Introduction to Hochschild (co)homology

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Abstract

We define Hochschild (co)homology, motivated by ideas from Algebraic Topology. We then prove it is equivalent to (Ext)Tor of bimodules. We prove that it is Morita invariant and finally give some geometrical meaning to the Hochschild (co)homology.

Contents

1 Introduction

The aim of this project is to introduce and study Hochschild (co)homology. We assume knowledge of algebras, rings and modules; tensor products appear everywhere. Most of the assumed topics are covered in [1]. In general we do not assume commutativity.

The idea or motivation for Hochschild (co)homology comes from algebraic topology. Given a geometric object (topological space) we can associate to it a collection of groups, these groups contain a lot of information about the underlying space. One collection is the homotopy groups π_n but we will take the simpler homology groups as our motivation. One way to define these is to give the topological space a delta complex structure and define maps d_n from n-simplices to $(n-1)$-simpliies It turns out that $d_n \circ d_{n+1} = 0$ for all n, but not all elements which are killed by d_n are in the image of d_{n+1}. Informally these elements represent the ”holes” in the space. We want to take this idea and abstract it to non-geometrical objects.
We will do this for k-algebras by defining an infinite collection of bimodules C_n and maps $d_n : C_n \to C_{n-1}$ such that $d_n \circ d_{n+1} = 0$ (called a complex). Next we take the homology of this complex, which is the quotient $\ker d_n / \operatorname{Im} d_{n+1}$. As in the topological case, these groups will contain a lot of information. We can also define related cohomology groups. It turns out that these groups will still contain ”geometrical” information.
The project is about defining and studying these groups, the basic outline is,

First: We define the Hochschild complex and Hochschild (co)homology and then show that it is equivalent to related (Ext) Tor groups. Some homological algebra is assumed, but we will recap Ext and Tor briefly. Using this description we can then calculate Hochschild (co)homology for polynomial algebras. The Koszul resolution is also covered in this section.

Second: We take a short diversion into basic category theory, and look at Morita equivalence for rings, proving necessary and sufficient conditions for two rings to be Morita equivalent. We then show that Hochschild (co)homology is a Morita invariant.

Third: We take a closer look at Hochschild homology and its relationship to Kähler differentials if R is commutative. We also show the relationship between deformations and Hochschild cohomology.

There is quite a lot of algebraic manipulation, especially in the proof that Hochschild homology is a Morita invariant, and in the section on differentials. This can be skimmed over without affecting understanding that much. A lot of this report is based on [9] and [6]. For a different viewpoint, [5] covers a lot of the theory but assumes more background knowledge. I will reference more closely in the actual sections.

2 Definition of Hochschild (co)homology

2.1 Basic Definitions

Motivated by n-simplices from Algebraic Topology we define,

Definition 2.1. A *presimplicial module* C is a collection of modules $C_n,\ n \geq 0$, with maps, $d_i : C_n \to C_{n-1}\ i = 0, \ldots, n$ such that $d_i d_j = d_{j-1} d_i$ for $0 \leq i < j \leq n$.

The d_i are the generalization of the maps from a simplex to its faces. One can also define a *simplical module*, for this we need extra maps $s_i : C_n \to C_{n+1}$ satisfying $s_i s_j = s_{j+1} s_i$ when $i \leq j$ and extra relationships between the s_i and d_j. These s_i are generalizations of the inclusion of a simplex into a higher dimensional one. However we will not require these maps so a presimplicial module is sufficient.

Lemma 2.2. *If* $d = \sum_{i=0}^{n} (-1)^i d_i$ *then* $d \circ d = 0$.

Proof. We have $d \circ d = \sum_i \sum_j (-1)^{i+j} d_i d_j$ and we can split it into two parts, $i < j$, and $i \geq j$. We have

$$\sum_{i<j} (-1)^{i+j} d_i d_j + \sum_{i \geq j} (-1)^{i+j} d_i d_j.$$

Then use $d_i d_j = d_{j-1} d_i$ for $i < j$ to get

$$\sum_{i<j} (-1)^{i+j} d_{j-1} d_i + \sum_{i \geq j} (-1)^{i+j} d_i d_j$$

now with $j-1 \geq i$ in the first sum, these two sums then differ by a sign so they cancel. $\square$

This shows that a presimplicial module is a chain complex. We also want maps between presimplicial modules and to know when two maps give the same maps on homology, these are straight forward generalizations from the normal definitions.

Definition 2.3. A map of *presimplicial modules* $f : C \to C'$ is a collection of maps $f_n : C_n \to C'_n$ such that $f_{n-1}d_i = d_i f_n$.
A *presimplicial homotopy* between two presimplicial maps $f, g : C \to C'$ is a collection of maps $h_i : C_n \to C'_{n+1}$ such that,

- $d_i h_j = h_{j-1} d_i \quad i < j$
- $d_i h_i = d_i h_{i-1} \quad 0 < i \leq n$
- $d_i h_j = h_j d_{i-1} \quad i > j+1$
- $d_0 h_0 = f, \quad d_{n+1} h_n = g.$

Lemma 2.4. *If h is a presimplicial homotopy between f and g then $\sum_{i=0}^{n}(-1)^i h_i$ is a chain homotopy between f and g, i.e. $dh + hd = f - g$.*

Proof. Very similar to Lemma 2.2 this time using the properties of h_i. $\square$

Now we can set up Hochschild homology.
Let k be a field and R be a k-algebra, M a bimodule over R. Consider the module $M \otimes R \otimes \cdots \otimes R = M \otimes R^{\otimes n}$, where $\otimes = \otimes_k$, and maps $d_i : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1}$, given by,

$$d_0(m \otimes r_1 \otimes \cdots \otimes r_n) = mr_1 \otimes r_2 \otimes \cdots \otimes r_n$$
$$d_i(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes r_1 \otimes r_2 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n$$
$$d_n(m \otimes r_1 \otimes \cdots \otimes r_n) = r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1}.$$

It is easy to check that this is a presimplicial module.

Definition 2.5. The *Hochschild complex* $C(R, M)$ is the complex

$$\cdots \xrightarrow{d} M \otimes R^{\otimes n} \xrightarrow{d} M \otimes R^{\otimes n-1} \xrightarrow{d} \cdots \xrightarrow{d} M \otimes R \xrightarrow{d} M \to 0$$

with $C_n(R, M) = M \otimes R^{\otimes n}$ and $d = \sum_{i=0}^{n}(-1)^i d_i$ called the *Hochschild boundary*.
If $M = R$ we write $C_n(R)$ instead of $C_n(R, R)$.

Definition 2.6. The *Hochschild homology* of R with coefficients in M, $H_*(R, M)$, is the homology of the Hochschild complex

$$H_n(R, M) = \ker d : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1} / \operatorname{Im}\ d : M \otimes R^{\otimes n+1} \to M \otimes R^{\otimes n}.$$

If $M = R$, we write $HH_n(R)$.

We can also define *Hochschild cohomology* using the complex

$$0 \to M \xrightarrow{\delta} \operatorname{Hom}_k(R, M) \xrightarrow{\delta} \operatorname{Hom}_k(R^{\otimes 2}, M) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \operatorname{Hom}_k(R^{\otimes n}, M) \xrightarrow{\delta} \operatorname{Hom}_k(R^{\otimes n+1}, M) \xrightarrow{\delta} \cdots$$

where $\delta = \sum_{i=0}^{n}(-1)^i \delta_i$ is given by

$$(\delta_0 f)(r_1 \otimes \cdots \otimes r_n) = r_1 f(r_2 \otimes \cdots \otimes r_n)$$
$$(\delta_i f)(r_1 \otimes \cdots \otimes r_n) = f(r_1 \otimes r_2 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n)$$
$$(\delta_n f)(r_1 \otimes \cdots \otimes r_n) = f(r_1 \otimes \cdots \otimes r_{n-1}) r_n.$$

Note that this is not the dual complex, however it is similar as it is composed of Hom modules. By taking the cohomology of this complex we get the *Hochschild cohomology* of R with coefficients in M, denoted $H^n(R, M)$, and again if $M = R$, we write $HH^n(R)$.
The most important case is $HH_n(R)$ and $HH^n(R)$ but the theory is not more complicated in the general case, so we will mainly work with $H_n(R, M)$ and $H^n(R, M)$. We will however do calculations for $HH_n(R)$ and $HH^n(R)$.

2.2 Basic properties

In general the Hochschild homology groups are not R-modules, but we can define an action of $Z(R)$ on $C_n(R, M)$ given by $z \cdot m \otimes r_1 \otimes \cdots \otimes r_n = zm \otimes r_1 \otimes \cdots \otimes r_n$. This action commutes with d as $z \cdot d_n(m \otimes r_1 \otimes \cdots \otimes r_n) = z r_{n-1} m \otimes r_1 \otimes \cdots \otimes r_{n-1} = r_{n-1} z m \otimes r_1 \otimes \cdots \otimes r_n = d_n(z \cdot m \otimes r_1 \otimes \cdots \otimes r_n)$. This makes $H_*(R, M)$ into a left $Z(R)$-module. We can also give it a right $Z(R)$ action by $m \otimes r_1 \otimes \cdots \otimes r_n \cdot z = mz \otimes r_1 \otimes \cdots \otimes r_n$.

Lemma 2.7. *The two $Z(R)$-module structures on $H_*(R, M)$ are equivalent.*

Proof. Define a map $h_i : C_n(R, M) \to C_n(R, M)$ by $m \otimes r_1 \otimes \cdots \otimes r_n \mapsto m \otimes r_1 \cdots r_i \otimes z \otimes r_{i+1} \cdots \otimes r_n$. These maps form a presimplicial homotopy as $d_i h_j = h_{j-1} d_i$ for $i < j$ and $d_i h_j = h_j d_{i-1}$ for $i > j + 1$ directly. The identity follows for $d_i h_i = d_i h_{i-1}$ as z is in the centre. Finally $d_0 h_0(m \otimes r_1 \otimes \cdots \otimes r_n) = mz \otimes r_1 \otimes \cdots \otimes r_n$ and $d_{n+1} h_n(m \otimes r_1 \otimes \cdots \otimes r_n) = zm \otimes r_1 \otimes \cdots \otimes r_n$. $\square$

In particular, if R is commutative then $H_*(R, M)$ is an R-module. Similarly $H^*(R, M)$ is also a $Z(R)$-module, via $z \cdot f = zf$. Again the two possible actions are equivalent. Another property is;

Lemma 2.8. *$H_*(R, -)$ and $H_*(-, M)$ are functors.*

Proof. Given a bimodule homomorphism $f : M \to N$, we get an induced map $f_* : H_*(R, M) \to H_*(R, N)$ given by $f_n(m \otimes r_1 \cdots \otimes r_n) = f(m) \otimes r_1 \cdots \otimes r_n$. It is clear that $(f \circ g)_* = f_* \circ g_*$ and that id_* is the identity map, so $H_*(R, -)$ is a covariant functor from R-bimodules to $Z(R)$-bimodules.
Now if we are given a k-algebra map $f : R \to S$ and an S-bimodule M, we can make M into a R-bimodule by $r \cdot m = f(r)m$. Then we can define $f_* : H_*(R, M) \to H_*(S, M)$ by $g_n(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes f(r_1) \otimes \cdots \otimes f(r_n)$. Again the two functor properties are satisfied, so we have a functor from k-algebras to groups. $\square$

Note that if $M = R$ then we can also get a functor $HH_*(-)$ from k-algebra to k-modules.
In a similar way, one can show that $H^*(R, -)$ is a covariant functor, but this time $H^*(-, M)$ is contravariant. Given $f : R \to S$, we define $f^* : H^*(S, M) \to H^*(R, M)$ by $f^n(\phi)(r_1 \otimes \cdots \otimes r_n) = \phi(f(r_1) \otimes \cdots \otimes f(r_n))$. For this reason $HH^*(-)$ is not a functor. (It wants to be covariant and contravariant at the same).
We can also calculate the low degree Hochschild (co)homology for general R and M.
Starting with homology, in degree 0 we have, $R \otimes M \xrightarrow{d} M \to 0$, where $d(m \otimes r) = mr - rm$. So $H_0(R, M) = M/\{mr - rm\} = M/[R, M]$, also known as the module of *coinvariants* of M. If R is commutative then $HH_0(R) = R$.

Next we have $M \otimes R^{\otimes 2} \xrightarrow{d} M \otimes R \xrightarrow{d} M$, with $d(m \otimes r_1 \otimes r_2) = mr_1 \otimes r_2 - m \otimes r_1 r_2 + r_2 m \otimes r_1$, so we have

$$H_1(R, M) = \{m \otimes r \mid mr - rm = 0\}/\{mr_1 \otimes r_2 - m \otimes r_1 r_2 + r_2 m \otimes r_1\}.$$

It is not clear if this represents anything useful, but note that if R is commutative, then $d : R \otimes R \to R$ is the zero map, so has full kernel, and the relation $r \otimes st = rs \otimes r + sr \otimes s$ looks like a product rule. For more details see Section 6.1.

