LOWER BOUNDS ON STEKLOV EIGENFUNCTIONS FOR A CLASS OF CURVILINEAR POLYGONS

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ABSTRACT. Steklov eigenfunctions, $\{\phi_{\sigma_j}\}_{j=0}^{\infty}$, are the eigenfunctions of the Dirichlet to Neumann map on a bounded domain with eigenvalue σ_j. This paper considers Steklov eigenfunctions and eigenvalues on a family of curvilinear polygons, Ω_α whose boundaries are smooth away from a single corner of angle α. We (i) prove optimal lower bounds on the decay of the harmonic extension, u_{σ_j}, of ϕ_{σ_j} away from the boundary of Ω_α and (ii) give optimal lower bounds on the degree to which the high energy Steklov spectrum of $\Omega_\alpha \setminus H_1$ may differ from that of $\Omega_\alpha \setminus H_0$ when H_1 and H_0 are isometric. This shows that the corresponding estimates in the recent work by the authors together with Malagutti are optimal [GMW26].

The proof in this paper was obtained through an interaction with ChatGPT 6 Astra: While writing the work [GMW26], the authors became curious about optimal error bounds for the difference between the Steklov spectra on a curvilinear polygon, $\mathcal{P}$, and the spectrum of *any* operator which knows only local information about $\partial\mathcal{P}$. The authors suggested to ChatGPT to remove two isometric, smooth domains from a curvilinear polygon. ChatGPT suggested the ideas for a proof based on a regularized trace and the singularity in the kernel of the DtN map near a corner. The authors then suggested a simpler, conformal approach, the method for handling small angles, and the idea to obtain lower bounds on Steklov eigenfunctions. The paper is entirely written by the authors and the argument has been significantly modified, clarified, and simplified from the ChatGPT ideas.

1. INTRODUCTION

Steklov eigenfunctions for $\Omega \Subset \mathbb{R}^d$ with Lipschitz boundary are non-zero $\phi_{\sigma_j} \in L^2(\partial\Omega)$ solving

$$-\Delta u_{\sigma_j} = 0 \text{ in } \Omega, \qquad \partial_\nu u_{\sigma_j} - \sigma_j u_{\sigma_j} = 0 \text{ on } \partial\Omega, \qquad u_{\sigma_j}|_{\partial\Omega} = \phi_{\sigma_j}.$$

That is, ϕ_{σ_j} are the eigenfunctions with eigenvalue σ_j for the Dirichlet-to-Neumann map, Λ^Ω satisfying $\Lambda^\Omega f := \partial_\nu \mathcal{H}^\Omega f$, where $\mathcal{H}^\Omega : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ satisfies

$$-\Delta \mathcal{H}^\Omega f = 0 \text{ in } \Omega, \qquad \mathcal{H}^\Omega f|_{\partial\Omega} = f.$$

It is well known that there are $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \dots$ such that $\{\phi_{\sigma_j}\}_{j=0}^{\infty}$ form an orthonormal basis for $L^2(\partial\Omega)$ [AM12].

This paper considers a family of domains in $\mathbb{R}^2 \cong \mathbb{C}$ given as follows. Let $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to \Gamma \subset \mathbb{C}$ be a smooth diffeomorphsim $0 \notin \operatorname{supp}(\gamma(s) - s)$ and such that

$$\Gamma \subset \{z \in \mathbb{C} \,:\, 0 \leq \arg z < 2\pi \min(\tfrac{\pi}{\alpha}, 1)\}.$$

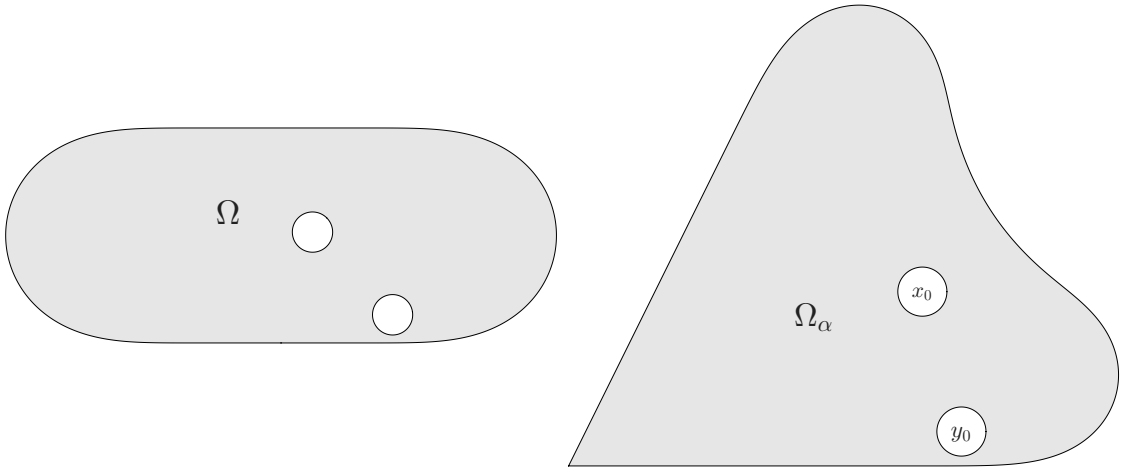


FIGURE 1. The domains Ω and Ω_α in an example where $\alpha = \frac{\pi}{2\sqrt{2}}$ with the holes used in Theorem 1.2 removed.

Let $\mathbb{C} \setminus \Gamma =: \Omega \sqcup \Omega_{\text{out}}$, where Ω is bounded, define

$$f_\alpha : \{z \in \mathbb{C} \,:\, 0 \leq \arg z < 2\pi \min(\tfrac{\pi}{\alpha}, 1)\} \to \mathbb{C},$$

$$f_\alpha(z) := \begin{cases} \exp\left(\frac{\alpha}{\pi} \log z\right) & z \neq 0 \\ 0 & z = 0 \end{cases}, \qquad \text{with the branch log chosen so that } 0 \leq \Im \log z < 2\pi \text{ on } \Omega.$$

and set $\Omega_\alpha := f_\alpha(\Omega)$. We denote by

$$\mathcal{E} := \left\{ \tfrac{\pi}{n} \,:\, 0 < n \in \mathbb{Z} \right\},$$

the set of *exceptional angles*.

Our first theorem provides a lower bound on u_{σ_j} in Ω_α.

Theorem 1.1. *Let $\alpha \in (0, 2\pi) \setminus \mathcal{E}$. Then for all $x_0 \in \Omega_\alpha$ and $r > 0$,*

$$\limsup_{j \to \infty} \sigma_j^{\frac{\pi}{\alpha}} \|u_{\sigma_j}\|_{L^2(B(x_0, r) \cap \Omega_\alpha)} > 0.$$

When a domain Ω has smooth boundary $u_{\sigma_j} = O(\sigma_j^{-\infty})$ in the interior of Ω [HL01, GT19, GPPS14] and this, perhaps surprisingly, persists for polygons all of whose angles lie in $\mathcal{E}$ [LPPS22]. Theorem 1.1 is the first result demonstrating that Steklov eigenfunctions need not decay super-polynomially into the interior. In another very recent work [DLTP26], the authors show that, on domains with C^2 boundary, harmonic Steklov eigenfunctions need not decay superpolynomially. Moreover, in a recent paper [GMW26], the authors together with Malagutti show that, at least modulo an ϵ loss in the power, Theorem 1.1 is optimal in the sense that on a curvilinear polygon all of whose angles are less than or equal to $\alpha_{\max}$ every eigenfunction decays like $C_\epsilon \sigma_j^{-\frac{\pi}{\alpha_{\max}} + \epsilon}$ away from the boundary.

On curvilinear polygons, $\mathcal{P}$, the article [GMW26] also proves an asymptotic formula for the Steklov spectrum of the form

$$|\sigma_j - \lambda_j(\mathcal{D})| \leq C_\epsilon \sigma_j^{-\frac{2\pi}{\alpha_{\max}}+\epsilon}, \tag{1.1}$$

where $\mathcal{D}$ is a Dirac operator on the boundary of $\mathcal{P}$ that is constructed using only local information about $\partial\mathcal{P}$. A second goal of this article is to show that (1.1) is optimal modulo an ϵ loss in the sense that there are two curvilinear polygons whose boundaries have identical local information and whose Steklov spectra differ by at least $\sigma_j^{-\frac{2\pi}{\alpha}}$.

To do this, we consider puncturing Ω_α by two different holes of the same radius, H_1 centered at x_0 and H_0 at y_0. We show that, despite the fact that *all* local information about $\Omega_\alpha \setminus H_1$ is the same as that for $\Omega_\alpha \setminus H_0$, the high frequency spectrum is substantially perturbed by the movement of H_1 to H_0.

Theorem 1.2. *Let $\alpha \in (0, 2\pi) \setminus \mathcal{E}$. Then for all $r_0 > 0$, there is $\epsilon_0 > 0$ such that for all $x_0 \in \Omega_\alpha$ with $d(x_0, \partial\Omega_\alpha) > r_0$ and all $0 < \epsilon < \epsilon_0$ there is $y_0 \in \Omega_\alpha$ with $B(y_0, \epsilon) \subset \Omega_\alpha$ such that*

$$\limsup_{j\to\infty} \left(\sigma_j\big(\Lambda^{\Omega_\alpha\setminus B(x_0,\epsilon)}\big) + \sigma_j\big(\Lambda^{\Omega_\alpha\setminus B(y_0,\epsilon)}\big)\right)^{\frac{2\pi}{\alpha}} \left|\sigma_j\big(\Lambda^{\Omega_\alpha\setminus B(x_0,\epsilon)}\big) - \sigma_j\big(\Lambda^{\Omega_\alpha\setminus B(y_0,\epsilon)}\big)\right| > 0.$$

Remark 1.1. It is easy to see that for any curvilinear polygon, $cj \leq \sigma_j \leq Cj$ (see e.g. Lemma 5.1) and hence that one may replace

$$\sigma_j\big(\Lambda^{\Omega_\alpha\setminus B(x_0,\epsilon)}\big) + \sigma_j\big(\Lambda^{\Omega_\alpha\setminus B(y_0,\epsilon)}\big)$$

in Theorem 1.2 and σ_j in Theorem 1.1 by j without changing the statement.

