True, False]
```

This line sorts the classes so that they are ordered by increasing codimension, it also sets the second dummy variable to True.

```
In[6]:= GW[r_,d_,x_,True, True,False] := If[x[[1]] ==1 && d> 0, d*GW[r,d,x[[2;;Length[x]]]],GW[r,d,x,True,True,True]]
```

This line checks if the first class is a hyperplane (we have already sorted the classes by codimension), if it is not, it sets the third dummy variable to True, if it is, the value is set to d times the GW invariant with the hyperplane class removed.

```
In[7]:= GW[r_,d_,x_,True,True,True] :=If[Length[x] ≥ 4,
        Sum[
        GW[r,a,Join[{1},{x[[2]]},x[[y]],{b}]]*
        GW[r,d-a,
        Join[{x[[1]]-1},{x[[3]]},{r-b}, x[[Complement[Range[4,Length[x]],y]]]]
        ]
        ,{b,0,r}, {y,Subsets[Range[4,Length[x]]]}, {a,0,d}] -
        Sum[ GW[r,a,Join[{1},{x[[1]]-1},x[[y]],{b}]]*
        GW[r,d-a,
        Join[{x[[2]]},{x[[3]]},{r-b}, x[[Complement[Range[4,Length[x]],y]]]]
        ]
        ,{b,0,r}, {y,Subsets[Range[4,Length[x]]]}, {a,1,d}]
        ,
        Sum[
        GW[r,a,Join[{1},{x[[2]]},{b}]]*GW[r,d-a, Join[{x[[1]]-1},{x[[3]]},{r-b}]]
        ,{b,0,r},{a,0,d}] - Sum[
        GW[r,a,Join[{1},{x[[1]]-1},{b}]]*GW[r,d-a, Join[{x[[2]]},{x[[3]]},{r-b}]]
        ,{b,0,r},{a,1,d}]

        ]
```

This is the most complicated "simplification". Once it has already tried to apply all the above simplifications it finally expresses the GW invariant in terms of simpler ones using the idea of intersecting with the two different boundary divisors and then comparing. This is actually not too bad, it sums over a, b and y, where a is the degree denoted d_A in the proof of recursion, b is the codimension of the extra class that comes from the gluing mark, and y is the information of which marks go to the C_A curve. The choice of α_1, α_2 is chosen as the hyperplane class, and a class with codimension one less then the class with least codimension. (That class must have codimension at least 2 as we have already dealt with classes with smaller codimension). When summing over the divisor $D(a_1, a_2|p_1, p_2)$, the term where the GW invariant associated to the situation where the C_A

curve has degree 0 gives us the wanted GW invariant, that is why that sum starts from a = 1. (This sum is subtracted from the sum gotten from the divisor $D(a_1, p_1|a_2, p_2)$).
The recursion theorem shows that this algorithm will always terminate.

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