Now turning to cohomology, in degree 0, we have $0 \to M \xrightarrow{\delta} M \otimes R$, where $(\delta m)(r) = mr - rm$, so $H^0(R, M) = \{m \mid mr - rm = 0 \quad \forall r \in R\}$ also know as the *invariants* of M. In particular $HH^0(R) = Z(R)$, the centre of R.
For $H^1(R, M)$, we have $(\delta f)(r \otimes s) = rf(s) - f(rs) + f(r)s$, so a function $f : R \to M$ is in the kernel if $f(rs) = rf(s) + f(r)s$, i.e. if f is a derivation. By the above work, the functions in the image are just functions such that $f(r) = mr - rm$, these are known as inner derivations, therefore $H^1(R, M) =$ Derivations/ Inner derivations.
There is also an interpretation for HH^2 and HH^3, see Section 7 for more details.

3 Relationship with Tor and Ext

While the Hochschild complex can be useful for general theory, it is not very useful if we want to calculate $H_i(R, M)$ for given R and M. In this section we will show that Hochschild homology is equivalent to a specific Tor group, similarly Hochschild cohomology is equivalent to a specific Ext group.
This section is mainly based on [9, Ch 1.1].

3.1 Derived functors basics

Recall that a functor F is *additive* if it induces a group homomorphism between $\hom(A, A')$ and $\hom(F(A), F(A'))$. From now on, all functors are additive.
It is clear that functors take complexes to complexes, as $F(\phi_n) \circ F(\phi_{n+1}) = F(\phi_n \circ \phi_{n+1}) =$

$F(0) = 0$ (as F is additive). However there is no guarantee that a functor will take an exact sequence to another exact one. It turns out to be sufficient to look only at how short exact sequences transform.

For example consider the functors $N \to \mathrm{Hom}_R(M, N)$ or $N \to \mathrm{Hom}_R(N, M)$, from left R-modules to the category of Abelian groups. They are covariant and contravariant respectively, we write $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, M)$. If

$$0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$$

is a short exact sequence of modules. Then,

$$0 \to \mathrm{Hom}_R(M, U) \xrightarrow{\alpha_*} \mathrm{Hom}_R(M, V) \xrightarrow{\beta_*} \mathrm{Hom}_R(M, W)$$

is exact and so is

$$0 \to \mathrm{Hom}_R(W, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(V, M) \xrightarrow{\alpha_*} \mathrm{Hom}_R(U, M).$$

So in general Hom is only *left exact*.

Similarly the tensor products $M \otimes_R -$ and $- \otimes_R M$ are covariant functors from right/left R-modules to Abelian groups. They are *right exact* i.e

$$U \otimes_R M \xrightarrow{\alpha_*} V \otimes_R M \xrightarrow{\beta_*} W \otimes_R M \to 0$$

is exact whenever we started from a short exact sequence,

$$0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0.$$

We want a way to turn this partial sequence into a long exact one. If we have a right exact covariant functor F then we want a series of functors $L_i F$ such that

$$\cdots \to L_i F(U) \to L_i F(V) \to L_i F(W) \to \cdots \to L_1 F(U) \to L_1 F(V) \to L_1 F(W) \to$$

$$F(U) \to F(V) \to F(W) \to 0$$

is exact. $L_i F$ is called the *left derived functor* of F.
The way that $L_i F$ is constructed is to take a projective resolution of M. This works as the category of modules has enough projectives.

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

Then apply F to get

$$\cdots \to F(P_n) \to F(P_{n-1}) \to \cdots \to F(P_1) \to (P_0) \to 0.$$

Note that we forget M at the end. This sequence is no longer exact in general, define $L_i F(M)$ as the i^{th} homology group. One can show that this does not depend on the choice of projective resolution and that a map $f : M \to M'$ induces a map on the projective resolutions and therefore a map $f_* : L_i F(M) \to L_i F(M')$. One can also show that we have the wanted long exact sequence. If we started with a left exact covariant functor G then we take an injective resolution instead and get the *right derived functor* of G. Similarly if we have a contravariant left/right exact functor then we take a projective/injective resolution.
In the special cases of Hom and $\otimes$ we write $\mathrm{Ext}^n_R(N, M)$ for $R^n \mathrm{Hom}(N, M)$ and $\mathrm{Tor}^R_n(N, M)$ for $L_n N \otimes_R M$. One can also show that it does not matter whether we take a resolution of N or M in these cases. In the case of Tor it is also sufficient to take a flat resolution. For more details see [6, Ch.2-3].

3.2 The bar complex

Let R be a k-algebra then R^{op} is the *opposite algebra* of R. (The underlying set is the same, but the multiplication is $r \cdot s = sr$). $R^e = R \otimes R^{op}$ is called the *enveloping algebra*. R is a left R^e-module by $(s \otimes t)r = srt$ and in general, given a bimodule M we get a left R^e-module by $(s \otimes t)m = smt$, and a right R^e-module by $m(r \otimes s) = smr$.

Definition 3.1. The *bar complex* $C^{bar}(R)$ is the complex

$$\cdots \xrightarrow{d'} R^{\otimes n+3} \xrightarrow{d'} R^{\otimes n+2} \xrightarrow{d'} R^{\otimes n+1} \xrightarrow{d'} \cdots\cdots \xrightarrow{d'} R^{\otimes 3} \xrightarrow{d'} R^{\otimes 2}.$$

where $C_n^{bar} = R^{\otimes n+2}$ and $d' = \sum_{i=0}^{n-1}(-1)^i d_i$ where the d_i is the map from Definition 2.5.

Lemma 3.2. *Let R be a k-algebra . Then $C^{bar}(R)$ is a resolution of the R^e module R.*

Proof. We have a map $d' = \mu : R \bigotimes R \to R,\ r \otimes s \mapsto rs$. Now define $S : R^{\otimes n} \to R^{\otimes n+1}$ by $r_1 \otimes \cdots r_n \mapsto 1 \otimes r_1 \otimes \cdots r_n$. We have $(d's + sd')(r_1 \otimes \cdots r_n) = d'(1 \otimes r_1 \otimes \cdots r_n) + s(\sum(-1)^{i+1} r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n) = r_1 \otimes \cdots \otimes r_n + \sum(-1)^i 1 \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n + \sum(-1)^{i+1} 1 \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n = r_1 \otimes \cdots \otimes r_n$.
So the identity map is homotopic to the zero map and therefore the complex is exact. □

This resolution is called the *bar resolution*.

Theorem 3.3. *For any R-module M, we have $H_n(R, M) \cong \mathrm{Tor}_n^{R^e}(M, R)$.*

Proof. We have that R is free as a k-module, so $R^{\otimes n}$ is also free, and therefore $R^{\otimes n+2} = R \bigotimes R^{\otimes n} \bigotimes R \cong R^{\otimes n} \bigotimes R^e$ is free as an R^e-module. Therefore the bar complex is a free, therefore projective resolution of R as an R^e-module. To calculate the homology groups $\mathrm{Tor}_n^{R^e}(M, R)$ we tensor the resolution with M as a right R^e-module over R^e. In degree n we have $M \bigotimes_{R^e} R^{\otimes n+2} \cong M \bigotimes R^{\otimes n}$ and the map $1_M \otimes d'$ becomes d. $\big(m \otimes_{R^e} (r_0 \otimes r_1 \otimes \cdots \otimes r_{n+1}) \mapsto r_{n+1} m r_0 \otimes r_1 \otimes \cdots \otimes r_n$ under the isomorphism.$\big)$ □

In a very similar way can show that $H^n(R, M) \cong \mathrm{Ext}^n_{R^e}(R, M)$.
This shows that we can define Hochschild (co)homology in two different ways,
1^{st} - using presimplicial modules, motivated by Algebraic Topology.
2^{nd} - as the (co)homology of bimodules using derived functors.
We therefore have two different ways of thinking about Hochschild (co)homology and we can use which ever is more useful for calculations or proofs, each way of thinking brings its own techniques and ideas.

3.3 Two examples of Hochschild (co)homology

Now that we have seen that we can use Tor and Ext to calculate Hochschild (co)homology we can calculate some examples. Note we do not have to use the Bar resolution, any projective resolution will do. In particular if we can find a finite or periodic resolution then we can actually find all the Hochschild groups.

First consider $k[x]$, we want a projective resolution of $k[x]$ as a $k[x]^e$ module. We have that $k[x] \otimes k[y] \cong k[x, y]$ and we have the natural map $\mu : k[x, y] \to k[x]$ from Lemma 3.2. This map has kernel generated by $x - y$. This gives us the projective resolution

$$0 \to k[x, y] \xrightarrow{x-y} k[x, y] \xrightarrow{\mu} k[x] \to 0.$$

Next we tensor with $k[x]$ over $k[x, y]$ and get

$$0 \to k[x] \xrightarrow{0} k[x] \to 0.$$

The map becomes 0 as x and y both have the same action on $k[x]$. This gives us that $HH_0(k[x]) = HH_1(k[x]) = k[x]$ and $HH_i(k[x]) = 0$ for $i > 1$. By applying $\mathrm{Hom}_{k[x]^e}(-, k[x])$ we get that $HH^0(k[x]) = HH^1(k[x]) = k[x]$ and $HH^i(k[x]) = 0$ for $i > 1$.

This can be generalized to deal with polynomials in n variables see Section 4.

Next, let $R = k[x]/(x^n)$, let $u = x \otimes 1 - 1 \otimes x$ and let $v = \sum_{i=0}^{n-1} x^{n-1-i} \otimes x^i$. Then we have a free resolution of R

$$\cdots \xrightarrow{v} R^e \xrightarrow{u} R^e \xrightarrow{v} R^e \xrightarrow{u} \cdots \xrightarrow{u} R^e \to R \to 0.$$

To see this note that $(x^n \otimes 1 - 1 \otimes x^n)/(x \otimes 1 - 1 \otimes x) = v$. Tensoring with R over R^e gives R in each degree and the maps are $\tilde{u} = x - x = 0$ and $\tilde{v} = nx^{n-1}$. Using Theorem 3.3 we have that $HH_0(R) = R$, this agrees with Section 2.2. As the rest of the resolution is 2-periodic, the homology will also be 2-periodic. We have $\mathrm{Im}\,\tilde{u} = 0$ and $\ker \tilde{u} = R$. Now assume that n is not a factor of $\mathrm{char}(k)$. Then as $\tilde{v} = nx^{n-1}$, the kernel is (x). The image is (x^{n-1}). Together these give $HH_{2i}(k[x]/(x^n)) = (x)$ and $HH_{2i-1} = R/(x^{n-1})$ for $i > 0$. These are isomorphic as R-modules by $\alpha : (x) \to R/(x^{n-1})$ which sends hx to the image of h under the quotient map.

(R is commutative, so $HH_*(R)$ is an R-module).
One can do a very similar thing to get the same result for $HH^*(k[x]/(x^n))$. This example is from [6, Ex 9.1.4].

4 Koszul resolution and polynomial calculations

We want to calculate $HH_i(k[x_1, \dots, x_n])$. To do this we will find a specific finite resolution of rings which can be used to explicitly calculate Tor and Ext, and therefore the Hochschild groups. Mainly based on [6, Ch 4.5].

4.1 Koszul complex

Let R be a k-algebra, let $\mathbf{x} = (x_1, \dots, x_n)$ be a sequence of central elements in R. (i.e. all the x_i are in the centre of R).

Definition 4.1. Set $K_p(\mathbf{x})$ to be the free R-module generated by the symbols $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$ for $1 \leq i_1 < i_2 < \cdots < i_p \leq n$.
Define $d_k : K_p(\mathbf{x}) \to K_{p-1}(\mathbf{x})$ by $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \mapsto x_k e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}$. Where, as usual, the hat means that that term is left out.

This is a presimplicial module (this follows from the fact that the x_i are central), so by setting $d = \sum_{k=1}^{n} (-1)^{k+1} d_k$ (the index is out by 1), we have that $d \circ d = 0$, so this is a complex, called the *Koszul complex* and denoted by $K(\mathbf{x})$. Note that $K_p(\mathbf{x}) \cong \bigwedge^p R^n$.
As an example, consider $K(x, y)$, this is the complex

$$0 \to R \xrightarrow{(y,-x)} R^2 \xrightarrow{\binom{x}{y}} R \to 0$$

Where the maps are written as matrices and the bases are $\{e_x \wedge e_y\}, \{e_x, e_y\}$ and $\{1\}$ respectively.
If we consider $K(x - y)$, we get $0 \to R \xrightarrow{x-y} R \to 0$, note that for $R = k[x, y]$ the first 3 terms are exactly a projective resolution of $k[x]$, so this is a generalization of that resolution.
In general the Koszul complex is not exact, but there is a type of sequence for which it is.

Definition 4.2. Let M be a finitely generated R-module, a *regular sequence* on M is a sequence of elements $x_1, \ldots, x_n$ such that x_1 is not a zero divisor on M, i.e. $x_1 m = 0$ implies $m = 0$, and x_i is not a zero divisor on $M/(x_1, \ldots, x_{i-1})M$.

Theorem 4.3. *Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a regular sequence on R, then $K(\mathbf{x})$ is exact except at the 0^{th} degree where the homology is $R/(x_1, \ldots, x_n)R$.*

For the proof see Appendix A, the proof is not difficult but it is long and does not use techniques that are used later.