Remark 1.2. It is not hard to see from the proof of Theorem 1.2 that the shape of the hole cut in Ω_α is unimportant. What is used in the proof is, in fact, that the hole has diameter $\sim \epsilon$ and there is $N > 0$ such that the hole has volume larger than ϵ^N.

Outline of the proof

The proofs of Theorem 1.1 and 1.2 both rely on a trace lower bound together with a variational principle that identifies the quadratic form associated with the difference between the DtN map on a domain Ω with and without a hole, $H \Subset \Omega$. The general philosophy that lower bounds on a relative trace can be used to understand spectra has many precedents (see e.g. [Dav13, SZ16, Chr17, MV09]).

In this paper, we consider the difference

$$\mathscr{D}^\flat(\Omega, H) := 1_{\partial\Omega}\Lambda^{\Omega\setminus H}1_{\partial\Omega} - \Lambda^\Omega,$$

and show in Lemma 2.1 that, given $f \in H^{\frac{1}{2}}(\partial\Omega)$, $\langle \mathscr{D}^\flat(\Omega, H)f, f\rangle$ is the $\mathcal{H}^\Omega f$ capacity of H in Ω (see e.g. [AFH$^+$19] for another use of this quantity in spectral estimates).

From this, we obtain two important consequences. First, we can control

$$\langle \mathscr{D}^\flat(\Omega, f_\alpha^{-1}(H))\delta_x, \delta_x\rangle \tag{1.2}$$

and second, if $f = \phi_{\sigma_j}$ is a Steklov eigenfunction on $\partial\Omega_\alpha$, then

$$\langle \mathscr{D}^\flat(\Omega_\alpha, H)\phi_{\sigma_j}, \phi_{\sigma_j}\rangle \leq C\|u_{\sigma_j}\|^2_{L^2(\tilde{H})}, \tag{1.3}$$

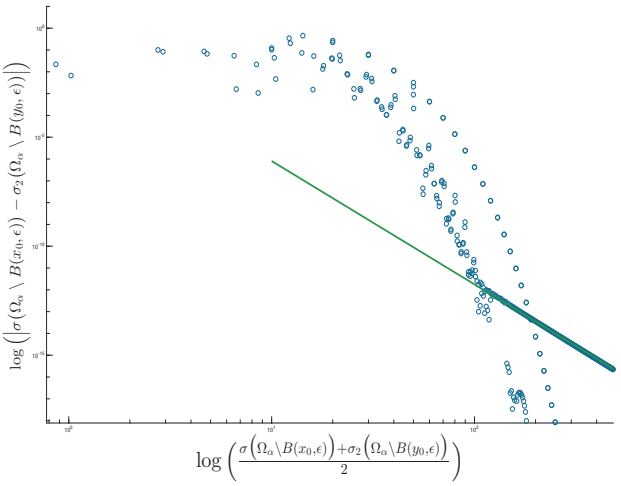


FIGURE 2. Log-log plot of the numerically computed differences $|\sigma_j\big(\Lambda^{\Omega_\alpha\setminus B(x_0,\epsilon)}\big) - \sigma_j\big(\Lambda^{\Omega_\alpha\setminus B(y_0,\epsilon)}\big)|$, with the domain Ω_α pictured in Figure 1, $y_0 = 0.7 + 0.25i$ and $x_0 = e^{i\pi/(3\sqrt{2})}y_0$ and $\epsilon = 0.1$. The solid green line is a line with slope $-4\sqrt{2} = -2\pi/\alpha$. This line has the same slope as the lower bound from Theorem 1.2 and, up to an ϵ loss, the upper bound from [GMW26, Theorem 1.1]

where $H \Subset \tilde{H}$.

With Theorem 1.1 in mind, the inequality (1.3) makes finding a lower bound on a certain trace of $\mathscr{D}^\flat(\Omega_\alpha, H)$ natural. Specifically, to obtain a lower bound on $\|u_{\sigma_j}\|^2_{L^2(\tilde{H})}$, we would ideally like to find a lower bound for

$$\sum_j 1_{[1,\infty)}(h\sigma_j)\langle \mathscr{D}^\flat(\Omega_\alpha, H)\phi_{\sigma_j}, \phi_{\sigma_j}\rangle, \qquad 0 < h < 1.$$

However, as is usually the case, it is difficult to control sharp spectral cutoffs and we instead use a smooth approximation to $1_{[1,\infty)}$.

Rather than convolving $1_{[1,\infty)}$ with a smooth function, it turns out to be simpler to consider for some $m > 0$ the quantity

$$\begin{aligned}\sum_j \psi((h\sigma_j)^m)\langle \mathscr{D}^\flat(\Omega_\alpha, H)\phi_{\sigma_j}, \phi_{\sigma_j}\rangle &= \sum_j \left\langle \psi((h\Lambda^{\Omega_\alpha})^m)\ \mathscr{D}^\flat(\Omega_\alpha, H)\phi_{\sigma_j}, \phi_{\sigma_j}\right\rangle \\ &= \operatorname{tr}\Big(\psi((h\Lambda^{\Omega_\alpha})^m)\mathscr{D}^\flat(\Omega_\alpha, H)\Big),\end{aligned}$$

with $\psi(x) := \frac{x}{1+x}$. This regularization is particularly convenient for proving a lower bound on the trace because it allows us to work directly with powers of Λ^{Ω_α} instead of using the full functional calculus. (This type of regularization is also known as a *scaled Yosida approximation of* $(h\Lambda^{\Omega_\alpha})^m$ [BB11, Definition before Proposition 7.2].)

To finish the proof, we show in Lemmas 4.8 and 4.9 that for certain m

$$ch^{\frac{2\pi}{\alpha}-1} \le \operatorname{tr}\Big(\psi((h\Lambda^{\Omega_\alpha})^m)\mathscr{D}^\flat(\Omega_\alpha, H)\Big). \tag{1.4}$$

This relies on (1.2) which, together with the conformal invariance of harmonic functions, allows us to show that the most singular part of the kernel of $\mathscr{D}^\flat(\Omega_\alpha, H)$ has the form $c|s|^{\frac{\pi}{\alpha}-1}|t|^{\frac{\pi}{\alpha}-1}$. By finding approximations to $\big(\Lambda^{\Omega_\alpha}\big)^m|s|^{\frac{\pi}{\alpha}-1}$ in Corollary 4.6, and understanding the domain of $(\Lambda^{\Omega_\alpha})^m$ sufficiently well, we are then able to prove (1.4) which, together with an elementary contradiction argument, proves Theorem 1.1.

The proof of Theorem 1.2 relies on a similar argument except that we replace $\psi((h\Lambda^{\Omega_\alpha)^m})$ by $\psi((h\Lambda_\theta)^m)$, with

$$\Lambda_\theta = \theta\Lambda^{\Omega_\alpha\setminus H_1} + (1-\theta)\Lambda^{\Omega_\alpha\setminus H_0},$$

and integrate the lower bound in θ to obtain

$$ch^{\frac{2\pi}{\alpha}} \le \operatorname{tr}\Big(\Phi_m\big(h\Lambda^{\Omega_\alpha\setminus H_1}\big) - \Phi_m\big(h\Lambda^{\Omega_\alpha\setminus H_0}\big)\Big) = \sum_j \Phi_m\big(h\sigma_j(\Lambda^{\Omega_\alpha\setminus H_1})\big) - \Phi_m\big(h\sigma_j(\Lambda^{\Omega_\alpha\setminus H_0})\big),$$

$$\Phi_m(x) := \int_0^x \psi(s^m)ds.$$

To prove Theorem 1.2 we then again proceed by contradiction, using that fact that $\Phi_m(x) \sim x$ for x large and $|\Phi_m(x+s) - \Phi_m(x)| \le Cx^m|s|$ for $0 < |s| < x$ small.

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2. Comparison of Dirichlet to Neumann Maps on domains with and without holes

In this section, we let $\Omega \subset \mathbb{R}^d$ be open, connected with Lipschitz boundary and $H \Subset \Omega$ open with Lipschitz boundary. We then compare the DtN map on Ω with the restriction of that on $\Omega \setminus H$ restricted to $\partial\Omega$. More precisely, we study the operator

$$\mathscr{D}^\flat(\Omega, H) := 1_{\partial\Omega}\Lambda^{\Omega\setminus H}1_{\partial\Omega} - \Lambda^\Omega,$$

where $1_{\partial\Omega}$ indicates restriction to $\partial\Omega$.

2.1. Variational formulae for $\mathscr{D}^\flat(\Omega, H)$.

Lemma 2.1. *For all $f \in H^{1/2}(\partial\Omega)$,*

$$\big\langle \mathscr{D}^\flat(\Omega,H)f, f\big\rangle = \min\Big\{ \int_\Omega |\nabla v|^2 dx \,:\, v \in H_0^1(\Omega),\, v = -\mathcal{H}^\Omega f \text{ on } H\Big\},$$

Proof. Let $f \in H^{1/2}(\partial\Omega)$ and set

$$w_f := 1_{\Omega\setminus H}\mathcal{H}^{\Omega_H}1_{\partial\Omega}f - \mathcal{H}^\Omega f.$$

Then, w is harmonic in $\Omega \setminus H$ and hence for all $g \in H_0^1(\Omega \setminus H)$,

$$\int_{\Omega\setminus H} \nabla w_f \cdot \nabla g dx = 0. \tag{2.1}$$

Now, notice that

$$\begin{aligned}\langle \mathscr{D}^\flat(\Omega, H) f, f\rangle &= \int_{\Omega_H} \nabla \mathcal{H}^{\Omega_H} 1_\Omega f \cdot \nabla \mathcal{H}^{\Omega_H} 1_\Omega f dx - \int_\Omega \nabla \mathcal{H}^\Omega f \cdot \nabla \mathcal{H}^\Omega f dx = \int_\Omega |\nabla w_f|^2 dx \\ &= \min\Big\{ \int_\Omega |\nabla v|^2 dx \,:\, v \in H_0^1(\Omega),\, v = -\mathcal{H}^\Omega f \text{ on } H\Big\},\end{aligned}$$

where the last equality follows from (2.1). □

Our next lemma specializes to the case of Ω and H having smooth boundary.