Corollary 4.4. *$K(\mathbf{x})$ is a free resolution of $R/(x_1, \ldots, x_n)R$.*

4.2 Hochschild (co)homology of polynomials

We now have enough tools to calculate the Hochschild (co)homology of polynomials in n variables. By Theorem 3.3, $HH_i(R) \cong \mathrm{Tor}_i^{R^e}(R, R)$. So we need to calculate $k[x_1, \ldots, x_n]^e$ and then find a projective resolution of $k[x_1, \ldots, x_n]$.
We have that $k[x_1, \ldots, x_n]^e = k[x_1, \ldots, x_n] \otimes_k k[x_1, \ldots, x_n] \cong k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Call this R. Now we have a natural surjective map $R \to k[x_1, \ldots, x_n]$ which has kernel generated by $x_i - y_i$. (We are identifying the x and y variables). This makes $k[x_1, \ldots, x_n]$ into a R-module and we have $k[x_1, \ldots, x_n] \cong R/(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)R$. It is clear that $(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$ is a regular sequence, so by Corollary 4.4

$$0 \to \bigwedge^n R^n \xrightarrow{d} \bigwedge^{n-1} R^n \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^1 R^n \xrightarrow{d} R \to k[x_1, \ldots, x_n] \to 0$$

is a free resolution of $k[x_1, \ldots, x_n]$. We then tensor the resolution by $k[x_1, \ldots, x_n]$ over R. We have $\bigwedge^p R^n \otimes_R k[x_1, \ldots, x_n] \cong \bigwedge^p k[x_1, \ldots, x_n]^n$. The maps all become 0 as $(x_i - y_i)$ acts as 0 on $k[x_1, \ldots, x_n]$, so the complex becomes

$$0 \to \bigwedge^n k[x_1, \ldots, x_n]^n \xrightarrow{0} \bigwedge^{n-1} k[x_1, \ldots, x_n]^n \xrightarrow{0} \cdots \xrightarrow{0} \bigwedge^1 k[x_1, \ldots, x_n]^n \xrightarrow{0} k[x_1, \ldots, x_n] \to 0.$$

The homology of this is clearly $\bigwedge^i k[x_1, \ldots, x_n]^n$ in degree i. In effectively the same way,
$\mathrm{Ext}^i_R(k[x_1, \ldots, x_n], k[x_1, \ldots, x_n]) = \bigwedge^i k[x_1, \ldots, x_n]^n$. (As $\mathrm{Hom}_R(R, M) \cong M$, and in a very

similar way all the maps become 0).

This shows that $HH_i(k[x_1, \dots, x_n]) = HH^i(k[x_1, \dots, x_n]) = \wedge^i k[x_1, \dots, x_n]^n$ for $0 \leq i \leq n$ and $HH_i(k[x_1, \dots, x_n]) = HH^i(k[x_1, \dots, x_n]) = 0$ for $i > n$.

This example is from [6, Ex 9.1.13]. Note that for polynomials $HH_i(R) = \wedge^i HH_1(R)$, this is a special case of a more general result, see Section 6 for more details.

5 Morita invariance

We want to know when two rings have the same collection of modules, i.e. when they have equivalent categories. We will then show that swapping a ring for another equivalent ring does not change the Hochschild (co)homology. This whole section is high on definitions, but is relatively straightforward, it is mainly a combination of [7, Ch. 7] and [2, Ch 6.4]. Basic category theory is assumed, see for example [8, Ch. 1] for more details.

5.1 Module categories

Recall that a *natural transformation* $\eta : F \to G$ between two functors is a collection of functions $\eta(A) : F(A) \to G(A)$ for each A, such that

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \eta(A)}\downarrow & & \downarrow{\scriptstyle \eta(B)} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

commutes for any $f : A \to B$.

If $\eta(A)$ is an isomorphism for each A, we call η a *natural isomorphism* and write $F \simeq G$.

Definition 5.1. Two categories $\mathscr{A}, \mathscr{B}$ are *equivalent*, denoted $\mathscr{A} \simeq \mathscr{B}$, if there exist two functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ such that $G \circ F \simeq \mathbf{1}_A, F \circ G \simeq \mathbf{1}_B$.

We only consider one type of category here, the category of all left R-modules and module homomorphisms, ${}_R\mathscr{M}$, and the category of all right R-modules and module homomorphisms, $\mathscr{M}_R$. The goal is to work out when are two such categories equivalent for different rings. There is a first case that we will do now. We write $M_n(R)$ is the ring of all $n \times n$ matrices with entries in R.

Theorem 5.2. *Let R be any ring, set $S = M_n(R)$, then ${}_R\mathscr{M} \simeq {}_S\mathscr{M}$ for any n.*

We will explain this proof in almost full detail to show that this result can be done without many technical results, but after this we explain the proof in a more general way and then return to more general theory rather than algebraic manipulations.

Proof. We need to construct two functors, $F : {}_R\mathscr{M} \to {}_S\mathscr{M}$ and $G : {}_S\mathscr{M} \to {}_R\mathscr{M}$, such that there is a natural transformation between their composition and the identity map.
Let $M \in {}_R\mathscr{M}$ and define $F(M) = M^{(n)}$ where $M^{(n)}$ is elements of the form $(m_1, \dots, m_n) \quad m_i \in M$. Then $M^{(n)}$ is a left S-module with the action being matrix multiplication on the left. Let $\phi : M \to N$ be a module homomorphisms, set $F(\phi)(m_1, \dots, m_n) = (\phi(m_1), \dots, \phi(m_n))$. It is easy to see that $F(\phi)$ is a S-module homomorphism and that F is a functor.
To go the other way, let $U \in {}_S\mathscr{M}$, set $G(U) = e_{11}U$, where e_{11} is the matrix with a 1 in the first row and first column and 0 everywhere else. Then $e_{11}U$ is a left R-module with action induced by multiplication by rI_n, as we have $rI_n e_{11}U = e_{11}rI_nU \subseteq e_{11}U$. For $\phi : U \to V$, we have $\phi(e_{11}U) = e_{11}\phi(U) \subseteq e_{11}V$. So set $G(\phi)$ to be the map induced by ϕ. Again G is a functor.
Consider now $(G \circ F)(M) = G(M^{(n)}) = e_{11}M^{(n)}$ which consists of elements of the form $(m, 0 \dots, 0) \quad m \in M$. This is clearly isomorphic to M in a natural way. The other way is slightly less clear.
Consider $(F \circ G)(U) = (e_{11}U)^{(n)}$. To show this is isomorphic to U, consider the map $\psi : U \to (e_{11}U)^{(n)}$ defined by $\psi(u) = (e_{11}u, e_{12}u, \dots, e_{1n}u)$. This is well defined as $e_{1i} = e_{11}e_{1i}$. It is an S-module homomorphism as $\psi(re_{ij}u) = (e_{11}re_{ij}u, e_{12}re_{ij}u, \dots, e_{1n}re_{ij}u) = (0, \dots, re_{1j}u, \dots, 0)$, with $re_{1j}u$ in the i^{th} position. However $re_{ij}(e_{11}u, \dots, e_{1n}u) = (0, \dots, re_{1j}u, \dots, 0)$ again with $re_{1j}u$ in the i^{th} position by simple matrix multiplication. Now as elements of the form re_{ij} generate S we are done.
To show ψ is injective, assume $\psi(u) = 0$. Then $e_{jj}u = (e_{j1}e_{1j})u = e_{j1}(e_{1j}u) = 0$, so $0 = \sum e_{jj}u = u$. For surjectivity, let $(e_{11}u_1, \dots, e_{11}u_n) \in (e_{11}U)^{(n)}$, then $\psi(e_{11}u_1 + e_{21}u_2 + \dots + e_{n1}u_n) = (e_{11}u_1, e_{11}u_2, \cdots, e_{11}u_n)$.
Finally, ψ is a natural transformation as if $f : U \to U'$ then we have $(\psi \circ f)(u) = (e_{11}f(u), \dots, e_{1n}f(u)$.

On the other hand, $(F(G(f)) \circ \psi)(u) = F(G(f))(e_{11}u, \dots e_{1n}u) = (f(e_{11}u), \dots, f(e_{1n}u)) = (e_{11}f(u), \dots, e_{1n}f(u))$, as f is a module homomorphism.
So we have $G \circ F \simeq \mathbf{1}_{{}_R\mathscr{M}}$ and $F \circ G \simeq \mathbf{1}_{{}_S\mathscr{M}}$ $\square$

Almost exactly the same proof will show that $\mathscr{M}_R \simeq \mathscr{M}_S$. This proof is based on [7, 17B]. The above proof seems very specific to the situation but it can be phrased in a way that gives us an idea for when two rings have equivalent module categories.
Consider $R^{\oplus n}$ as a column vector, it is a left S-module and right R-module, and F is effectively $R^{\oplus n} \otimes_R -$. We can also consider $R^{\oplus n}$ as a row vector, it is a left R-module and right S-module, then G is effectively $R^{\oplus n} \otimes_S -$. (As $e_{11}S$ picks out the top row). We have $R^{\oplus n} \otimes_R R^{\oplus n} \cong S$ and $R^{\oplus n} \otimes_S R^{\oplus n} \cong R$. To see this think about multiplying a row vector with a column vector both possible ways.
This shows that $R^{\oplus n} \otimes_R -$ and $R^{\oplus n} \otimes_S -$ are inverse category equivalences.

Definition 5.3. If ${}_R\mathscr{M} \simeq {}_S\mathscr{M}$ for two rings R and S, we call the rings *Morita Equivalent*.
If a property is preserved by Morita equivalence it is called *Morita invariant*.

Note: We will show later that if ${}_R\mathscr{M} \simeq {}_S\mathscr{M}$, then also $\mathscr{M}_R \simeq \mathscr{M}_S$, that is the reason why there is no need to talk about left (right) Morita equivalence. Before going on to the more general case, we need a few more definitions.

Definition 5.4. Let N be a left R-module, then it is a *generator* if $\mathrm{Hom}(N, -)$ is faithful, i.e. if it does not kill non-zero morphisms.

We have that R and $R^{\oplus k}$ are examples of generators.

Definition 5.5. Let M be a left R-module, let $M^* = \mathrm{Hom}(M, R)$ be the dual module. Then the *trace module*, denoted tr(M), is the submodule of R generated by

$$\{f(m) \text{ for } m \in M, f \in M^*\}.$$

There is a nice equivalent definition of being a generator that uses the trace module.

Proposition 5.6. *Let N be a left R-module, then the following are equivalent,* *1) N is a generator.*

2) $tr(N) = R$.

3) R is a direct summand of $\bigoplus_I P$ for some indexing set I.

4) every $M \in {}_R\mathscr{M}$ is a surjective image of $\bigoplus_I P$.

Proof. $1) \implies 2)$
Assume $\mathrm{tr}(N) \neq R$. Then $\pi : R \to R/\mathrm{tr}(N)$ is non zero, and then as N is a generator, there exists a $\phi \in \mathrm{Hom}(N, R)$ such that $\pi \circ \phi$ is non zero, but then $\phi(N) \not\subset \mathrm{tr}(N)$ which can not be true.
$2) \implies 3)$
By $2)$, there are $g_i \in N^*$ such that $\sum g_i(N) = R$ (in fact we can pick a finite number of g_i). This gives us a map $\phi = (g_1, \ldots, g_n) : P \oplus \cdots \oplus P \to R$ that is surjective. Then as R is projective, we have a map $\psi : R \to P \oplus \cdots \oplus P$ such that $\phi \circ \psi = \mathbf{1}_R$, therefore ψ is a section and R is a direct summand of $P \oplus \cdots \oplus P$.
$3) \implies 4)$
We have that M is the surjective image of a free module, so compose that map with some "power" of the map into R.
$4) \implies 1)$
Let $f : M \to N$ be non zero, then as M is some surjective image of $\bigoplus_I P$, we must have that the composition is non-zero on some factor. □

Condition $4)$ gives some justification to the name generator, and the next Lemma explains their significance for us.

Lemma 5.7. *Let $F, G : {}_R\mathscr{M} \to {}_S\mathscr{M}$ be two functors that are right exact and preserve direct sums. If there exists a natural transformation η, between them, such that $\eta(N) : F(N) \to G(N)$ is an isomorphism for N a generator, then η is a natural isomorphism.*

Proof. Let M be any left R-module, then by Proposition 5.6 we can construct an exact sequence $\bigoplus_i N \xrightarrow{\alpha} M \to 0$. In fact we can extend this by one term to the left by considering $\bigoplus_j N \xrightarrow{\beta} \ker(\alpha) \to 0$. Combining we get $\bigoplus_j N \to \bigoplus_i N \to M \to 0$. Apply F and G to get a commutative diagram

$$\begin{array}{ccccccc} F(\bigoplus_j N) & \longrightarrow & F(\bigoplus_i N) & \longrightarrow & F(M) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta(M)} & & \\ G(\bigoplus_j N) & \longrightarrow & G(\bigoplus_i N) & \longrightarrow & G(M) & \longrightarrow & 0 \end{array}$$

Now as F, G both preserve direct sums, and $\eta(N)$ is an isomorphism, we get that η_1 and η_2 are both isomorphisms, then apply the 5-lemma to get that $\eta(M)$ is also an isomorphism. □

5.2 Morita Equivalence

We have already seen that any ring is Morita equivalent to all of its matrix rings, we want to find a way to describe the other ways two rings can be Morita equivalent.