Lemma 2.2. *Suppose that Ω and H have smooth boundary. Then $\mathscr{D}^\flat(\Omega, H)$ is smoothing and its kernel, $K_{\mathscr{D}^\flat(\Omega,H)}$, satisfies*

$$K_{\mathscr{D}^\flat(\Omega,H)}(x,x) = \min\Big\{ \int_\Omega |\nabla v|^2 dx \,:\, v \in H_0^1(\Omega),\, v|_H = -P_\Omega(\cdot, x)|_H\Big\}, \tag{2.2}$$

where P_Ω denotes the kernel of $\mathcal{H}^\Omega$.

Proof. The smoothing property of $\mathscr{D}^\flat(\Omega, H)$ follows from the fact that

$$\begin{gathered}-\Delta(\mathcal{H}^{\Omega\setminus H} f - \mathcal{H}^\Omega f) = 0 \text{ in } \Omega \setminus H, \\ (\mathcal{H}^{\Omega\setminus H} - \mathcal{H}^\Omega) f|_{\partial\Omega} = 0, \qquad (\mathcal{H}^{\Omega\setminus H} - \mathcal{H}^\Omega) f|_{\partial H} = -\mathcal{H}^\Omega f|_{\partial H}\end{gathered}$$

and, by elliptic regularity $\mathcal{H}^\Omega f|_{\partial H} \in C^\infty(\partial H)$ and hence, again using elliptic regularity $\mathcal{H}^{\Omega\setminus H} f - \mathcal{H}^\Omega f) \in H^N(\Omega \setminus H)$ for any N.

The lemma now follows from letting $f_\epsilon \in H^{1/2}(\partial\Omega)$ with $f_\epsilon \to \delta_x$ (using the fact that, since $H \Subset \Omega$, $\mathcal{H}^\Omega f_\epsilon|_H \to P_\Omega(\cdot, x)|_H$ in H^1). □

2.2. **Holes with large and small difference kernels.** We first show that if the hole H is close to the boundary, then $K_{\mathscr{D}^\flat(\Omega,H)}(x_0, x_0)$ is small.

Lemma 2.3. *Let Ω have smooth boundary and $x_0 \in \partial\Omega$ and $c_0 > 0$. Then there is $C > 0$ such that for all $0 < \epsilon$ and H with smooth boundary satisfying*

$$\begin{gathered}H \Subset \{x \in \Omega \,:\, d(x, x_0) > c_0,\, d(x, \partial\Omega) < 2\epsilon\}, \\ 0 \le K_{\mathscr{D}^\flat(\Omega,H)}(x_0, x_0) \le C\epsilon.\end{gathered}$$

Proof. Let $\chi \in C^\infty(\Omega; [0,1])$ with $\operatorname{supp}(1-\chi) \cap H = \emptyset$, $d(\operatorname{supp}\chi, x_0) > c_0/2$ and

$$\operatorname{supp}\chi \subset \mathcal{C}_\epsilon, \qquad \|\nabla\chi\|_{L^\infty} \le C\epsilon^{-1}$$

where

$$\mathcal{C}_\epsilon := \{y \in \Omega \,:\, d(y, x_0) \ge c_0/2,\, d(y, \partial\Omega) \le 3\epsilon\}.$$

(It is easy to construct such a function using Fermi normal coordinates near $\partial\Omega$). Then, observe that $-\chi P_\Omega(\cdot, x_0)$ is a candidate for (2.2) and

$$\int |\nabla \chi P_\Omega(y, x_0)|^2 dy \leq C\epsilon^{-2} \int_{\mathcal{C}_\epsilon} |P_\Omega(y, x_0)|^2 dy + C \int \chi^2 |\nabla P_\Omega(y, x_0)|^2 dy \leq C\epsilon.$$

Here, we use that since $d(\operatorname{supp} \chi, x_0) \geq c_0/2$, $P_\Omega(y, x_0)$ is smooth in y and $P(\cdot, x_0)|_{\partial\Omega \setminus \{x_0\}} = 0$ which implies that, $\|P_\Omega(\cdot, x_0)\|_{L^\infty(\mathcal{C}_\epsilon)} \leq C\epsilon$. □

Next, we show that if the hole H is not close to the boundary, then $K_{\mathscr{D}^\flat(\Omega,H)}(x_0, x_0)$ is controlled from below.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^2$ have smooth boundary, $x_0 \in \partial\Omega$, $c_0 > 0$, $1 \leq p < \infty$. Then there is $c > 0$ such that for all H with smooth boundary satisfying*

$$H \subset \{d(x, \partial\Omega) > c_0\},$$

we have

$$K_{\mathscr{D}^\flat(\Omega,H)}(x_0, x_0) \geq c(\operatorname{vol}(H))^{2/p}.$$

Proof. Suppose that v is a candidate for (2.2). Then,

$$v|_H = -P_\Omega(\cdot, x_0)|_H$$

and, hence, since $P_\Omega(\cdot, x_0) > 0$ in Ω there is $c > 0$ satisfying

$$c(\operatorname{vol}(H))^{1/p} < \inf_{x \in H} P_\Omega(x, x_0) \operatorname{vol}(H)^{1/p} \leq \|v\|_{L^p(H)} \leq \|v\|_{L^p(\Omega)} \leq C\|v\|_{H^1(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}.$$

Hence, the lemma follows from (2.2). □

3. Kernels of DtN differences in Ω_α

From now on, we identify $\partial\Omega_\alpha$ with $\mathbb{S}^1$ using a constant speed, bijective map $\gamma_\alpha : \mathbb{S}^1 \to \partial\Omega_\alpha$ such that $\gamma_\alpha(0) = 0$. Then, let

$$w_\alpha : \mathbb{S}^1 \to\to \mathbb{R}, \qquad x_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$$

such that x_α, w_α are smooth away from 0, $w_\alpha > 0$ away from 0, x_α is a homeomorphism, smooth away from 0, and, in a neighborhood of 0 are given by

$$x_\alpha(s) = \operatorname{sgn}(s)|s|^{\frac{\pi}{\alpha}}, \qquad w_\alpha(s) = |s|^{\frac{\pi}{\alpha}-1}.$$

Lemma 3.1. *Let $H \Subset \Omega_\alpha$ have smooth boundary. Then, the kernel of $\mathscr{D}^\flat(\Omega_\alpha, H)$ has the form*

$$k_{\mathscr{D}^\flat(\Omega_\alpha,H)}(s, t) = w_\alpha(t) k(x_\alpha(t), x_\alpha(s)) w_\alpha(s), \tag{3.1}$$

where $k \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$ and $k(0, 0) = c k_{\mathscr{D}^\flat(\Omega, f_\alpha^{-1}(H))}(0, 0)$ and $c > 0$

Proof. Observe that $\mathscr{D}^\flat(\Omega, f_\alpha^{-1}(H))$ is smoothing. Therefore, since

$$\mathscr{D}^\flat(\Omega_\alpha, H) = \frac{\pi}{\alpha} |z|^{\frac{\pi}{\alpha}-1} f_\alpha^* \mathscr{D}^\flat(\Omega, f_\alpha^{-1}(H))(f_\alpha^*)^{-1},$$

the lemma follows. □

Next let $H_1, H_0 \Subset \Omega_\alpha$ be simply connected and equal length L. We again identify the boundaries of H_i with $\mathbb{S}^1$ using maps $\gamma_i : \mathbb{S}^1 \to \partial H_i$ so that $\partial(\Omega_\alpha \setminus H_i) \cong \mathbb{S}^1 \sqcup \mathbb{S}^1$ and hence $L^2(\partial(\Omega_\alpha \setminus H_i)) \cong L^2(\mathbb{S}^1; \mathbb{C}^2)$, where we identify the first entry with ∂H_i and the second with $\partial\Omega_\alpha$.

An argument almost identical to that for Lemma 3.1 yields the following

Lemma 3.2. *Let $H_i \Subset \Omega_\alpha$, $i = 1, 2$ be open, simply connected, possibly empty, with smooth boundaries both having length $L \geq 0$. Then the kernel of $\mathscr{D}(H_1, H_0) := \Lambda^{\Omega_\alpha \setminus H_1} - \Lambda^{\Omega_\alpha \setminus H_0}$ has the form*

$$K_{\mathscr{D}(H_1,H_0)} = \begin{pmatrix} K_{HH}(s_H, t_H) & K_{H\Omega}(s_H, x_\alpha(t))w_\alpha(t) \\ \overline{K}_{H\Omega}(t_H, x_\alpha(s))w_\alpha(s) & K_{\Omega\Omega}(x_\alpha(t), x_\alpha(s))w_\alpha(s)w_\alpha(t) \end{pmatrix}, \tag{3.2}$$

where K_{HH}, $K_{H\Omega}$, $K_{\Omega\Omega} \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$ and $K_{\Omega\Omega}(0,0) = \frac{\pi^2}{\alpha^2}(k_{\mathscr{D}^\flat(\Omega, f_\alpha^{-1}(H_1))}(0,0) - k_{\mathscr{D}^\flat(\Omega, f_\alpha^{-1}(H_0))}(0,0))$.

4. Trace class estimates on DtN differences in the plane

In this section, we study regularised traces of $\mathscr{D}^\flat(\Omega, H)$ and $\mathscr{D}(H_1, H_0)$.

4.1. **Trace class properties of operators with kernels of the form** (3.2)**.** We first show that these operators are, in fact, trace class. To do this, we recall the following basic fact:

Lemma 4.1. *Let $\beta > -\frac{1}{2}$, $\alpha > 0$ and $f \in C_c^\infty(-1, 1)$, then*

$$|t|^\beta f(\operatorname{sgn}(t)|t|^\alpha) \in H^r, \qquad r < \beta + \frac{1}{2}, \qquad f(\operatorname{sgn}(t)|t|^\alpha) \in H^r, \qquad r < \alpha + \frac{1}{2}$$

Lemma 4.2. *Suppose that $k : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{C}$ is smooth. Then the operators B_i $i = 1, 2, 3$ given by*

$$B_1 u(t) = \int k(x_\alpha(t), x_\alpha(s))w_\alpha(s)w_\alpha(t)u(s)ds$$

$$B_2 u(t) = \int k(x_\alpha(t), x_\alpha(s))w_\alpha(s)u(s)ds$$

$$B_3 u(t) = \int k(x_\alpha(t), x_\alpha(s))w_\alpha(t)u(s)ds$$

are trace class.

Proof. We consider B_1, the other cases being similar. Taylor expanding k at $(0, \tilde{s})$, we obtain

$$k(t, s) = \sum_{j=0}^{M-1} t^j a_j(s) + t^M e_M(t, s),$$

where a_j and e_M are smooth. Therefore,

$$B_1 = \sum_{j=0}^{M-1} x_\alpha^j w_\alpha \otimes a_j(x_\Omega)w_\alpha + E_M,$$

where

$$\|E_M\|_{L^2 \to H^{(M+1)\frac{\pi}{\alpha} - \frac{1}{2} - 0}} < \infty.$$

Taking M large enough that $(M+1)\frac{\pi}{\alpha} - \frac{1}{2} > 1$ implies that E_M is trace class and hence, since $E_M - B_1$ is finite rank, so is B_1. □

4.2. **The action of DtN maps on space of functions with b-regularity.** We first recall a direct consequence of [BNZ05, Proposition 3.9, Theorem 5.4] that allows us to understand the action of DtN maps on functions with b-regularity.

Lemma 4.3. *Let $H \Subset \Omega_\alpha$ be open, possibly empty, with smooth, connected boundary, $\frac{1}{2} - \frac{\pi}{\alpha} < t < \frac{\pi}{\alpha} + \frac{1}{2}$. Then*

$$\||s|^{m-t}\partial_s^m g\|_{L^2} < \infty \text{ for all } m \geq 0 \quad \Rightarrow \quad \||s|^{m-t+1}\partial_s^m \Lambda^{\Omega_\alpha \setminus H} 1_{\partial\Omega_\alpha} g\|_{L^2} < \infty, \text{ for all } m \geq 0. \tag{4.1}$$

A simple application of Lemma 4.3 together with the characterisation of the domains of small powers of $\Lambda^{\Omega_\alpha \setminus H}$ in Corollary A.4 allows us to apply the following Lemma to DtN maps and hence to show that functions with b-regularity are in the domain of high powers of DtN maps.

Lemma 4.4. *Suppose that $\Lambda \geq 0$ is self-adjoint with $\Lambda : H^1 \to L^2$ and satisfies (4.1) for $0 \leq t < \frac{\pi}{\alpha} + \frac{1}{2}$. Then for all $0 \leq t < \frac{\pi}{\alpha} + \frac{1}{2}$, g satisfying*

$$\||s|^{m-t}\partial_s^m g\|_{L^2} < \infty, \qquad m \geq 0,$$

we have $\Lambda^t g \in L^2$.

Proof. Let $J := \lfloor t \rfloor$. Then, by (4.1)

$$\||s|^{m-t+J}\partial_s^m \Lambda^J 1_{\partial\Omega_\alpha} g\|_{L^2} < \infty.$$

Therefore, it is enough to show that if $0 < t < 1$ and for all $m \geq 0$,

$$\||s|^{m-t}\partial_s^m g\|_{L^2} < \infty$$

then, $\Lambda^t g \in L^2$. Since $\Lambda : H^1 \to L^2$, we have by interpolation that $\left(\Lambda^{\Omega_\alpha \setminus H}\right)^t : H^t \to L^2$.

We now show that $g \in H^t$ and hence $(\Lambda^{\Omega_\alpha \setminus H})^t g \in L^2$ as claimed.

Since $\psi g \in C^\infty$ for any $\psi \in C^\infty$ supported away from 0, we may assume that g is supported in a small neighborhood of 0. Therefore,

$$\|g\|_{H^t}^2 = \|g\|_{L^2} + \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{|g(x) - g(y)|^2}{|x-y|^{2t+1}} dxdy$$

We now estimate

$$\begin{aligned}
&\iint \frac{|g(x) - g(y)|^2}{|x-y|^{2t+1}} dxdy \\
&= \iint_{|x-y| \leq \frac{|y|}{2}} \frac{|g(x) - g(y)|^2}{|x-y|^{2t+1}} dxdy + \iint_{\frac{|y|}{2} < |x-y|} \frac{|g(x) - g(y)|^2}{|x-y|^{2t+1}} dxdy \\
&\leq \iint_{|x-y| \leq \frac{|y|}{2}} \frac{\int_0^1 |g'(y + s(x-y))|^2 ds}{|x-y|^{2t-1}} dxdy + 2\iint_{\frac{|y|}{2} < |x-y|} \frac{|g(x)|^2 + |g(y)|^2}{|x-y|^{2t+1}} dxdy \\
&\leq \int_0^1 \iint_{|x-y| \leq \frac{|y|}{2}} \frac{|g'(y + s(x-y))|^2}{|x-y|^{2t-1}} dxdyds + C\int |g(x)|^2 |x|^{-2t} dx.
\end{aligned}$$

where the last line follows from the fact that $|x-y| \geq \frac{|y|}{2}$ implies $|x-y| \geq c|x|$.

Now, for $0 \leq s \leq 1$

$$\begin{aligned}\iint_{|x-y|\leq \frac{|y|}{2}} \frac{|g'(y+s(x-y))|^2}{|x-y|^{2t-1}} dxdy &= \iint_{|v|\leq \frac{|w-sv|}{2}} \frac{|g'(w)|^2}{|v|^{2t-1}} dvdw \\ &= \int \operatorname{sgn}(w) \int_{-\frac{w}{2-s}}^{\frac{w}{2+s}} \frac{|g'(w)|^2}{|v|^{2t-1}} dvdw \\ &\leq C \int |g'(w)|^2 |w|^{2-2t} dw,\end{aligned}$$

and hence $g \in H^t$ as claimed. □

In our lower bound on the regularized trace of difference of DtN maps, we need a more precise formula for the action of DtN maps on one particular function.

Lemma 4.5. *Let $H \Subset \Omega_\alpha$ be open, possibly empty, with smooth, connected boundary. Then there is $\epsilon > 0$ such that for $\psi \in C_c^\infty(-\epsilon, \epsilon)$ with $\psi \equiv 1$ near 0 and $q > \frac{1}{2}$, $q\alpha \notin \pi(2\mathbb{Z}+1)$. Then, for any $|t| < \frac{\pi}{\alpha} - \frac{1}{2}$ and $m \geq 0$,*

$$\| |s|^{m-t} \partial_s^m (\Lambda^{\Omega_\alpha \setminus H} 1_{\partial\Omega_\alpha} \psi(|s|)|s|^q + q\psi(|s|)|s|^{q-1} \tan(q\alpha/2)) \|_{L^2} < \infty, \tag{4.2}$$

Proof. It will be convenient to rotate coordinates so that, with $\gamma : [-L/2, L/2) \to \partial\Omega_\alpha$, a unit speed parametrisation satisfying $\gamma(0) = 0$, we have $\gamma'(0+) = e^{-i\alpha/2}$ and $\gamma'(0-) = -e^{i\alpha/2}$. Note that, by assumption there is $\epsilon > 0$ such that $\gamma(\pm s) = \pm s\gamma'(0\pm)$ on $0 \leq s \leq \epsilon$.

Let $g := \psi(|s|)|s|^q \in L^2(\partial\Omega_\alpha)$, and define

$$u_q := \frac{r^q}{2\cos(q\alpha/2)} (e^{iq\theta} + e^{-iq\theta}).$$

Then,

$$-\Delta(\mathcal{H}^{\Omega_\alpha \setminus H} 1_{\partial\Omega_\alpha} g - \psi(r)u_q) = [\Delta, \psi(r)]u_q \text{ in } \Omega_\alpha, \qquad (g - \psi u_q)|_{\partial(\Omega_\alpha \setminus H)} = 0$$

Therefore, by [BNZ05, Theorems 3.7, 5.4], for $|t| < \frac{\pi}{\alpha}$ and any $m \geq 1$,

$$\sum_{|\alpha| \leq m} \| r^{|\alpha|-t-1} \partial^{|\alpha|} (\mathcal{H}^{\Omega_\alpha \setminus H} 1_{\partial\Omega_\alpha} g - \psi(r)u_q) \|_{L^2} \leq C_m$$

Hence, for $|t| < \frac{\pi}{\alpha} - \frac{1}{2}$ and $m \geq 0$, we have

$$\| |s|^{m-t} \partial_s^m (\Lambda^{\Omega_\alpha \setminus H} 1_{\partial\Omega_\alpha} g + q\psi(|s|)|s|^{q-1} \tan(q\alpha/2)) \|_{L^2} < \infty$$

as claimed. □

As a corollary of Lemmas 4.5 and 4.3, and their analogs on $\Omega_\alpha \setminus H$, we obtain

Corollary 4.6. *Let $\psi \in C_c^\infty(-1,1)$ with $\psi \equiv 1$ near 0, $\frac{1}{2} < q < \frac{\pi}{\alpha}$, and $m \in \mathbb{Z}$ with $0 \le m < q + \frac{1}{2}$. Then, for all $H \Subset \Omega_\alpha$ with smooth boundary (possibly empty), and all j and $t < \frac{\pi}{\alpha} + \frac{1}{2}$*

$$\Big\| |s|^{j-t+m} \partial_s^j \Big((\Lambda^{\Omega_\alpha \setminus H})^m 1_{\partial\Omega_\alpha \setminus H} 1_{\partial\Omega_\alpha} \psi(|s|)|s|^q - (-1)^m \psi(|s|)|s|^{q-m} \prod_{j=0}^{m-1} (q-j) \tan\big((q-j)\alpha/2\big) \Big) \Big\|_{L^2} < \infty.$$

Remark 4.7. It is possible to extend the range of q's to which Lemma 4.5 and Corollary 4.6 apply, but to do so requires one to solve the Dirichlet problem in spaces worse than $H^{\frac{1}{2}}$ and the use of the distributional normal derivative. To avoid these technicalities we restrict ourselves to the case $q > \frac{1}{2}$ used in this paper.