In general given a bimodule we can get a functor (⊗ or Hom), it turns out that the converse is partially true, by adding a condition on the functor we can realize it using a bimodule. One of these situations is right exact functors preserving direct sums in module categories.

Theorem 5.8 (Eilenberg-Watts)**.** *Let $F = N \otimes_R - : {}_R\mathscr{M} \to {}_S\mathscr{M}$ for some (S, R)-bimodule N. Then F is right exact and preserves direct sums. The converse also holds, any right exact functor that preserves direct sums is naturally isomorphic to tensoring by a bimodule.*

Proof. The first property is a well known fact about the tensor product, the important part is the converse.

The idea that is key to this proof is simple, give $F(R)$ a right R-module structure, this is done by looking at the endomorphisms of R and sending them through F and using the fact that R has a natural right action. The actual details which follow are slightly technical.

Let $F : {}_R\mathscr{M} \to {}_S\mathscr{M}$ be right exact and preserve direct sums.

First: define $\phi_m : R \to M$ by $\phi_m(r) = rm$. This is a homomorphism of left R-modules. We get an induced map $F(\phi_m) : F(R) \to F(M)$. Now define $\alpha^M : F(R) \times M \to F(M)$ by $\alpha^M(\bar{r}, m) = F(\phi_m)(\bar{r})$. If we set $M = R$, then we get a right R-module structure on $F(R)$ as $\bar{r}(r_1 r_2) = \alpha^R(\bar{r}, r_1 r_2) = F(\phi_{r_1 r_2})(\bar{r})$ and $(\bar{r} r_1) r_2 = (F(\phi_{r_1})(\bar{r}) r_2 = F(\phi_{r_2}) \circ F(\phi_{r_1})(\bar{r}) = F(\phi_{r_2} \circ \phi_{r_1})(\bar{r})$. Now $\phi_{r_2} \circ \phi_{r_1}(t) = t r_1 r_2 = \phi_{r_1 r_2}(t)$ so the result follows. It is a bimodule as $(s\bar{r})r' = F(\phi_{r'})(s\bar{r}) = sF(\phi_{r'})(\bar{r}) = s(\bar{r}r')$.

So we have a map $\alpha^M : F(R) \times M \to F(M)$, this map is clearly bilinear and $\alpha^M(\bar{r}r', m) =$

$\alpha^M(F(\phi_{r'})(\bar{r}), m) = F(\phi_m) \circ F(\phi_{r'})(\bar{r}) = F(\phi_{r'm})(\bar{r}) = \alpha^M(\bar{r}, r'm)$ so it lifts to a map α^M : $F(R) \otimes_R M \to F(M)$. This is a natural transformation as given $f : M \to M'$ we have $\bar{r} \otimes m \mapsto \bar{r} \otimes f(m) \mapsto F(\phi_{f(m)})(\bar{r})$. Going the other way round we have $\bar{r} \otimes m \mapsto F(\phi_m)(\bar{r}) \mapsto F(f) \circ F(\phi_m)(\bar{r}) = F(\phi_{f(m)})(\bar{r})$. The final step follows as $f(\phi_m(r)) = f(rm) = rf(m)$.
As this map is clearly an isomorphism for $M = R$ and R is a generator, by Lemma 5.7, we get that $F \simeq F(R) \otimes_R -$. □

This proof is based on [11], in this paper there is also a similar condition for when functors are equivalent to $\mathrm{Hom}(M, -)$ for a bimodule M.

Corollary 5.9. *For any rings R, S the following are equivalent,*

1) *There exists functors $F : {}_R\mathscr{M} \to {}_S\mathscr{M}, G : {}_S\mathscr{M} \to {}_R\mathscr{M}$ which are right exact and preserve direct sums, and such that $G \circ F \simeq \mathbf{1}_{{}_R\mathscr{M}}$.*

2) *There exist two bimodules, ${}_RQ_S, {}_SP_R$ such that $Q \otimes_S P \cong R$.*

Proof. Given the functors F and G, by Elienberg-Watts, we have that $F \simeq P \otimes_R -, G \simeq Q \otimes_S -$, for some P, Q. Consider $G \circ F \simeq (Q \otimes_S P) \otimes_R - : {}_R\mathscr{M} \to {}_R\mathscr{M}$. We have that $G \circ F \simeq \mathbf{1}_{{}_R\mathscr{M}}$, so $(Q \otimes_S P) \otimes_R R \cong R$, which shows that $Q \otimes_S P \cong R$.
Conversely, define $F = P \otimes_R -, G = Q \otimes_S -$. Then by Elienberg-Watts, we have that both F, G are right exact and preserve direct sums, also as $Q \otimes_S P \cong R$ we have that $G \circ F \simeq \mathbf{1}_{{}_R\mathscr{M}}$. □

Note: In the above situation, we also get functors $F' = - \otimes_R Q : \mathscr{M}_R \to \mathscr{M}_S$ and $G' = - \otimes_S P : \mathscr{M}_S \to \mathscr{M}_R$, such that $G' \circ F' \simeq \mathbf{1}_{\mathscr{M}_R}$. We now have enough results to prove conditions for two rings to be Morita Equivalent.

Theorem 5.10. *Let R, S be two rings, then the following are equivalent,*

1) *a) ${}_R\mathscr{M} \simeq {}_S\mathscr{M}$*

   a') $\mathscr{M}_R \simeq \mathscr{M}_S$.

2) *There exist a pair of bimodules ${}_RQ_S, {}_SP_R$ such that $Q \otimes_S P \cong R$, and $P \otimes_R Q \cong S$.*

Proof. 2) $\implies$ 1)
Define $F : {}_R\mathscr{M} \to {}_S\mathscr{M}$ by $F(M) = P \otimes_R M$ and $G : {}_S\mathscr{M} \to {}_R\mathscr{M}$ by $G(N) = Q \otimes_S N$. Then

$(F \circ G)(N) = P \otimes_R (Q \otimes_S N) \cong S \otimes_S N \cong N$, and $(G \circ F)(M) = Q \otimes_S (P \otimes_R M) \cong R \otimes_R M \cong M$. Similarly $F' = - \otimes_R Q$, and $G' = - \otimes_S P$ give category equivalences between $\mathscr{M}_R$ and $\mathscr{M}_S$.

$1)\, a) \implies 2)$

There exist $F : {}_R\mathscr{M} \to {}_S\mathscr{M}$ and $G : {}_S\mathscr{M} \to {}_R\mathscr{M}$ such that $F \circ G \simeq \mathbf{1}_{{}_S\mathscr{M}}$ and $G \circ F \simeq \mathbf{1}_{{}_R\mathscr{M}}$. Category equivalences preserve categorical definitions so are exact (therefore right exact) and preserve direct sums so we can apply Corollary 5.9 twice to get $2)$. (By following the proof of Corollary 5.9 it is clear that the same P, Q will work for both directions).

The note gives us that $a)$ and $a')$ are equivalent. $\square$

5.3 The Morita Context

Having seen the conditions for when two rings are Morita equivalent, we wish to describe P and Q more concretely, in doing so, we will also set up what we need to show that Hochschild (co)homology is a Morita invariant.

Let R be any ring, P any right R-module, then we can define $Q = \mathrm{Hom}_R(P, R)$, it is the dual of P, and another ring $S = \mathrm{End}_R(P)$, the R-endomorphisms of P. By letting S act on the left of P, we get a left S-module which is in fact a (S, R)-bimodule as $s(pr) = (sp)r$, because s is a R-homomorphism.

We can also turn Q into a (R, S)-bimodule, first let $(rq)p = r(qp)$, this works as R is a (R, R)-bimodule, then set $(qs)p = q(sp)$, again as P as a (S, R)-bimodule.

Note: $qs \in Q$ as $(qs)(pr) = q(s(pr)) = q((sp)r) = q(sp)r = ((qs)(p))r$, and we have the bimodule property, $((rq)s)p = (rq)(sp) = r(q(sp)) = r((qs)(p)) = (r(qs))p$.

Lemma 5.11. *In the above notation, we have well defined homomorphisms $\alpha : Q \otimes_S P \to R$, and $\beta : P \otimes_R Q \to S$.*

Proof. First, note that as we have ${}_SP_R$ and ${}_RQ_S$ both tensor products make sense, and are (R, R) and (S, S)-bimodules respectively.

For α define a map $Q \times P \to R, \quad (q, p) \mapsto qp$, this map is clearly linear in both arguments and $(qs)p = q(sp)$, so this map lifts to a map from $Q \otimes_S P$, this map is a (R, R)-homomorphism as $(rq)p = r(qp)$, and $q(pr) = (qp)r$.

For β, the same argument works, as pq defines an element of S by $(pq)p' = p(qp')$. We have $((pr)q)(p') = (pr)(qp') = p(r(qp')) = p((rq)(p')) = (p(rq))(p')$, so the map lifts. Finally one can show that the map is an (S, S)-homomorphism using a similar argument. $\square$

We can sum up all the above properties by saying that we have a ring $R \oplus Q \oplus P \oplus S$ which we think of as $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ with the standard matrix operations. This ring is called the *Morita Ring* associated with P_R, we call $(R, P, Q, S; \alpha, \beta)$ the *Morita Context* associated with P_R.
For an example consider $P = R^{\oplus n}$, then $Q = R^{\oplus n}$. We also have $S = \mathrm{End}_R(R^{\oplus n}) = M_n(R)$; this is exactly the set-up used in Theorem 5.2.
In the rest of this section, we work in some fixed Morita Context.

Proposition 5.12.

1) *P_R is a generator iff α is surjective.*

2) *P_R is finitely generated and projective iff β is surjective.*

Proof. 1) The map α has image $q(p)$ for $p \in P$, $q \in Q = \mathrm{Hom}_R(P, R)$. In other words, Im$(\alpha)$ is the trace module, so the result follows by Proposition 5.6.
2) We have that β is surjective iff $1_S = \sum p_i q_i$. This implies that $p = (\sum p_i q_i)\, p = \sum p_i q_i(p)$ for all $p \in P$. This is true iff P is finitely generated and projective by the Dual Basis Lemma [1]. $\square$

Corollary 5.13.

1) *α surjective implies that α is an isomorphism.*

2) *β surjective implies that β is an isomorphism.*

Proof. 1) Assume $\sum q_i p_i = 0$, as α is surjective, we have $1_R = \sum q_j' p_j'$. Consider $\sum q_i \otimes_S p_i = \sum (q_j' p_j') q_i \otimes_S p_i = \sum q_j' (p_j' q_i) \otimes_S p_i = \sum q_j' \otimes_S (p_j' q_i) p_i = \sum q_j' \otimes_S p_j'(q_i p_i) = 0$.
2) Again, we have by assumption $1_S = \sum p_i' q_i'$, assume $\sum p_j q_j = 0$. Exactly the same argument as above shows that $\sum p_j \otimes_R q_j = 0$. $\square$

In light of this result we make a definition,

[1] P is projective if and only if there exists $\{a_i \in P | i \in I\}$ and $\{f_i \in P^* | i \in I\}$ such that for all $a \in P$, $f_i(a) = 0$ for all but finitely many i and $a = \sum_{i \in I} a_i f_i(a)$, see [7, Sec. 2B]

Definition 5.14. If M, a right R-module is finitely generated, projective and a generator, it is called a *progenerator*.

So combining Corollary 5.13 and Theorem 5.10, we see that given a progenerator P, we get Morita equivalent rings in the Morita Context. In this case, Q is also a progenerator as it is the image of S under a category equivalence, and clearly S is a progenerator (category equivalences preserve all "category theory" definitions). Note that $R^{\oplus n}$ from the example is a progenerator. If we start with a category equivalence, by Theorem 5.10, we can take our two functors to be tensor products, and we get modules P and Q which are progenerators as they are the images of S and R, these modules also have nice properties.

Proposition 5.15. *Let P, Q be as in Theorem 5.10. Then we have,*

1) $P \cong \mathrm{Hom}_R(Q, R) \cong \mathrm{Hom}_S(Q, S)$

2) $Q \cong \mathrm{Hom}_R(P, R) \cong \mathrm{Hom}_S(P, S)$

3) $R \cong \mathrm{End}_S(P) \cong \mathrm{End}_S(Q)$

4) $S \cong \mathrm{End}_R(P) \cong \mathrm{End}_R(Q)$.