4.3. **A lower bound on the regularized trace.** Next, we prove the key technical estimate used to prove Theorems 1.1 and 1.2. It shows that certain regularized traces of kernels of the form (3.2) can be controlled from below. To state it, we define for $m \ge 0$,

$$\psi_m(x) := \frac{x^m}{1+x^m}.$$

Lemma 4.8. *Let $\alpha \in (0, 2\pi) \setminus \mathcal{E}$, $L > 0$, and $m \in \mathbb{R}$ with $\frac{2\pi}{\alpha} - 1 < m < \frac{2\pi}{\alpha}$. Let $H_i \Subset \Omega_\alpha$, $i = 1, 2$ be open with smooth, connected boundary of length L. Then $\mathscr{D}(H_1, H_0)$ is trace class and there is $c > 0$ such that if $\mathscr{D}(H_1, H_0)$ has kernel of the form (3.2) with $K_{\Omega\Omega}(0,0) > 0$, then*

$$\inf_{\theta \in [0,1]} \operatorname{tr}\big(T_{h,m}^\theta \mathscr{D}(H_1, H_0)\big) \ge c K_{\Omega\Omega}(0,0) h^{\frac{2\pi}{\alpha} - 1} + o(h^{\frac{2\pi}{\alpha} - 1}), \tag{4.3}$$

$$T_{h,m}^\theta := \psi_m(h\Lambda_\theta), \qquad \Lambda_\theta := I + \theta \Lambda^{\Omega_\alpha \setminus H_1} + (1-\theta)\Lambda^{\Omega_\alpha \setminus H_0}.$$

Proof. The fact that $\mathscr{D}(H_1, H_0)$ is trace class follows from Lemma 4.2. To reduce the notation below, we write $\Lambda_i := 1 + \Lambda^{\Omega_\alpha \setminus H_i}$, $i = 1, 2$.

Step 1: We claim that for $0 \le t < \min(\frac{\pi}{\alpha} + \frac{1}{2}, \frac{3}{2})$,

$$\sup_{\theta \in [0,1]} \|\Lambda_\theta^t\|_{H^t \to L^2} + \|\Lambda_\theta^{-t}\|_{L^2 \to H^t} < \infty. \tag{4.4}$$

First notice that, since by Corollary A.4 $\Lambda_1^{2t} : H^t \to H^{-t}$ and $\Lambda_1 \ge 0$, we have

$$\|\Lambda_1^t u\|_{L^2}^2 = \langle \Lambda_1^{2t} u, u \rangle_{L^2} \le C \|u\|_{H^t}^2.$$

Hence, $\Lambda_1^t : H^t \to L^2$ and, since $\Lambda_1^{2t} : H^t \to H^{-t}$ is invertible,

$$\|u\|_{H^t} \le C \|\Lambda_1^{2t} u\|_{H^{-t}} \le C \|\Lambda_1^t\|_{L^2 \to H^{-t}} \|\Lambda_1^t u\|_{L^2} \le C \|\Lambda_1^t u\|_{L^2}.$$

In particular, $\Lambda_1^t : H^t \to L^2$ is invertible.

After applying the same argument to Λ_θ, it remains to show that

$$\sup_{\theta \in [0,1]} \|\Lambda_\theta^{2t}\|_{H^t \to H^{-t}} + \|\Lambda_\theta^{-2t}\|_{H^{-t} \to H^t} < \infty. \tag{4.5}$$

For this, observe that by Lemma A.1, for $-t+1 \leq s \leq t$,

$$\|\Lambda_\theta\|_{H^s \to H^{s-1}} \leq C(\|\Lambda_1\|_{H^s \to H^{s-1}} + \|\Lambda_2\|_{H^s \to H^{s-1}}) \leq C.$$

First, notice that since $\Lambda_i \geq 0$, we have for all $\theta \in [0,1]$, $\Lambda_\theta^{-1} : H^{-\frac{1}{2}} \to H^{\frac{1}{2}}$. Hence,

$$\Lambda_\theta^{-1} = \Lambda_1^{-1} + \Lambda_1^{-1}(1-\theta)(\Lambda_1 - \Lambda_2)\Lambda_\theta^{-1}.$$

Therefore, for $\frac{1}{2} \leq s \leq t$, since $(\Lambda_2 - \Lambda_1) : H^{\frac{1}{2}} \to H^{s-1}$,

$$\|\Lambda_\theta^{-1}\|_{H^{s-1} \to H^s} \leq C.$$

In particular, dualtiy, interpolation and composition imply (4.5).

Step 2: An intermediate quadratic form.

Let $0 \leq t < \min(\frac{2\pi}{\alpha} + 1, 3)$

$$Q_{h,t}(u) := \langle h^t \Lambda_\theta^t (I + h^t \Lambda_\theta^t)^{-1} u, u \rangle.$$

Then, since

$$\langle (1 + ch^t(1+|D|)^t)u, u\rangle \leq \|u\|_{L^2}^2 + ch^t \|u\|_{H^{t/2}}^2 \leq \langle (1 + h^t\Lambda_\theta^t)u, u\rangle,$$

we have $0 \leq (1+h^t\Lambda_\theta^t)^{-1} \leq (1+ch^t|D|^t)^{-1}$ and hence

$$\begin{aligned} Q_{h,t}(v) &= \|v\|_{L^2}^2 - \langle (1+h^t\Lambda_\theta^t)^{-1} v, v\rangle \\ &\geq \|v\|_{L^2}^2 - \langle (1+ch^t(1+|D|)^t)^{-1} v, v\rangle \\ &\geq c\langle h^t(1+|D|)^t(1+ch^t(1+|D|)^t)^{-1} v, v\rangle. \end{aligned} \tag{4.6}$$

Similarly, since for

$$0 \leq \langle (1 + h^t\Lambda_\theta^t)u, u\rangle \leq \|u\|_{L^2}^2 + Ch^t \|u\|_{H^{t/2}}^2 \leq \langle (1 + Ch^t(1+|D|)^t)u, u\rangle,$$

we have

$$Q_{h,t}(v) \leq C\langle h^t(1+|D|)^t(1+Ch^t(1+|D|)^t)^{-1} v, v\rangle. \tag{4.7}$$

We also claim that

$$Q_{h,t}(1_{h\epsilon\Lambda_\theta \geq 1} v) \leq 2Q_{h\epsilon,t}(v). \tag{4.8}$$

Indeed, this follows from the fact the functional calculus and the fact that

$$\frac{x^t}{1+x^t} 1_{x \geq \epsilon^{-1}} \leq 2\frac{\epsilon^t x^t}{1+\epsilon^t x^t}.$$

Step 3: A lower bound for the weighted trace

The goal of this step is to prove (4.3), which follows from the fact that

$$\operatorname{tr}(T_{h,m}^\theta \mathscr{D}(H_1, H_0)) \geq cK_{\Omega\Omega}(0,0) h^{\frac{2\pi}{\alpha}-1} + o(h^{\frac{2\pi}{\alpha}-1}). \tag{4.9}$$

Let B_0 be the operator with kernel

$$K_{B_0} = \begin{pmatrix} 0 & 0 \\ 0 & K_{\Omega\Omega}(0,0) w_\alpha(s) w_\alpha(t) \end{pmatrix}.$$

We claim that

$$\operatorname{tr}(T_{h,m}^\theta B_0) \geq cK_{\Omega\Omega}(0,0) h^{\frac{2\pi}{\alpha}-1} + o(h^{\frac{2\pi}{\alpha}-1}), \qquad |\operatorname{tr}(T_{h,m}^\theta(\mathscr{D}(H_1,H_0) - B_0)| = o(h^{\frac{2\pi}{\alpha}-1}) \tag{4.10}$$

Step 3a: the principal term

We start by proving the first part of (4.10). Observe that

$$\operatorname{tr}(T^\theta_{h,m}B_0) = K_{\Omega\Omega}(0,0)\langle T^\theta_{h,m}w_\alpha, w_\alpha\rangle$$

Now, since $w_\alpha - c\psi(s)|s|^{\frac{\pi}{\alpha}-1} =: r \in C^\infty_c(\mathbb{S}^1\setminus\{0\})$,

$$\langle T^\theta_{h,m}w_\alpha, w_\alpha\rangle \geq cQ_{h,m}\big(\psi(|s|)|s|^{\frac{\pi}{\alpha}-1}\big) - h^m\|\Lambda_\theta^{m/2}r\|^2_{L^2}.$$

Now, by Lemmas 4.4 and 4.3, $\Lambda_\theta^{m/2}r \in L^2$. Therefore, we need only show that

$$Q_{h,m}\big(\psi(s)|s|^{\frac{\pi}{\alpha}-1}\big) \geq ch^{\frac{2\pi}{\alpha}-1}.$$

For this, let $J := \lceil \frac{\pi}{\alpha} - \frac{3}{2}\rceil$. Then, by Lemma 4.5 and (4.8),

$$\begin{aligned}
Q_{h,m}\big(\psi(|s|)|s|^{\frac{\pi}{\alpha}-1}\big) &= \langle (1+h^m\Lambda_\theta^m)^{-1}h^m\Lambda_\theta^{m-2J}\Lambda_\theta^J\psi(|s|)|s|^{\frac{\pi}{\alpha}-1}, \Lambda_\theta^J\psi(|s|)|s|^{\frac{\pi}{\alpha}-1}\rangle\\
&\geq c_\epsilon h^{2J}Q_{h,m-2J}(1_{\epsilon h\Lambda_\theta\leq 1}\Lambda_\theta^J\psi(|s|)|s|^{\frac{\pi}{\alpha}-1})\\
&\geq c_\epsilon h^{2J}\big(Q_{h,m-2J}(\Lambda_\theta^J\psi(|s|)|s|^{\frac{\pi}{\alpha}-1}) - 2Q_{h\epsilon,m-2J}(\Lambda_\theta^J\psi(|s|)|s|^{\frac{\pi}{\alpha}-1})\big)\\
&= c_{\epsilon,J}h^{2J}\big(Q_{h,m-2J}(C_J\psi(|s|)|s|^{\frac{\pi}{\alpha}-1-J}+g) - 2Q_{h\epsilon,m-2J}(C_J\psi(|s|)|s|^{\frac{\pi}{\alpha}-1-J}+g)\big)\\
&\geq c_\epsilon h^{2J}\big(Q_{h,m-2J}(C_J\psi(|s|)|s|^{\frac{\pi}{\alpha}-1-J}) - CQ_{h\epsilon,m-2J}(C_J\psi(|s|)|s|^{\frac{\pi}{\alpha}-1-J})\big)\\
&\quad - Ch^{2J}(Q_{h,m-2J}(g)+Q_{h\epsilon,m-2J}(g)),
\end{aligned}$$

where for all $n\geq 0$ and $t < \frac{\pi}{\alpha}+\frac{1}{2}$, $|s|^{n-t+J}\partial_s^n g \in L^2$ and

$$C_J := (-1)^J\prod_{j=0}^{J-1}\Big(\frac{\pi}{\alpha}-j-1\Big)\tan\Big((\frac{\pi}{\alpha}-j-1)\frac{\alpha}{2}\Big) \neq 0.$$

Where the fact that $C_J\neq 0$ follows from $J<\frac{\pi}{\alpha}$ and $0<(\frac{\pi}{\alpha}-j-1)\frac{\alpha}{2}<\pi$.

Now, by Lemma 4.4, since $0\leq\frac{m-2J}{2} = \frac{m}{2} - \lceil\frac{\pi}{\alpha}-\frac{3}{2}\rceil < \frac{\pi}{\alpha}+\frac{1}{2}$, we have

$$Q_{h,m-2J}(g) \leq Ch^{m-2J}$$

Now, to finish the first step, observe that

$$\begin{aligned}
&|\xi|^{\frac{\pi}{\alpha}-J}\widehat{(\psi(|\cdot|)|\cdot|^{\frac{\pi}{\alpha}-1-J})}(\pm\xi)\\
&= \int e^{\mp is}\psi(|\xi|^{-1}|s|)|s|^{\frac{\pi}{\alpha}-1-J}ds\\
&= \int e^{is}\psi(s)|s|^{\frac{\pi}{\alpha}-1-J}ds + \int e^{is}(-D_s)^N\big(\psi(|\xi|^{-1}|s|)-\psi(|s|)\big)|s|^{\frac{\pi}{\alpha}-1-J}\big)ds\\
&\to \int e^{is}\psi(s)|s|^{\frac{\pi}{\alpha}-1-J}ds + \int e^{is}(D_s)^N\Big((1-\psi(|s|))|s|^{\frac{\pi}{\alpha}-1-J}\Big)ds = \widehat{|\cdot|^{\frac{\pi}{\alpha}-1-J}}(1)\neq 0,
\end{aligned}$$

where the last inequality follows from the fact that $\frac{\pi}{\alpha} \notin \mathbb{Z}$. In particular,

$$\begin{aligned}
&Q_{h,m-2J}(\psi(|s|)|s|^{\frac{\pi}{\alpha}-1-J}) \\
&\simeq \langle h^{m-2J}(I+|D|)^{m-2J}(I+h^{m-2J}(I+|D|)^{m-2J})^{-1}\psi(|s|)|s|^{\frac{\pi}{\alpha}-1-J}, \psi(|s|)|s|^{\frac{\pi}{\alpha}-1-J}\rangle \\
&= \sum |\widehat{\psi(|\cdot|)|\cdot|^{\frac{\pi}{\alpha}-1-J}}(n)|^2 h^{m-2J}(1+|n|)^{m-2J}/(1+h^{m-2J}(1+|n|)^{m-2J}) \\
&= \sum_{|n|\geq M} |n|^{-\frac{2\pi}{\alpha}+2J} h^{m-2J}(1+|n|)^{m-2J}/(1+h^{m-2J}(1+|n|)^{m-2J}) + O(h^{m-2J}) \\
&\simeq h^{\frac{2\pi}{\alpha}-1-2J}
\end{aligned}$$

since $m > \frac{2\pi}{\alpha} - 1$. This implies the first estimate in (4.10).

Step 3b: the error term

To obtain the second estimate, we consider operators R_1, R_2, R_3 with kernels of the form

$$\begin{aligned}
K_{R_1}(s,t) &= w_\alpha(t)w_\alpha(s)x_\alpha(t)k_1(x_\alpha(t),x_\alpha(s)) \\
K_{R_2}(s,t) &= w_\alpha(s)k_2(x_\alpha(t),x_\alpha(s)), \\
K_{R_3}(s,t) &= w_\alpha(t)k_3(x_\alpha(t),x_\alpha(s)),
\end{aligned}$$

with k_i smooth.

We consider R_1, the others being similar. To analyze R_1, we Taylor expand the kernel to obtain for any M

$$\begin{aligned}
K_{R_1}(t,s) = w_\alpha(t)w_\alpha(s)x_\alpha(t)\Big(&\sum_{j=0,\ell}^{M-1} c_{j,\ell}x_\alpha(t)^j x_\alpha(s)^\ell + \sum_{j=0}^{M-1} x_\alpha(s)^M x_\alpha(t)^j a_{j,M}(x_\alpha(s)) \\
&+ x_\alpha(t)^M \sum_{\ell=0}^{M-1} x_\alpha(s)^\ell b_{\ell,M}(x_\alpha(t)) + x_\alpha(s)^M x_\alpha(t)^M e_M(x_\alpha(t),x_\alpha(s))\Big),
\end{aligned}$$

with a_j, b_ℓ, and e smooth.

Hence, for any $M \geq 1$,

$$\begin{aligned}
R_1 = &\sum_{j,\ell=0}^{M-1} c_{j,\ell}\Big(\operatorname{sgn}(t)^{j+1}\psi(|t|)|t|^{\frac{(2+j)\pi}{\alpha}-1}\Big) \otimes \Big(\psi(|s|)\operatorname{sgn}(s)^\ell |s|^{\frac{(1+\ell)\pi}{\alpha}-1}\Big) \\
&+ \sum_{j=0}^{M-1}\Big(\operatorname{sgn}(t)^{j+1}\psi(|t|)|t|^{\frac{(2+j)\pi}{\alpha}-1}\Big) \otimes g_{j,M} + \sum_{\ell=0}^{M-1} \tilde{g}_{\ell,M} \otimes \Big(\operatorname{sgn}(s)^\ell \psi(|s|)|s|^{\frac{(1+\ell)\pi}{\alpha}-1}\Big) + R'_{1,M}
\end{aligned}$$

where, with $K_{R'_{1,M}}$, we have for all n, n', that

$$\begin{gathered}
|t|^{-\frac{(M+1)\pi}{\alpha}+\frac{1}{2}+0+n}|s|^{-\frac{(M+1)\pi}{\alpha}+\frac{1}{2}+0+n'}\partial_t^n\partial_s^{n'}K_{R'_{1,M}} \in L^2, \\
|s|^{-\frac{(M+1)\pi}{\alpha}+\frac{1}{2}+0+n}\partial_s^n g_{j,M} \in L^2, \quad |s|^{-\frac{(M+2)\pi}{\alpha}+\frac{1}{2}+0+n}\partial_s^n \tilde{g}_{\ell,M} \in L^2.
\end{gathered}$$

The same analysis as in Step 2, using Lemma 4.4 then shows that

$$|\operatorname{tr}(T^\theta_{h,m}R_1)| = o(h^{\frac{2\pi}{\alpha}-1}) + |\operatorname{tr}(T^\theta_{h,m}R'_{1,M})|$$

Now, let $s>1$, $m_1', m_2' < \min(\frac{\pi}{\alpha}+\frac{1}{2}, \frac{3}{2})$ and $J_1+J_2+m_1'+m_2' \le m$. Then, using cyclicity of the trace and boundedness of $\Lambda_\theta^{m_i'} : L^2 \to H^{-m_i'}$ and $\Lambda_\theta^{m_i'} : H^{m_i'} \to L^2$, we have

$$\begin{aligned}
&|\operatorname{tr}(T^\theta_{h,m}R'_{1,M})| \\
&= |\operatorname{tr}((1+(h\Lambda_\theta)^m)^{-(m-J_2-m_2')/m}(h\Lambda_\theta)^{m-m_2'-J_2}R'_{1,M}(h\Lambda_\theta)^{m_2'+J_2}(1+(h\Lambda_\theta)^m)^{-(m_2'+J_2)/m})| \\
&\le h^{m'+J_1+J_2}\|\Lambda_\theta^{m_1'}\Lambda_\theta^{J_1}R_{1,M'}\Lambda_\theta^{J_2}\Lambda_\theta^{m_2'}\|_{\mathrm{tr}} \\
&\le h^{m_1'+m_2'+J_1+J_2}\|(1+|D|)^{m_1'}\Lambda_\theta^{J_1}R_{1,M'}\Lambda_\theta^{J_2}(1+|D|)^{m_2'}\|_{\mathrm{tr}} \\
&\le h^{m_1'+m_2'+J_1+J_2}\|(1+|D|)^{m_1'+s}\Lambda_\theta^{J_1}R_{1,M'}\Lambda_\theta^{J_2}(1+|D|)^{m_2'}\|_{L^2\to L^2}
\end{aligned}$$

Now, by Lemma 4.4, for $m_1'+s<\frac{3}{2}$, $J_1+m_1'+s<\frac{\pi}{\alpha}+\frac{1}{2}$, and $J_2+m_2'<\frac{\pi}{\alpha}+\frac{1}{2}$, we have

$$\|(1+|D|)^{m_1'+s}\Lambda_\theta^{J_1}R_{1,M'}\Lambda_\theta^{J_2}(1+|D|)^{m_2'}\|_{L^2\to L^2} \le C,$$

and hence

$$|\operatorname{tr}(T^\theta_{h,m}R'_{1,M})| = o(h^{\frac{2\pi}{\alpha}-1}),$$

which proves the second estimate in (4.10) and hence the lemma.

□

An almost identical proof yields

Lemma 4.9. *Let $\alpha \in (0,2\pi)\setminus\mathcal{E}$ and $m\in\mathbb{R}$ with $\frac{2\pi}{\alpha}-1<m<\frac{2\pi}{\alpha}$. Let $H \Subset \Omega_\alpha$ be open with smooth, connected boundary. Then, $\mathscr{D}^\flat(\Omega_\alpha, H)$ is trace class and there is $c>0$ such that if $\mathscr{D}^\flat(\Omega_\alpha,H)$ has kernel of the form (3.1) with $k(0,0)\ge 0$, then*

$$\operatorname{tr}\left(T_{h,m}\mathscr{D}^\flat(\Omega_\alpha,H)\right) \ge ck(0,0)h^{\frac{2\pi}{\alpha}-1} + o(h^{\frac{2\pi}{\alpha}-1}),$$
$$T_{h,m} := \psi_m(h(1+\Lambda^{\Omega_\alpha})).$$

5. Proofs of the main theorems

Before proving our main theorem, we prove a basic estimate on σ_j. This estimate follows easily from the Weyl law [Roz23, KLP23], but we choose to present a self-contained proof.