Proof. All eight statements follow from the fact that $\mathrm{Hom}_R(M, F(N)) \cong \mathrm{Hom}_S(G(M), N)$ and other similar results. This is because we have an induced map $\mathrm{Hom}_R(M, M') \to \mathrm{Hom}_S(F(M), F(M'))$ which is an isomorphism as it has an inverse induced by G, by setting $M' = F(N)$ the fact follows. Similarly we also have $\mathrm{Hom}_R(G(A), B) \cong \mathrm{Hom}_S(A, F(B))$ and similar statements for $F' = - \otimes_R Q$, and $G' = - \otimes_S P$.
1) $P \cong \mathrm{Hom}_S(S, P) \cong \mathrm{Hom}_S(S, F(R)) \cong \mathrm{Hom}_R(G(S), R) \cong \mathrm{Hom}_R(Q, R)$.
2) $Q \cong \mathrm{Hom}_R(R, Q) \cong \mathrm{Hom}_R(R, G(S)) \cong \mathrm{Hom}_S(F(R), S) \cong \mathrm{Hom}_S(P, S)$.
3) $\mathrm{End}_S(P) \cong \mathrm{Hom}_S(P, P) \cong \mathrm{Hom}_S(F(R), F(R)) \cong \mathrm{Hom}_R(R, G(F(R))) \cong R$.
4) $\mathrm{End}_R(Q) \cong \mathrm{Hom}_R(Q, Q) \cong \mathrm{Hom}_R(G(S), G(S)) \cong \mathrm{Hom}_S(S, F(G(S))) \cong S$.
The other statements following using F' and G'. $\square$

This shows that given the functors, we can create a Morita Context, it also show the full symmetry between R and S, and between P and Q. We can also get a category equivalence between the category of bimodules using $P \otimes_R - \otimes_R Q$.

5.4 Hochschild (co)homology is a Morita invariant

Note that any categorical property is preserved by Morita Equivalence, including a module being projective, injective, faithful, finitely generated and Noetherian. Some of the ring properties preserved include semisimple, left(right) Noetherian and left(right) Artinian. As seen above in Theorem 5.2, commutativity is not preserved, however the centre of the ring is.

Lemma 5.16. *The centre is a Morita invariant.*

Proof. To prove this we want a way of describing the centre of ${}_R\mathscr{M}$ categorically. To do this, call the set of all natural transformations from the identity functor to itself C. These natural transformations can be added and composed, turning C into a ring. We claim that this ring is isomorphic to the centre of the ring R.
Construct $\mu : Z(R) \to C$ as follows, given $r \in Z(R)$ and $M \in {}_R\mathscr{M}$, let $\mu^M(r) : M \to M$ be the left action by r. This is an element of C as given any $f : M \to M'$ we have that $rf(m) = f(rm)$ clearly. This map is injective as if $\mu(r) = 0$, then consider $M = R$ and the image of 1, we get $0 = r$. Now let γ be any element in C. As $\gamma(s) = \gamma(1)s$ and for any $r \in R$ we have an endomorphism $t \mapsto rt$ and therefore $\gamma(1)r = r\gamma(1)$ (consider the image of 1 in the commutative diagram). This shows that $\gamma(1) \in Z(R)$. Now let $M \in {}_R\mathscr{M}$, we have a map $R \to M$ given by the R action. As γ is a natural transformation we get $\gamma(1 \cdot m) = \gamma(1) \cdot m$, this shows that $\gamma = \mu(\gamma(1))$. □

Remember that $HH^0(R) = Z(R)$, so if R and S are equivalent then $HH^0(R) \cong HH^0(S)$. In fact this result generalizes massively.

Theorem 5.17. *Let P and Q be as above. Then $H_i(R, M) \cong H_i(S, P \otimes_R M \otimes_R Q)$ and $H^i(R, M) \cong H^i(S, P \otimes_R M \otimes_R Q)$ for any R-bimodule M.*

Proof. From the above work we have isomorphisms $\phi : P \otimes_R Q \to S$ and $\psi : Q \otimes_S P \to R$. These maps satisfy $\phi(p \otimes q)p' = (pq)p' = p(qp') = p\psi(q \otimes p')$, see Lemma 5.11, and also $q\phi(p \otimes q') = \psi(q \otimes p)q'$. This is because $(q(pq'))p'' = q((pq')p'') = q(p(q'p'')) = (qp)(q'p'') = ((qp)q')p''$, see the start of Section 5.3. Call these properties †.
We also have elements $p_1, \dots p_t, q_1 \dots q_t$ such that $\psi(\sum q_i \otimes p_i) = 1_R$ and elements $p'_1, \dots p'_s, q'_1, \dots q'_s$ such that $\phi(\sum p'_i \otimes q'_i) = 1_S$.

The idea for the rest of the proof is simple, use the special elements p_i, q_i, p'_i, q'_i to create maps between the complexes. Then define a presimplicial homotopy between the maps. The details are messy but none of the individual steps are difficult. We have tried to keep things as simple as possible but at the same time we want to have a complete proof, therefore not all the steps are shown, just the key ones.
We define $\alpha_n : M \otimes R^{\otimes n} \to (P \otimes_R M \otimes_R Q) \otimes S^{\otimes n}$ by

$$\alpha_n(m \otimes r_1 \otimes \cdots r_n) = \sum p_{k_0} \otimes_R m \otimes_R q_{k_1} \otimes \phi(p_{k_1} \otimes r_1 q_{k_2}) \otimes \cdots \otimes \phi(p_{k_n} \otimes r_n q_{k_0}).$$

The sum is over all sets of indices $(k_0, \ldots k_n)$ such that $1 \leq k_* \leq t$. We also have β_n : $(P \otimes_R M \otimes_R Q) \otimes S^{\otimes n} \to M \otimes R^{\otimes n}$ by,

$$\beta_n(p \otimes_R m \otimes_R q \otimes s_1 \otimes \cdots s_n) = \sum \psi(q'_{l_0} \otimes p) m \psi(q \otimes p'_{l_1}) \otimes \psi(q'_{l_1} \otimes s_1 p'_{l_2}) \otimes \cdots \otimes \psi(q'_{l_n} \otimes s_n p'_{l_0}).$$

The sum is over all sets of indices $(l_0, \ldots l_n)$ such that $1 \leq l_* \leq s$. These are maps between complexes as $d_i \alpha_n = \alpha_{n-1} d_i$ and $d_i \beta_n = \beta_{n-1} d_i$. This follows by applying †, the argument is very similar to ones later on in this proof which are covered in more depth.
Define a map $h_i : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n+1}$ given by

$$h_i(m \otimes r_1 \otimes \cdots r_n) = \sum m \psi(q_{k_0} \otimes p'_{l_0}) \otimes \psi(q'_{l_0} \otimes p_{k_0}) r_1 \psi(q_{k_1} \otimes p'_{l_1}) \otimes \cdots$$

$$\cdots \otimes \psi(q'_{l_{i-1}} \otimes p_{k_{i-1}}) r_i \psi(q_{k_i} \otimes p'_{l_i}) \otimes \psi(q'_{l_i} \otimes p_{k_i}) \otimes r_{i+1} \otimes \cdots \otimes r_n.$$

Again the sum is over all sets $(k_0, \ldots k_n)$ such that $1 \leq k_* \leq t$ and $(l_0, \ldots l_n)$ such that $1 \leq l_* \leq s$.
Claim: The h_i are a presimplicial homotopy between $\beta_n \circ \alpha_n$ and the identity map.
First: $d_0 h_0 = id$.
We have $d_0 h_0(m \otimes r_1 \otimes \cdots r_n) = d_0 \left(\sum m \psi(q_{k_0} \otimes p'_{l_0}) \otimes \psi(q'_{l_0} \otimes p_{k_0}) \otimes r_1 \cdots \otimes r_n \right) = \sum m \psi(q_{k_0} \otimes p'_{l_0}) \psi(q'_{l_0} \otimes p_{k_0}) \otimes r_1 \cdots \otimes r_n$. Now ψ is an R-homomorphism so $\psi(q_{k_0} \otimes p'_{l_0}) \psi(q'_{l_0} \otimes p_{k_0}) = \psi(q_{k_0} \otimes p'_{l_0} \psi(q'_{l_0} \otimes p_{k_0})) = \psi(q_{k_0} \otimes \phi(p'_{l_0} \otimes q'_{l_0}) p_{k_0})$. The second equality is †. By taking the sum over l_0 then over k_0 we get the identity map.
Second: $d_{n+1} h_n = \beta_n \circ \alpha_n$.

We have $\beta_n \circ \alpha_n(m \otimes r_1 \otimes \cdots r_n) = \beta_n \left(\sum p_{k_0} \otimes_R m \otimes_R q_{k_1} \otimes \phi(p_{k_1} \otimes r_1 q_{k_2}) \otimes \cdots \otimes \phi(p_{k_n} \otimes r_n q_{k_0})\right) = \sum \psi(q'_{l_0} \otimes p_{k_0}) m \psi(q_{k_1} \otimes p'_{l_1}) \otimes \psi(q'_{l_1} \otimes \phi(p_{k_1} \otimes r_1 q_{k_2}) p'_{l_2}) \otimes \cdots \otimes \psi(q'_{l_n} \otimes \phi(p_{k_n} \otimes r_n q_{k_0}) p'_{l_0})$.

Now $\psi(q'_{l_{i-1}} \otimes \phi(p_{k_{i-1}} \otimes r_i q_{k_i}) p'_{l_i}) = \psi(q'_{l_{i-1}} \otimes p_{k_{i-1}} \psi(r_i q_{k_i} \otimes p'_{l_i}) = \psi(q'_{l_{i-1}} \otimes p_{k_{i-1}}) r_i \psi(q_{k_i} \otimes p'_{l_i})$.

(Using † and R-hom.) (1)

We also have that $d_{n+1} h_n(m \otimes r_1 \otimes \cdots r_n) = d_{n+1}(\sum m \psi(q_{k_0} \otimes p'_{l_0}) \otimes \psi(q'_{l_0} \otimes p_{k_0}) r_1 \psi(q_{k_1} \otimes p'_{l_1}) \otimes \cdots \cdots \otimes \psi(q'_{l_{n-1}} \otimes p_{k_{n-1}}) r_n \psi(q_{k_n} \otimes p'_{l_n}) \otimes \psi(q'_{l_n} \otimes p_{k_n})) = \sum \psi(q'_{l_n} \otimes p_{k_n}) m \psi(q_{k_0} \otimes p'_{l_0}) \otimes \psi(q'_{l_0} \otimes p_{k_0}) r_1 \psi(q_{k_1} \otimes p'_{l_1}) \otimes \cdots \otimes \psi(q'_{l_{n-1}} \otimes p_{k_{n-1}}) r_n \psi(q_{k_n} \otimes p'_{l_n})$. Using the rearrangement in (1) and relabelling the indices we get the desired result.

Third: $d_i h_j = h_j d_{i-1}$ when $i > j+1$.

This follows immediately without any need for rearrangement.

Fourth: $d_i h_j = h_{j-1} d_i$ for $i < j$.

This follows from the fact that $\psi(q'_{l_{i-1}} \otimes p_{k_{i-1}}) r_i \psi(q_{k_i} \otimes p'_{l_i}) \psi(q'_{l_i} \otimes p_{k_i}) r_{i+1} \psi(q_{k_{i+1}} \otimes p'_{l_{i+1}}) = \psi(q'_{l_{i-1}} \otimes p_{k_{i-1}}) r_i r_{i+1} \psi(q_{k_{i+1}} \otimes p'_{l_{i+1}})$. Again by using † and R-hom.

Fifth: $d_i h_i = d_i h_{i-1}$ for $0 < i \leq n$.

This follows in a very similar way to the fourth one. We have $\psi(q'_{l_{i-1}} \otimes p_{k_{i-1}}) r_i \psi(q_{k_i} \otimes p'_{l_i}) \psi(q'_{l_i} \otimes p_{k_i}) = \psi(q'_{l_{i-1}} \otimes p_{k_{i-1}}) r_i$.

As all these hold, we have by Lemma 2.4 that $\beta_n \circ \alpha_n$ is homotopic to the identity map.

We also have a presimplicial homotopy from $\alpha_n \circ \beta_n$ to the identity map given by

$$\bar{h}_i(p \otimes_R m \otimes_R q \otimes s_1 \otimes \cdots s_n) = \sum p \otimes_R m \otimes_R q \phi(p'_{k_0} \otimes q_{l_0}) \otimes \phi(p_{l_0} \otimes q'_{k_0}) s_1 \phi(p'_{k_1} \otimes q_{l_1}) \otimes \cdots$$

$$\cdots \otimes \phi(p_{l_{i-1}} \otimes q'_{k_{i-1}}) s_i \phi(p'_{k_i} \otimes q_{l_i}) \otimes \phi(p_{l_i} \otimes q'_{k_i}) \otimes s_{i+1} \otimes \cdots \otimes s_n.$$

This is a presimplicial homotopy for the same reasons as h_i by symmetry. The only difference is that to show that $d_{n+1} h_n = \alpha_n \circ \beta_n$ we need to use the fact that $pr \otimes_R m \otimes_R q = p \otimes_R rm \otimes_R q$ and $p \otimes_R m \otimes_R rq = p \otimes_R mr \otimes_R q$.

Together these give the result. □

This proof is based on [9, Thm. 1.2.7]. There is a different proof using bicomplexes in [6, Thm. 9.5.6]. One can do something very similar to get the Morita invariance of Hochschild cohomology.

6 Kähler Differentials

When we are working with commutative k-algebras we can give an interpretation for Hochschild homology which uses language and ideas from geometry, especially manifolds. Here it is purely algebraic. This section is based on [9, Ch. 1.3].

6.1 Derivations and Differentials

We have already seen derivations in Section 2.2, but will define them here. They are an algebraic generalization of the derivative operation.

Definition 6.1. A derivation of R with values in M is a k-linear map $D : R \to M$ such that $D(rs) = rD(s) + D(r)s \quad \forall r, s \in R$.
The module of all derivations is denoted $\mathrm{Der}(R, M)$, or $\mathrm{Der}(R)$ when $M = R$.