Lemma 5.1. *Let $\mathcal{P}$ be a curvilinear polygon. Then there is $c>0$ such that*

$$cj \le \sigma_j \le C^{-1}j.$$

Proof. First, notice that $\sigma_0 = 0$ and $\sigma_1 > 0$. Therefore, it is enough to prove that

$$c(j+1) \le 1+\sigma_j \le C(j+1).$$

For a compact operator A, let $s_j(A)$ denote the singular values of A. By Lemma A.3 $(1+\Lambda^{\mathcal{P}})^{1/2} : L^2 \to H^{-1/2}$ is invertible and hence, using [DZ19, Proposition B.15], we have

$$\begin{aligned}(1+cj)^{-1/2} &\leq s_j((1+|D|)^{-1/2}))\\ &= s_j((1+|D|)^{-1/2}(1+\Lambda^{\mathcal{P}})^{1/2}(1+\Lambda^{\mathcal{P}})^{-1/2})\\ &\leq C s_j((1+\Lambda^{\mathcal{P}})^{-1/2}) = C(1+\sigma_j)^{-1/2}\end{aligned}$$

Similarly,

$$\begin{aligned}(1+\sigma_j)^{-1/2} &= s_j((1+\Lambda^{\mathcal{P}})^{-1/2})\\ &= s_j((1+\Lambda^{\mathcal{P}})^{-1/2}(1+|D|)^{1/2}(1+|D|)^{-1/2})\\ &\leq C s_j((1+|D|)^{-1/2}) \leq C(1+cj)^{-1/2}.\end{aligned}$$

□

We can now prove the main theorems.

Proof of Theorem 1.1. Shrinking r if necessary, we may assume that $B(x_0,r) \Subset \Omega_\alpha$. We apply Lemma 2.2 to see that, in Lemma 4.9 with $H = B(x_0, r/2)$, $k(0,0) > 0$ and hence $\frac{2\pi}{\alpha} - 1 < m < \frac{2\pi}{\alpha}$

$$\operatorname{tr} T_{h,m}\mathscr{D}^\flat(\Omega_\alpha, H) \geq c h^{\frac{2\pi}{\alpha}-1}.$$

Now, let $\{\phi_{\sigma_j}\}_{j=0}^\infty \subset L^2(\partial\Omega_\alpha)$ be an orthonormal basis of eigenfunctions for Λ^{Ω_α} with eigenvalue σ_j and u_{σ_j} the corresponding Harmonic extensions. Then, let $\chi \in C_c^\infty(B(x_0,r))$ with $\operatorname{supp}(1-\chi) \cap B(x_0, r/2) = \emptyset$. Then, by Lemma 2.1,

$$\begin{aligned}c h^{\frac{2\pi}{\alpha}-1} \leq \operatorname{tr} T_{h,m}\mathscr{D}^\flat(\Omega_\alpha, H) &= \sum_j \langle T_{h,m}\mathscr{D}^\flat(\Omega_\alpha, H))\phi_j, \phi_j\rangle\\ &= \sum_j \frac{h^m(1+\sigma_j)^m}{1+h^m(1+\sigma_j)^m}\langle \mathscr{D}^\flat(\Omega_\alpha, H)\phi_j, \phi_j\rangle\\ &\leq \sum_j \frac{h^m(1+\sigma_j)^m}{1+h^m(1+\sigma_j)^m}\int_{\Omega_\alpha} |\nabla \chi u_{\sigma_j}|^2 dx\\ &\leq \sum_j C\frac{h^m(1+\sigma_j)^m}{1+h^m(1+\sigma_j)^m}\|u_{\sigma_j}\|^2_{L^2(B(x_0,r))},\end{aligned}$$

where we use that u_{σ_j} is harmonic in the last line.

Suppose now that $\|u_{\sigma_j}\|_{L^2(B(x_0,r))} = o(j^{-\frac{\pi}{\alpha}})$. Since by Lemma 5.1 there is $c > 0$ such that $cj \leq \sigma_j \leq Cj$, for any $\epsilon > 0$, we have

$$c h^{\frac{2\pi}{\alpha}-1} \leq \sum_j C\frac{h^m(1+j)^m}{1+h^m(1+j)^m} o(j^{-\frac{2\pi}{\alpha}}) \leq C_\epsilon h^m + C\epsilon \sum_j \frac{h^m(1+j)^m}{1+h^m(1+j)^m} j^{-\frac{2\pi}{\alpha}} \leq Ch^m + C\epsilon h^{\frac{2\pi}{\alpha}-1},$$

which is a contradiction for $\epsilon > 0$ and h small enough. □

Proof of Theorem 1.2. Let $x_0 \in \Omega_\alpha$ with $d(x_0, \partial\Omega_\alpha) > r_0$ and $y_\epsilon \in \Omega_\alpha$ $|y_\epsilon| > r_0$ and

$$\epsilon < d(y_0, \partial\Omega_\alpha) < 2\epsilon.$$

Let $H_1 = B(x_0, \epsilon)$ and $H_0 = B(y_0, \epsilon)$. Then, by Lemmas 2.3, 2.4 (with $p > 4$), and 3.2, for ϵ small enough, $\mathscr{D}(H_1, H_0)$ has a kernel of the form (3.2) with $K_{\Omega\Omega}(0,0) > 0$. Lemma 4.8 then implies that for $\frac{2\pi}{\alpha} - 1 < m < \frac{2\pi}{\alpha}$

$$\inf_{\theta\in[0,1]} \operatorname{tr} T^\theta_{h,m} \mathscr{D}(H_1, H_0) \geq c h^{\frac{2\pi}{\alpha}-1}. \tag{5.1}$$

Define

$$\Phi_m(x) := \int_0^x \psi_m(t)dt.$$

To prove Theorem 1.2, we first show that

$$\text{“}\operatorname{tr}''(\Phi_m(h\Lambda_1) - \Phi_m(h\Lambda_0)) \geq \int_0^1 \operatorname{tr} T^\theta_{h,m} h\mathscr{D}(H_1, H_0)d\theta \geq c h^{\frac{2\pi}{\alpha}}, \tag{5.2}$$

where “tr'' is an appropriately regularized trace, $\Lambda_1 := 1 + \Lambda^{\Omega_\alpha\setminus H_1}$, and $\Lambda_0 := 1 + \Lambda^{\Omega_\alpha\setminus H_0}$.

To dot his, let q_n by polynomials with $q_n \to \Phi_m$ in C^1 uniformly on compact sets. Let $\{e_j\}_{j=0}^\infty \subset L^2(\mathbb{S}^1) \times L^2(\mathbb{S}^1)$ be an orthonormal basis of eigenvectors of Λ_1 with eigenvalues $1 + \sigma_j(\Lambda^{\Omega_\alpha\setminus H_1})$. Let Π_M denote the orthogonal projector onto $\{e_j\}_{j=0}^M$. Then, by cyclicity of the trace

$$\begin{aligned}\operatorname{tr}\big(\Pi_M\big(q_n(\Pi_M h\Lambda_1\Pi_M) - q_n(\Pi_M h\Lambda_0\Pi_M)\big)\Pi_M\big) &= \int_0^1 \operatorname{tr}(q_n'(\Pi_M h\Lambda_\theta\Pi_M)\Pi_M h\mathscr{D}(H_1, H_0)\Pi_M)d\theta \\ &\underset{n\to\infty}{\longrightarrow} \int_0^1 \operatorname{tr}\psi_m(\Pi_M h\Lambda_\theta\Pi_M)\Pi_M h\mathscr{D}(H_1, H_0)\Pi_M d\theta.\end{aligned}$$

On the other hand

$$\operatorname{tr}\Pi_M q_n(\Pi_M h\Lambda_j\Pi_M)\Pi_M \to \operatorname{tr}\Pi_M \Phi_m(\Pi_M h\Lambda_j\Pi_M)\Pi_M,$$

so

$$\operatorname{tr}\big(\Pi_M\big(\Phi_m(\Pi_M h\Lambda_1\Pi_M) - \Phi_m(\Pi_M h\Lambda_0\Pi_M)\big)\Pi_M\big) = \int_0^1 \operatorname{tr}\psi_m(\Pi_M h\Lambda_\theta\Pi_M)\Pi_M h\mathscr{D}(H_1, H_0)\Pi_M d\theta.$$

Since $\mathscr{D}(H_1, H_0)$ is trace class,

$$\|\Pi_M h\mathscr{D}(H_1, H_0)\Pi_M - h\mathscr{D}(H_1, H_0)\|_{\operatorname{tr}} \underset{M\to\infty}{\longrightarrow} 0. \tag{5.3}$$

Next, observe that

$$\psi_m(\Pi_M h\Lambda_\theta\Pi_M) = I - (I + (\Pi_M h\Lambda_\theta\Pi_M)^m)^{-1}. \tag{5.4}$$

Then, since $h\Lambda_\theta \geq h$, letting $\Gamma = \{\frac{h}{2} - it + m^2|t| \,:\, t \in \mathbb{R}\}$ be a contour around $[h,\infty)$, we have

$$
\begin{aligned}
&(I + (\Pi_M h\Lambda_\theta \Pi_M)^m)^{-1} - (I + (h\Lambda_\theta)^m)^{-1} \\
&= \frac{h}{2\pi i}\int_\Gamma \frac{1}{1+z^m}(h\Lambda_\theta - z)^{-1}(\Pi_M\Lambda_\theta\Pi_M - \Lambda_\theta)(\Pi_M h\Lambda_\theta\Pi_M - z)^{-1}dz + (I-\Pi_M) \\
&= -\frac{h}{2\pi i}\int_\Gamma \frac{1}{1+z^m}(h\Lambda_\theta - z)^{-1}(I-\Pi_M)\Lambda_1(I-\Pi_M)(\Pi_M h\Lambda_\theta\Pi_M - z)^{-1}dz \\
&\quad - (1-\theta)\frac{h}{2\pi i}\int_\Gamma \frac{1}{1+z^m}(h\Lambda_\theta - z)^{-1}(\Pi_M\mathscr{D}(H_1,H_0)\Pi_M - \mathscr{D}(H_1,H_0))(\Pi_M h\Lambda_\theta\Pi_M - z)^{-1}dz \\
&\quad + (I-\Pi_M) \\
&= -\frac{h}{2\pi i}\int_\Gamma \frac{1}{1+z^m}(h\Lambda_\theta - z)^{-1}(h\Lambda_1 - z)(h\Lambda_1 - z)^{-1}(I-\Pi_M)\Lambda_1(I-\Pi_M)(\Pi_M h\Lambda_\theta\Pi_M - z)^{-1}dz \\
&\quad - \frac{h}{2\pi i}(1-\theta)\int_\Gamma \frac{1}{1+z^m}(h\Lambda_\theta - z)^{-1}(\Pi_M\mathscr{D}(H_1,H_0)\Pi_M - \mathscr{D}(H_1,H_0))(\Pi_M h\Lambda_\theta\Pi_M - z)^{-1}dz \\
&\quad + (I-\Pi_M) \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{1}{1+z^m}(h\Lambda_\theta - z)^{-1}(h\Lambda_1 - z)(I + z(h\Lambda_1 - z)^{-1})z^{-1}(I-\Pi_M)dz + (I-\Pi_M) \\
&\quad - \frac{h}{2\pi i}(1-\theta)\int_\Gamma \frac{1}{1+z^m}(h\Lambda_\theta - z)^{-1}(\Pi_M\mathscr{D}(H_1,H_0)\Pi_M - \mathscr{D}(H_1,H_0))(\Pi_M h\Lambda_\theta\Pi_M - z)^{-1}dz \\
&=: A(I-\Pi_M) + B_M.
\end{aligned}
$$

B_M converges to zero since $\mathscr{D}(H_1,H_0)$ is trace class, $m > 0$, and the resolvents are bounded by $d(z,[h,\infty))^{-1}$. On the other hand A is bounded since $(h\Lambda_\theta - z)^{-1}(h\Lambda_1 - z)$ is bounded and $(h\Lambda - z)^{-1} \leq d(z,[h,\infty))^{-1}$.