Any $m \in M$ defines an *inner derivation*, $\text{ad}(m)$, given by $\text{ad}(m)r = [m, r] = mr - rm$. We can also define a similar map to act on $C_n(R, M)$

$$\text{ad}(s)(m_0 \otimes r_1 \otimes \cdots \otimes r_n) = \sum_{i=0}^{n} m_0 \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes [s, r_i] \otimes r_{i+1} \otimes \cdots \otimes r_n.$$

It is easy to check that $\text{ad}(s)$ commutes with d, the Hochschild boundary. We can also calculate what map it induces on Hochschild homology.

Lemma 6.2. *Define $h(s) : C_n(R, M) \to C_{n+1}(R, M)$ by*

$$h(s)(m_0 \otimes r_1 \otimes \cdots \otimes r_n) = \sum_{i=0}^{n} (-1)^i m_0 \otimes r_1 \otimes \cdots \otimes r_i \otimes s \otimes r_{i+1} \otimes \cdots \otimes r_n.$$

Then we have $dh(s) + h(s)d = -ad(s)$.

Proof. Set $h_i(s)(m_0 \otimes r_1 \otimes \cdots \otimes r_n) = m_0 \otimes r_1 \otimes \cdots \otimes r_i \otimes s \otimes r_{i+1} \otimes \cdots \otimes r_n$. Then $h(s) = \sum (-1)^i h_i(s)$. We have $d_i h_j(s)(m_0 \otimes r_1 \otimes \cdots \otimes r_n) = d_i(m_0 \otimes r_1 \otimes \cdots \otimes r_j \otimes s \otimes r_{j+1} \otimes \cdots \otimes r_n) = (m_0 \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_j \otimes s \otimes r_{j+1} \otimes \cdots \otimes r_n)$ for $i < j$. This is equal to $h_{j-1} d_i$, and it is also easy to check that $d_i h_j = h_j d_{i-1}$ for $i > j + 1$. By the same argument as Lemma 2.4, we get $dh(s) + h(s)d = d_0 h_0 - d_{n+1} h_n + \sum_i (d_i h_i - d_i h_{i-1})$.

We have $(d_0 h_0 - d_{n+1} h_n)(m_0 \otimes r_1 \otimes \cdots \otimes r_n) = m_0 s \otimes r_1 \otimes \cdots \otimes r_n - s m_0 \otimes r_1 \otimes \cdots \otimes r_n$ and $(d_i h_i - d_i h_{i-1})(m_0 \otimes r_1 \otimes \cdots \otimes r_n) = m_0 \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes r_i s \otimes r_{i+1} \otimes \cdots \otimes r_n - m_0 \otimes r_1 \otimes \cdots \otimes r_{i-1} \otimes s r_i \otimes r_{i+1} \otimes \cdots \otimes r_n$. So $dh(s) - h(s)d = -\mathrm{ad}(s)$. $\square$

So we get that $\mathrm{ad}(s) : H_i(R, M) \to H_i(R, M)$ is homotopic to the zero map, this is clear when R is commutative and M is such that $mr = rm$, called *symmetric*. We will need this identity later.

Now let R be commutative.

Definition 6.3. A derivation $d : R \to M$ is *universal*, if for any other derivation $\delta : R \to N$ there is a unique linear map $\phi : M \to N$ such that $\delta = \phi \circ d$. In diagram form,

$$\begin{tikzcd} R \arrow[d, "d"'] \arrow[dr, "\delta"] & \\ M \arrow[r, dashed, "\exists! \phi"] & N \end{tikzcd}$$

Definition 6.4. The module of *Kähler differentials* is the module $\Omega^1_{R|k}$ such that $d : R \to \Omega^1_{R|k}$ is a universal derivation.

As usual for objects defined universally, $\Omega^1_{R|k}$ is unique. We can describe it explicitly.
Consider the module generated by the symbols dr for $r \in R$, add the relations $dc = 0$ for $c \in k$, $d(r+s) = dr + ds, d(rs) = sdr + rds$ for $r, s \in R$, and define d by $s \mapsto ds$. By construction d is a derivation, and if we have $\delta : R \to N$ any other derivation, define ϕ by $dr \mapsto \delta(r)$.
There is also another more concrete way of describing $\Omega^1_{R|k}$, consider $\mu : R \otimes_k R \to R$ where $\mu(r \otimes s) = rs$. Let I be the kernel of μ, and consider I/I^2. I is generated by $r \otimes 1 - 1 \otimes r$ as an R-module. Define $\delta : R \to I/I^2$ by $\delta(x)$ equals the class of $x \otimes 1 - 1 \otimes x$. It is a derivation as $rd(s) + d(r)s = rs \otimes 1 - r \otimes s + r \otimes s - 1 \otimes rs = d(rs)$. We have a unique map $\phi : \Omega^1_{R|k} \to I/I^2$, where $\phi(dr)$ equals the class of $r \otimes 1 - 1 \otimes r$.
We can also define a map $\psi : I/I^2 \to \Omega^1_{R|k}$, given by $\psi(r \otimes 1 - 1 \otimes r) = dr$, so ϕ and ψ are inverses. This map is well defined as $\phi((r \otimes 1 - 1 \otimes r)(s \otimes 1 - 1 \otimes s)) = \phi(rs \otimes 1 - r \otimes s - s \otimes r + 1 \otimes rs) = \phi(s(r \otimes 1 - 1 \otimes r) - (r \otimes 1 - 1 \otimes r)s) = sdr - (dr)s = 0$.
The above calculation also shows that I/I^2 is a symmetric bimodule.
Note: One can show that Der is a functor, and by definition of $\Omega^1_{R|k}$ we have an isomorphism

$\mathrm{Hom}_A(\Omega^1_{R|k}, M) \cong \mathrm{Der}(R, M)$. This shows that Der is a representable functor, represented by $\Omega^1_{R|k}$.

As an example, consider $R = k[x_1, \dots, x_n]$, polynomials in n variables. Consider the map $\delta : R \to R^n$, given by $\delta(f) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$. This map is a derivation as the product rule holds. This mean we have a unique map $\phi : \Omega^1_{R|k} \to R^n$ sending df to $(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$. Consider the map $\psi : R^n \to \Omega^1_{R|k}$ defined by $\psi(f_1, \dots, f_n) = \sum_{i=1}^n f_i dx_i$.
Claim: this map is an inverse to ϕ. We have,
$\phi \circ \psi(f_1, \dots, f_n) = \phi(\sum f_i dx_i) = \sum \phi(f dx_i) = (f_1, \dots, f_n)$ by linearity, and $\psi \circ \phi(df) = \psi(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}) = \sum \frac{\partial f}{\partial x_i} dx_i$. So sufficient to show that $df = \sum \frac{\partial f}{\partial x_i} dx_i$.
This holds as $d(x^2) = xdx + xdx = 2xdx$ and by induction $d(x^n) = nx^{n-1}dx = \frac{\partial x^n}{\partial x} dx$. We can then repeatedly use $d(gx_i) = gdx_i + x_i dg$ to write everything in terms of the dx_i.
This shows that $\Omega^1_{R|k} \cong \bigoplus_{i=1}^n R dx_i$, when $R = k[x_1, \dots, x_n]$.
We can now give a description of $H_1(R, M)$.

Lemma 6.5. *Let R be commutative and M symmetric, then $H_1(R, M) \cong M \otimes_R \Omega^1_{R|k}$.*

Proof. By earlier work in Section 2.2, we have that $H_1(R, M) = M \otimes R/\{mr_1 \otimes r_2 - m \otimes r_1 r_2 + r_2 m \otimes r_1\}$. We have a map $m \otimes r \mapsto m \otimes_R dr$, which is well defined as $mr_1 \otimes_R dr_2 - m \otimes_R d(r_1 r_2) + r_2 m \otimes_R dr_1 = 0$ (swap r_2 and m, use $mr \otimes_R ds = m \otimes_R rds$ and use the relations in $\Omega^1_{R|k}$). This map has an inverse, $m \otimes_R dr \mapsto m \otimes r$, again it is well defined by the quotient condition. $\square$

This agrees with our earlier work with polynomials in Section 4.2 where we saw that $HH_1(k[x_1, \dots, x_n]) = k[x_1, \dots, x_n]^n$. We can also find $\Omega^1_{R|k}$ for $R = k(x)/(x^n)$. By Section 3.3 we have $HH_1(R) = (x)$. One can describe/write this as $k[x]dx/(x^n, x^{n-1}dx)$.

We can also define higher degree differentials.

Definition 6.6. The R-module of *differential n-forms* is $\Omega^n_{R|k} = \bigwedge^n_R \Omega^1_{R|k}$.

As an example, for $R = k[x_1, \dots, x_n]$, we have $\Omega^i_{R|k} = \bigwedge^i R^n$, and for $R = k(x)/(x^n)$, we have $\Omega^i_{R|k} = 0$ for $i > 1$.

6.2 Relationship between Differentials and Hochschild Homology

We have already seen that $H_1(R,M) = M \otimes_R \Omega^1_{R|k}$ and for polynomial rings, $HH_n(R) = \Omega^n_{R|k}$, by Section 4.2. We will look at the relationship between $HH_i(R)$ and $\Omega^n_{R|k}$ in general.
For now R is not assumed to be commutative.

Definition 6.7. Let $\sigma \in S_n$ then σ acts on $C_n(R,M)$ as

$$\sigma(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes r_{\sigma^{-1}(1)} \otimes \cdots \otimes r_{\sigma^{-1}(n)}.$$

Extend linearly to get an action of $k[S_n]$ on $C_n(R,M)$ and define the *antisymmetrization element* ϵ_n as

$$\epsilon_n = \sum_{\sigma \in s_n} \text{sgn}(\sigma)\sigma.$$

Definition 6.8. Let $\epsilon_n : M \bigotimes \bigwedge^n R \to C_n(R,M)$ be the map given by,

$$m \otimes r_1 \wedge \cdots \wedge r_n \mapsto \epsilon_n(m \otimes r_1 \otimes \cdots \otimes r_n).$$

This is a misuse of notation but it is clear what we mean. We want to know how ϵ_n and d, the Hochschild boundary, interact. To do this we need a another map.

Definition 6.9. The *Chevalley-Eilenberg map*, $\delta : M \bigotimes \bigwedge^n R \to M \bigotimes \bigwedge^{n-1} R$, is given by

$$\delta(m \otimes r_1 \wedge \cdots \wedge r_n) = \sum_{i=1}^{n} (-1)^{i+1} [m, r_i] \otimes r_1 \wedge \cdots \wedge \widehat{r_i} \wedge \cdots \wedge r_n$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} m \otimes [r_i, r_j] \wedge r_1 \wedge \cdots \wedge \widehat{r_i} \wedge \cdots \wedge \widehat{r_j} \wedge \cdots \wedge r_n.$$

Proposition 6.10. *Let R be any k-algebra , M any R-bimodule then the following square commutes,*

$$\begin{array}{ccc} M \bigotimes \bigwedge^n R & \xrightarrow{\epsilon_n} & C_n(R,M) \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle d} \\ M \bigotimes \bigwedge^{n-1} R & \xrightarrow{\epsilon_{n-1}} & C_{n-1}(R,M) \end{array}$$

Proof. By induction on n,
for $n = 1$, we have $\epsilon_1 = 1$ and $d(m \otimes r) = mr - rm$. We also have $\epsilon_0 = 1$ and $\delta(m \otimes r) = mr - rm$,

so $d \circ \epsilon_1 = \epsilon_0 \circ \delta$.

Now we claim that $\epsilon_{n+1}(m \otimes r_1 \wedge \cdots \wedge r_n \wedge s) = (-1)^n h(s)\epsilon_n(m \otimes r_1 \wedge \cdots \wedge r_n)$, where $h(s)$ is the map from Lemma 6.2. To see this calculate both sides

$$\epsilon_{n+1}(m \otimes r_1 \wedge \cdots \wedge r_n \wedge s) = \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma)\sigma(m \otimes r_1 \otimes \cdots \otimes r_n \otimes s$$

$$(-1)^n h(s)\epsilon_n(m \otimes r_1 \wedge \cdots \wedge r_n) = \sum_{\sigma \in S_n} (-1)^n \text{sgn}(\sigma) h(s)\sigma(m \otimes r_1 \otimes \cdots \otimes r_n)$$

$$= \sum_{\sigma \in S_n} (-1)^n \text{sgn}(\sigma) \sum_i (-1)^i (m \otimes r_{\sigma^{-1}(1)} \otimes \cdots \otimes r_{\sigma^{-1}(i)} \otimes s \otimes r_{\sigma^{-1}(i+1)} \otimes \ \cdots \otimes r_{\sigma^{-1}(n)}).$$

Now to move the s to the first position (straight after the m) requires i swaps, so a sign change of $(-1)^i$, then to move back to the end requires another n swaps, a sign change of $(-1)^n$. So both extra signs cancel out, and we get the desired relationship.

Now assume the result for n, consider $n+1$. We have

$$\begin{aligned} d\epsilon_{n+1}(m \otimes r_1 \wedge \cdots \wedge r_n \wedge s) &= (-1)^n dh(s)\epsilon_n(m \otimes r_1 \wedge \cdots \wedge r_n) \\ &= (-1)^n(-\text{ad}(s) - h(s)d)\epsilon_n(m \otimes r_1 \wedge \cdots \wedge r_n) \\ &= -(-1)^n \text{ad}(s)\epsilon_n(m \otimes r_1 \wedge \cdots \wedge r_n) - (-1)^n h(s)\epsilon_{n-1}\delta(m \otimes r_1 \wedge \cdots \wedge r_n) \\ &= -(-1)^n \text{ad}(s)\epsilon_n(m \otimes r_1 \wedge \cdots \wedge r_n) + \epsilon_n(\delta(m \otimes r_1 \wedge \cdots \wedge r_n) \wedge s). \end{aligned}$$

Where the 1^{st} and 4^{th} lines come from the earlier claim. The 2^{nd} line follows from Lemma 6.2 and the 3^{rd} line follows from induction assumption.

Now consider $\epsilon_n(\delta(m \otimes r_1 \wedge \cdots \wedge r_n \wedge s)) = \epsilon_n(\delta(m \otimes r_1 \wedge \cdots \wedge r_n) \wedge s) + (-1)^n \epsilon_n([m, s] \otimes r_1 \wedge \cdots \wedge r_n) + \sum_{i=1}^n (-1)^{i+n+1} \epsilon_n(m \otimes [r_i, s] \wedge r_1 \wedge \cdots \wedge \widehat{r_i} \wedge \cdots \wedge r_n)$. It is therefore sufficient to prove that $-\text{ad}(s)\epsilon_n(m \otimes r_1 \wedge \cdots \wedge r_n) = \epsilon_n([m, s] \otimes r_1 \wedge \cdots \wedge r_n) + \sum_{i=1}^n (-1)^{i+1} \epsilon_n(m \otimes [r_i, s] \wedge r_1 \wedge \cdots \wedge \widehat{r_i} \wedge \cdots \wedge r_n)$. We have

$$\begin{aligned} -\text{ad}(s)\epsilon_n(m \otimes r_1 \wedge \cdots \wedge r_n) &= -\sum_{\sigma \in S_n} \text{sgn}(\sigma)\text{ad}(s)\sigma(m \otimes r_1 \otimes \cdots \otimes r_n) \\ &= -\sum_{\sigma \in S_n} \text{sgn}(\sigma)[s, m] \otimes r_{\sigma^{-1}(1)} \otimes \cdots \otimes r_{\sigma^{-1}(n)} \end{aligned}$$

$$- \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \sum_{i=1}^{n} m \otimes r_{\sigma^{-1}(1)} \otimes \cdots \otimes r_{\sigma^{-1}(i-1)} \otimes [s, r_{\sigma^{-1}(i)}] \otimes \cdots \otimes r_{\sigma^{-1}(n)}$$

$$= \epsilon_n([m,s] \otimes r_1 \wedge \cdots \wedge r_n)$$
$$+ \sum_{i=1}^{n} (-1)^{i+1} \epsilon_n(m \otimes [r_i, s] \wedge r_1 \wedge \cdots \wedge \widehat{r_i} \wedge \cdots \wedge r_n).$$

□

Note that if R is commutative and M is symmetric, then clearly $\delta = 0$, and therefore $d \circ \epsilon_n = 0$.

Proposition 6.11. *For R a commutative k-algebra and a symmetric bimodule M we have a map*

$$\epsilon_n : M \bigotimes_R \Omega^n_{R|k} \to H_n(R, M).$$

Proof. Take the homology on both sides, as $\delta = 0$ and by Proposition 6.10, $d \circ \epsilon_n = 0$, we get a map $\epsilon_n : M \otimes \bigwedge^n R \to H_n(M, R)$. It is sufficient to see that $m \otimes xy \wedge \cdots \ - mx \otimes y \wedge \cdots \ - my \otimes x \wedge \cdots$ gets sent to an element in the image of d. It turns out that $-\sum_\sigma \mathrm{sgn}(\sigma)\sigma(m \otimes x \otimes y \otimes \cdots)$ where σ is such that $\sigma(1) < \sigma(2)$ is in the preimage of d, so the map is well defined. □

We can also construct a map going the other way.

Lemma 6.12. *Let $\pi_n : C_n(R, M) \to M \otimes_R \Omega^n_{R|k}$ be given by, $\pi_n(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes dr_1 dr_2 \ldots dr_n$. Then $\pi_n \circ d = 0$.*

Proof. $\pi_n d(m \otimes r_1 \otimes \cdots \otimes r_n) = \pi_n(mr_1 \otimes r_2 \otimes \cdots \otimes r_n - m \otimes r_1 r_2 \otimes \cdots \otimes r_n + \cdots + (-1)^n r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1}) = mr_1 dr_2 \ldots dr_n - md(r_1 r_2) \ldots dr_n + mdr_1 d(r_2 r_3) \ldots dr_n + \cdots + r_n m dr_1 \ldots dr_{n-1}$.
Now apply the derivation property to get
$mr_1 dr_2 \ldots dr_n - mr_1 dr_2 \ldots dr_n - mr_2 dr_1 dr_3 \ldots dr_n + mr_2 dr_1 dr_3 \ldots dr_n + \ldots$
It is clear that all the terms cancel. □

Proposition 6.13. *For R a commutative k-algebra and a symmetric bimodule M we have a well defined map*

$$\pi_n : H_n(R, M) \to M \bigotimes_R \Omega^n_{R|k}.$$

Proof. This follows from the above Lemma directly. □

Note that π_n is surjective, in fact we can say more than that.

Theorem 6.14. *We have $\pi_n \circ \epsilon_n = n!id$.*

Proof. Consider $\pi_n \epsilon_n(m \otimes dr_1 \dots dr_n) = \pi_n \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma(m \otimes r_1 \otimes \dots \otimes r_n)\right) =$
$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) m dr_{\sigma^{-1}(1)} dr_{\sigma^{-1}(2)} \dots dr_{\sigma^{-1}(n)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)^2 dr_1 \dots dr_n$.
The result now follows as $|S_n| = n!$. □

This shows that if $\mathbb{Q} \subset k$, then we have that $M \otimes_R \Omega^n_{R|k}$ is a direct summand of $H_n(R, M)$. (As $\epsilon_n/n!$ is a section of π_n). In the polynomial case the direct summand is in fact the whole of $HH_n(R)$, but for the case of truncated polynomials we only get the zero module as a direct summand.
By adding an extra condition one can strengthen this result.

Theorem 6.15 (Hochschild-Kostant-Rosenberg)**.** *Let R be a smooth k-algebra, then the antisymmetrisation map $\epsilon_n : \Omega^n_{R|k} \to HH_n(R)$ is an isomorphism for all n.*

A commutative k-algebra is *smooth*[2] if it is flat over k and if for any maximal ideal $\mathfrak{m}$ of R, the kernel of the localized map $\mu_{\mathfrak{m}} : (R \otimes_k R)_{\mu^{-1}(\mathfrak{m})} \to R_{\mathfrak{m}}$ is generated by a regular sequence in $(R \otimes_k R)_{\mu^{-1}(\mathfrak{m})}$.
Note: This also shows that $k(x)/(x^n)$ can not be smooth.

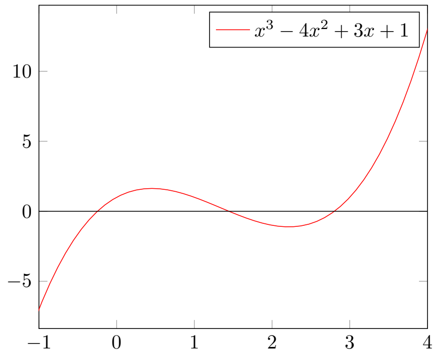
7 Deformations

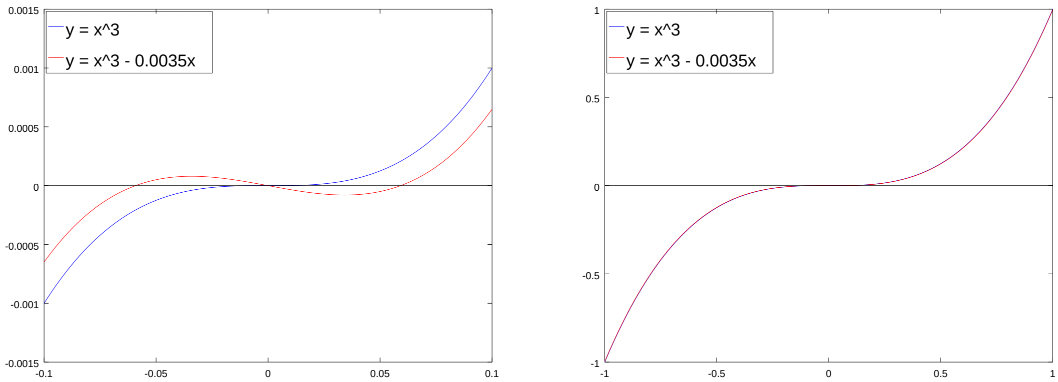
In this section we want to describe an interpretation for $HH^2(R)$. This is based on [4] and [10, 3].

[2]This definition of smooth and the statement and proof of the HKR theorem come from [9, 101-102]

7.1 Motivating example

Consider a general cubic $ax^3 + bx^2 + cx + d$. One would expect it to have 3 roots, but if we pick the cubic x^3 then it only has the single root $x = 0$. There is a way to fix that, we slightly deform the cubic by adding a small term.





The effect of subtracting $0.0035x$ to x^3.

By deforming the cubic by ϵx for any small value of ϵ, the geometry has been changed locally around $x = 0$, so that there are now 3 roots, but on a large scale, the geometry is still the same.

7.2 Deformations of algebras

Let R be a k-algebra, we will 'deform' it by 'deforming' the multiplication. Let $r \star s = rs + \epsilon f(r, s)$ be some new multiplication, we want to think of ϵ being small, the way we will do that is by declaring that $\epsilon^2 = 0$. One can make this more formal.

Definition 7.1. The *dual numbers* are $k[\epsilon]/(\epsilon^2)$.

Now consider $R \bigotimes_k k[\epsilon]/(\epsilon^2) \cong R \bigoplus \epsilon R$, we want a multiplication on $R \bigoplus \epsilon R$ that can be thought of as a deformation of R, so set $(r_1 + \epsilon r_2)(s_1 + \epsilon s_2) = r_1 s_1 + \epsilon r_1 s_2 + \epsilon r_2 s_1 + \epsilon f(r_1, s_1)$. There is no need to consider a $r_2 s_2$ term as $\epsilon^2 = 0$. It follows that $f : R \bigotimes_k R \to R$ determines the new multiplication.

Definition 7.2. An *infinitesimal deformation* of R is a k-algebra $R \otimes_k k[\epsilon]/(\epsilon^2)$ such that $r \star s = rs \mod \epsilon$.

By the above, an infinitesimal deformation is determined by $f : R \otimes R \to R$, however not all such functions will give an associative multiplication.

Lemma 7.3. *f as above gives an associative multiplication $r \star s$, if $f \in \ker \delta : \mathrm{Hom}(R \otimes R, R)$, where δ is the Hochschild coboundary map.*

Proof. We calculate $(r \star s) \star t$ and $r \star (s \star t)$.
$(r \star s) \star t = (rs + \epsilon f(r,s)) \star t = rst + \epsilon f(r,s)t + \epsilon f(rs,t)$. We also have $r \star (s \star t) = r \star (st + \epsilon f(s,t)) = rst + \epsilon r f(s,t) + \epsilon f(r,st)$. For these two to be equal we need that $f(r,s)t + f(rs,t) = rf(s,t) + f(r,st)$. Rearranging we get $rf(s,t) - f(rs,t) + f(r,st) - f(r,s)t = 0$ which is exactly the condition that $\delta f = 0$. $\square$

Definition 7.4. A infinitesimal deformation is *trivial* if there exists a $k[\epsilon]/(\epsilon^2)$ automorphism ϕ of $R \otimes k[\epsilon]/(\epsilon^2)$ such that $\phi = id \mod \epsilon$ and the following diagram commutes

$$\begin{array}{ccc} \left(R \otimes k[\epsilon]/(\epsilon^2)\right)^2 & \xrightarrow{(\phi,\phi)} & \left(R \otimes k[\epsilon]/(\epsilon^2)\right)^2 \\ \downarrow{\star} & & \downarrow{\cdot} \\ R \otimes k[\epsilon]/(\epsilon^2) & \xrightarrow{\phi} & R \otimes k[\epsilon]/(\epsilon^2) \end{array}$$

Where $\cdot$ is the multiplication with $f = 0$.

Definition 7.5. A k-algebra R is *rigid* if there are no non-trivial deformations.

Assume f is a trivial deformation, then there exists ϕ such that $\phi(r_1 + \epsilon r_2)\phi(s_1 + \epsilon s_2) = \phi((r_1 + \epsilon r_2) \star (s_1 + \epsilon s_2))$. Expanding the LHS, we get,
$\phi(r_1 + \epsilon r_2)\phi(s_1 + \epsilon s_2) = (r_1 + \epsilon\phi_1(r_1) + \epsilon r_2)(s_1 + \epsilon\phi_1(s_1) + \epsilon s_2) = r_1 s_1 + \epsilon r_1 \phi_1(s_1) + \epsilon r_1 s_2 + \epsilon\phi_1(r_1)s_1 + \epsilon r_2 s_1$.
Expanding the RHS we get
$\phi(r_1 s_1 + \epsilon(r_1 s_2 + r_2 s_1) + \epsilon f(r_1, s_1)) = r_1 s_1 + \epsilon\phi_1(r_1 s_1) + \epsilon(r_1 s_2 + r_2 s_1) + \epsilon f(r_1, s_1)$.
Equating the two we get
$f(r_1, s_1) = r_1\phi_1(s_1) - \phi_1(r_1 s_1) + \phi_1(r_1)s_1 = (\delta\phi_1)(r_1, s_1)$.
This, along with Lemma 7.3 proves;

Theorem 7.6. *$HH^2(R)$ is the group of all infinitesimal deformations of R, quotiented by trivial deformations.*

Corollary 7.7. *If $HH^2(R) = 0$ then R is rigid.*

WRONG!!!! As an example, consider $R = k[x,y]$, by Section 4.2 we have that $HH^2(k[x,y]) = k[x,y]$. So every polynomial $p \in k[x,y]$ defines a deformation. The multiplication is $f \star_p g = fg + \epsilon p(f,g)$.
We also know by Section 3.3 that $HH^2(k[x]/(x^n)) = k[x]/(x^{n-1})$ (as a module). So in the case when $n = 3$, we have the deformed multiplication given by,

- $1 \star x^i = x^i$
- $x \star x = x^2$
- $x \star x^2 = \epsilon(ax + b)$
- $x^2 \star x^2 = \epsilon(ax^2 + bx)$

Where $\star$ is $k[\epsilon]/(\epsilon^2)$-linear and commutative. This gives us $k[x,\epsilon]/(\epsilon^2, x^3 - \epsilon(a + bx))$ instead of $k[x]/(x^3)$. Note that this is very similar to the motivational example given at the beginning of this section. In general we have that $k[x,\epsilon]/(\epsilon^2, x^n - \epsilon f)$ is a deformation of $k[x]/(x^n)$ for f a polynomial of degree $n - 2$.

7.3 Higher order deformations

The infinitesimal deformations that were dealt with above were of the form $r \star s = rs + \epsilon f_1(r,s)$, they are also known as *first order deformations*.

Definition 7.8. A n^{th} *order deformation* is an associative multiplication on $R \otimes k[e]/(\epsilon^{n+1})$ of the form $r \star s = rs + \sum_{i=1}^{n} \epsilon^i f_i(r,s)$, where $f_i : R \otimes R \to R$.

Given a first order deformation we want to know when it extends to a second order one. (Note: changing $\epsilon^2 = 0$ to $\epsilon^3 = 0$ and adding $\epsilon^2 f_2(r,s)$ to the multiplication does not change the associativity for the ϵ terms).
Let $r \star s = rs + \epsilon f_1(r,s) + \epsilon^2 f_2(r,s)$, then $(r \star s) \star t = (rs + \epsilon f_1(r,s) + \epsilon^2 f_2(r,s)) \star t = rst + \epsilon f_1(r,s)t + \epsilon^2 f_2(r,s)t + \epsilon f_1(rs,t) + \epsilon^2 f_1(f_1(r,s),t) + \epsilon^2 f_2(rs,t)$. We also have that $r \star (s \star t) =$

$r\star(st+\epsilon f_1(s,t)+\epsilon^2 f_2(s,t)) = rst+\epsilon r f_1(s,t)+\epsilon^2 r f_2(s,t)+\epsilon f_1(r,st)+\epsilon^2 f_1(r,f_1(s,t))+\epsilon^2 f_2(r,st)$.
As noted above the ϵ terms already satisfy the associativity condition, so for this multiplication to be associative we require $rf_2(s,t)-f_2(rs,t)+f_2(r,st)-f_2(r,s)t = f_1(f_1(r,s),t)-f_1(r,f_1(s,t))$.
The LHS is δf_2 so f_2 exists if the RHS is a Hochschild coboundary. If we set $h(r_1,r_2,r_3) = f_1(f_1(r_1,r_2),r_3) - f_1(r_1,f_1(r_2,r_3))$ then one can check that $\delta h = 0$, this follows by using the properties of f_1 (use the fact that $\delta f_1 = 0$).
This shows that we can use any first order deformation of R to get an element of $HH^3(R)$, this element is called an *obstruction*. If this obstruction is zero then this deformation extends to a second order one. We have therefore proved.

Theorem 7.9. *If $HH^3(R) = 0$ then all first order deformations extend to second order ones.*

We can go further and consider extending an n^{th} order deformation.
Given an n^{th} order deformation $\sum_{i=0}^{n} \epsilon^i f_i$, where $f_0(r,s) = rs$, we want to know when it extends to a $(n+1)^{th}$ order deformation.
Assume it extends to $\sum_{i=0}^{n+1} \epsilon^i f_i$ we need this to be associative, as before we only need to consider the terms with coefficient ϵ^{n+1}. After doing the calculations and rearranging we get $\delta f_{n+1} = F(f_1,\ldots,f_n)$ where F is some function. One can show that $\delta F = 0$, for a proof see [4, Section 5]. This shows that again we have that an obstruction to extending an n^{th} order deformation to a $(n+1)^{th}$ one is a class in $HH^3(R)$.
For a general introduction on infinitesimal deformations and deformations in general see [3].

A Kozsul Exactness Proof

Proof of Theorem 4.3. It is clear that the image of $d : R^n \to R$ is $(x_1, \ldots, x_n)M$. This gives us the result for degree 0. For $\mathbf{x} = x_1$, we have $0 \to R \xrightarrow{x_1} R \to 0$, and as we have assumed that x_1 is not a zero divisor, the x_1 action on R is injective. This is the base case.

Now assume that $K(x_1, \ldots, x_{n-1})$ satisfies the induction assumption.

Consider $K(x_1, \ldots, x_{n-1}) \otimes K(x_n)$ this is the double complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R \otimes R & \xrightarrow{d'\otimes 1} & \left(\bigwedge^{n-2} R^{n-1}\right) \otimes R & \xrightarrow{d'\otimes 1} & \left(\bigwedge^{n-3} R^{n-1}\right) \otimes R & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle (-1)^{n-1}\otimes x_n} & & \Big\downarrow{\scriptstyle (-1)^{n-2}\otimes x_n} & & \Big\downarrow{\scriptstyle (-1)^{n-3}\otimes x_n} & & \\
0 & \longrightarrow & R \otimes R & \xrightarrow{d'\otimes 1} & \left(\bigwedge^{n-2} R^{n-1}\right) \otimes R & \xrightarrow{d'\otimes 1} & \left(\bigwedge^{n-3} R^{n-1}\right) \otimes R & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \left(\bigwedge^{2} R^{n-1}\right) \otimes R & \xrightarrow{d'\otimes 1} & \left(\bigwedge R^{n-1}\right) \otimes R & \xrightarrow{d'\otimes 1} & R \otimes R & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (-1)^{2}\otimes x_n} & & \Big\downarrow{\scriptstyle -1\otimes x_n} & & \Big\downarrow{\scriptstyle 1\otimes x_n} & & \\
\cdots & \longrightarrow & \left(\bigwedge^{2} R^{n-1}\right) \otimes R & \xrightarrow{d'\otimes 1} & \left(\bigwedge R^{n-1}\right) \otimes R & \xrightarrow{d'\otimes 1} & R \otimes R & \longrightarrow & 0
\end{array}$$

Above and from now on in this proof all tensor products are over R, d' is the map from $K(x_1, \ldots, x_{n-1})$. The homology of this double complex is the homology of the left-right diagonals $\left(\bigwedge^i R^{n-1}\right) \otimes R \oplus \left(\bigwedge^{i+1} R^{n-1}\right) \otimes R$. (It is easy to show that left-right diagonals form a complex, the signs of the vertical maps are chosen for this reason).

Claim: the homology of the double complex is the homology of $K(x_1, \ldots, x_n)$.

Consider $\bigwedge^i (R^{n-1} \oplus R) \cong \oplus_j \bigwedge^{i-j} R^{n-1} \otimes \bigwedge^j R \cong (\bigwedge^{i-1} R^{n-1}) \otimes R \oplus (\bigwedge^i R^{n-1}) \otimes R$. Under this identification, the first $\otimes R$ has basis dx_n and the second one has basis 1. The map

$(\bigwedge^{i-1} R^{n-1}) \otimes R \oplus (\bigwedge^i R^{n-1}) \otimes R \to (\bigwedge^{i-2} R^{n-1}) \otimes R \oplus (\bigwedge^{i-1} R^{n-1}) \otimes R$ is given by

$(d' \otimes 1, (-1)^{i-1} \otimes x_n + d' \otimes 1)$. Now this is the same as the map d from $K(x_1, \ldots, x_n)$, as the first $d' \otimes 1$ and $(-1)^{i-1} \otimes x_n$ is for when the i-form contains dx_n and the second $d' \otimes 1$ is for when the i-form does not contain dx_n.

Now in general if the rows are exact, then the homology is zero. To see this consider the sub diagram

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & C_{i+1,1} & \xrightarrow{h_{i+1,1}} & C_{i,1} & \xrightarrow{h_{i,1}} & C_{i-1,1} \\
 & & \Big\downarrow{\scriptstyle v_{i+1,1}} & & \Big\downarrow{\scriptstyle v_{i,1}} & & \\
C_{i+2,0} & \xrightarrow{h_{i+2,0}} & C_{i+1,0} & \xrightarrow{h_{i+1,0}} & C_{i,0} & \longrightarrow & \cdots
\end{array}
$$

Where h_i are the horizontal maps and v_j are the vertical maps, note that the squares anticommute. Now consider $(c_{i,1}, c_{i+1,0})$ and assume this is sent to zero, i.e. $(h_{i,1}(c_{i,1}), v_{i,1}(c_{i,1}) + h_{i+1,0}(c_{i+1,0}) = (0,0)$. As the rows are exact, we can lift $c_{i,1}$ to $e_{i+1,1}$. Then we have $v_{i,1}(c_{i,1}) + h_{i+1,0}(c_{i+1,0}) = v_{i,1}(h_{i+1,1}(e_{i+1,1})) + h_{i+1}(c_{i+1,0}) = h_{i+1,0}(c_{i+1,0} - v_{i,1}(e_{i+1,1}))$ by anticommuting squares. So we can again lift to $e_{i+2,0}$ such that $h_{i+2,0}(e_{i+2,0}) + v_{i,1}(e_{i+1,1}) = c_{i+1,0}$. This shows that $(e_{i+1,1}, e_{i+2,0})$ is a lift of $(c_{i,1}, c_{i+1,0})$.

By assumption $K(x_1, \ldots, x_{n-1})$ is exact everywhere apart from degree 0. So we get that $K(x_1, \ldots, x_n)$ is exact everywhere apart from degree 1 and 0. For degree 0, the image is clearly generated by $(x_1, \ldots, x_n)$. So we only need to show exactness at degree 1.

The proof of exactness above can be adapted to show that if the rows are exact apart from the end, then one can replace the final term in the rows by the cokernel, and that the degree 1 homology depends on the final vertical map only. In our case we have $R/(x_1, \ldots, x_{n-1})R \xrightarrow{x_n} R/(x_1, \ldots, x_{n-1})R$, by the fact that $(x_1, \ldots, x_n)$ is a regular sequence, this map is injective, so the degree 1 homology is 0. $\square$

One can adapt the proof of the exactness of the double complex if the rows are exact to a general bounded bicomplex. The same result holds for columns.

References

[1] Cohn, P.M. *Algebra Vol. 2 - Second Edition*, John Wiley and Sons

[2] Cohn, P.M. *Algebra Vol. 3 - Second Edition*, John Wiley and Sons

[3] Fox, T. F. *An Introduction To Algebraic Deformation Theory*, Journal of Pure and Applied Algebra, 84, no. 1 (1993): 17-41. doi:10.1016/0022-4049(93)90160-U. (accessed June 12, 2017)

[4] Gerstenhaber, M. *On the Deformation of Rings and Algebras.* Annals of Mathematics, Second Series, 79, no. 1 (1964): 59-103. doi:10.2307/1970484. (accessed June 12, 2017)

[5] Ginzburg, V. *Lectures on Noncommutative Geometry*, arXiv:math/0506603v1 (accessed June 12, 2017)

[6] Weibel, C. *An introduction to Homological algebra*, Cambridge University Press

[7] Lam, T.Y. *Lectures on Modules and Rings*, Springer (Graduate Texts in Mathematics 189)

[8] Leinster, T. *Basic Category Theory*, Cambridge University Press

[9] Loday, J-L. *Cyclic Homology - Second Edition*, Springer

[10] Schedler, T. *Deformations of Algebras in Noncommutative algebraic geometry*, Lecture Notes, `http://math.mit.edu/~trasched/deformations.pdf` (accessed June 12, 2017)

[11] Watts, C. E. *Intrinsic Characterizations of Some Additive Functors.* Proceedings of the American Mathematical Society 11, no. 1 (1960): 5-8. doi:10.2307/2032707. (accessed June 12, 2017)