In particular, since $\|(I-\Pi_M)\mathscr{D}(H_1,H_0)\|_{\mathrm{tr}} \to 0$, we have

$$
\|\big((I + (\Pi_M h\Lambda_\theta\Pi_M)^m)^{-1} - (I + (h\Lambda_\theta)^m)^{-1}\big)\mathscr{D}(H_1,H_0)\|_{\mathrm{tr}} \underset{M\to\infty}{\longrightarrow} 0 \tag{5.5}
$$

and hence by (5.1), (5.3), (5.4), and (5.5), we have

$$
\lim_{M\to\infty} \mathrm{tr}\,\big(\Phi_m(\Pi_M h\Lambda_1\Pi_M) - \Phi_m(\Pi_M h\Lambda_0\Pi_M)\big) = \int_0^1 \mathrm{tr}\, T^\theta_{h,m} h\mathscr{D}(H_1,H_0)d\theta \geq ch^{\frac{2\pi}{\alpha}}. \tag{5.6}
$$

Now, observe that, since Φ_m is increasing and

$$
\sigma_j(h\Lambda_0) = \min_{\dim V = j+1}\max_{v\in V,\,\|v\|=1}\langle h\Lambda_0 v, v\rangle,
$$

we have

$$
\sum_{j=0}^M \Phi_m(\sigma_j(h\Lambda_0)) \leq \mathrm{tr}\,\big(\Phi_m(\sigma_j(\Pi_M h\Lambda_0\Pi_M))\big),
$$

and hence

$$ch^{\frac{2\pi}{\alpha}} \leq \lim_{M\to\infty} \operatorname{tr}\left(\Phi_m(\Pi_M h\Lambda_1\Pi_M) - \Phi_m(\Pi_M h\Lambda_0\Pi_M)\right)$$
$$\leq \liminf_{M\to\infty} \sum_{j=0}^{M} \Phi_m(h(1+\sigma_j(\Lambda^{\Omega_\alpha\setminus H_1}))) - \Phi_m(h(1+\sigma_j(\Lambda^{\Omega_\alpha\setminus H_0}))). \quad (5.7)$$

Now that we have proved (5.2) in the sense of (5.7) suppose by contradiction that

$$|\sigma_j(\Lambda^{\Omega_\alpha\setminus H_1}) - \sigma_j(\Lambda^{\Omega_\alpha\setminus H_0})| = o(j^{-\frac{2\pi}{\alpha}}).$$

Then, since by Lemma 5.1 $cj \leq \sigma_j(\Lambda^{\Omega_\alpha\setminus H_1}) \leq Cj$ and $cj \leq \sigma_j(\Lambda^{\Omega_\alpha\setminus H_0}) \leq Cj$, for all $\epsilon > 0$ we have

$$\begin{aligned} ch^{\frac{2\pi}{\alpha}} &\leq \liminf_{M\to\infty} \sum_{j=0}^{M} \Phi(h(1+\sigma_j(\Lambda^{\Omega_\alpha\setminus H_1}))) - \Phi(h(1+\sigma_j(\Lambda^{\Omega_\alpha\setminus H_0})) \\ &\leq h \liminf_{M\to\infty} \sum_{j=1}^{M} \sup_{x\in h[1+cj,1+Cj]} |\psi_m(x)||\sigma_j(\Lambda^{\Omega_\alpha\setminus H_1}) - \sigma_j(\Lambda^{\Omega_\alpha\setminus H_0})| \\ &\leq Ch \sum_j \min(h^m(1+j)^m, 1) o(j^{-\frac{2\pi}{\alpha}}) \\ &= C_\epsilon h^{m+1} + C\epsilon h \sum_j \min(h^m(1+j)^m, 1) j^{-\frac{2\pi}{\alpha}} \leq C_\epsilon h^{m+1} + C\epsilon h^{\frac{2\pi}{\alpha}}. \end{aligned}$$

Taking ϵ small enough and then h small enough, we obtain a contradiction with (5.7). □

Appendix A. Domains of the DtN map on Curvilinear polygons

In this section, we study the DtN map on curvilinear polygons. In particular, we show that they satisfy the assumptions of Lemma 4.8.

Lemma A.1. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a curvilinear polygon with maximal angle α. Then for all $\max(-\frac{\pi}{\alpha}+\frac{1}{2}, -\frac{1}{2}) < s < \min(\frac{\pi}{\alpha}+\frac{1}{2}, \frac{3}{2})$, $\Lambda^{\mathcal{P}} : H^s(\partial\mathcal{P}) \to H^{s-1}(\partial\mathcal{P})$.*

Proof. We prove the statement for $\frac{1}{2} \leq s < \min(\frac{\pi}{\alpha}+\frac{1}{2}, \frac{3}{2})$ with the other indeces following by duality.

Let $1 \leq t < \frac{\pi}{\alpha}+1$. We recall some elliptic regularity results from [Dau88, Theorem 23.3]. Then for all $u \in H_0^1(\mathcal{P})$, $f \in H^{t-2}(\mathcal{P})$ such that

$$\langle \nabla u, \nabla v\rangle = \langle f, v\rangle \text{ for all } v \in H_0^1(\mathcal{P}), \quad (A.1)$$

we have $u \in H^t(\mathcal{P})$ and

$$\|u\|_{H^t(\mathcal{P})} \leq C(\|f\|_{H^{t-2}(\mathcal{P})} + \|u\|_{H^1(\mathcal{P})}).$$

In particular, since (A.1) together with the Poincaré inequality implies

$$\|u\|^2_{H^1(\mathcal{P})} \leq C\|\nabla u\|^2_{L^2(\mathcal{P})} = \langle f, u\rangle,$$

we have for all $u \in H^1_0(\mathcal{P})$ satisfying (A.1),

$$\|u\|_{H^t(\mathcal{P})} \leq C\|f\|_{H^{t-2}(\mathcal{P})}, \tag{A.2}$$

which implies that for all $f \in H^{t-2}$ there is a unique solution $u \in H^t(\mathcal{P}) \cap H^1_0(\mathcal{P})$ of (A.1).

Let $1 < s < \min(\frac{\pi}{\alpha}+\frac{1}{2}, \frac{3}{2})$. Then, by [Gri11, Theorem 1.5.2.8], there is $E : H^s(\partial\mathcal{P}) \to H^{s+\frac{1}{2}}(\mathcal{P})$ satisfying

$$(Ef)|_{\partial\mathcal{P}} = f$$

and $\|E\|_{H^{s-\frac{1}{2}}(\partial\mathcal{P})\to H^s(\mathcal{P})} < \infty$.

Next, let $w \in H^1_0(\mathcal{P})$ satisfy

$$\langle \nabla w, \nabla v\rangle = \langle \nabla Ef, \nabla v\rangle \text{ for all } v \in H^1_0(\mathcal{P}).$$

Then,

$$\|\Lambda^{\mathcal{P}} f\|_{H^{s-1}(\partial\mathcal{P})} = \|\partial_\nu(Ef-w)\|_{H^{s-1}(\partial\mathcal{P})} \leq C(\|w\|_{H^{s+\frac{1}{2}}(\mathcal{P})}+\|Ef\|_{H^{s+\frac{1}{2}}(\mathcal{P})}) \leq C\|Ef\|_{H^{s+\frac{1}{2}}(\mathcal{P})} \leq C\|f\|_{H^s(\partial\mathcal{P})}.$$

In particular,

$$\|\Lambda^{\mathcal{P}}\|_{H^s(\partial\mathcal{P})\to H^{s-1}(\partial\mathcal{P})} < \infty.$$

The estimates for $\frac{1}{2} \leq s \leq 1$ follow by interpolation with the energy bound $\Lambda^{\mathcal{P}} : H^{\frac{1}{2}}(\partial\mathcal{P}) \to H^{-\frac{1}{2}}(\partial\mathcal{P})$. □

We will use the following analog of the Poincaré inequality

Lemma A.2. *There is $C > 0$ such that for all $u \in H^1(\mathcal{P})$,*

$$\|u\|_{H^1(\mathcal{P})} \leq C(\|\nabla u\|_{L^2(\mathcal{P})} + \|u|_{\partial\mathcal{P}}\|_{L^2(\partial\mathcal{P})}). \tag{A.3}$$

Proof. We prove (A.3) by contradiction. Suppose (A.3) fails. Then there are $u_n \in H^1(\mathcal{P})$ with $\|u_n\|_{H^1(\mathcal{P})} = 1$, $\|\nabla u_n\|_{L^2(\mathcal{P})} \to 0$ and $\|u_n|_{\partial\mathcal{P}}\|_{L^2(\partial\mathcal{P})} \to 0$. In particular, extracting a subsequence, we may assume that $u_n \rightharpoonup u$ in $H^1(\mathcal{P})$ and $u_n \to u$ in $H^{\frac{3}{4}}(\mathcal{P})$. First, notice that since the trace map is bounded from $H^{3/4}(\mathcal{P}) \to L^2(\partial\mathcal{P})$, we have $u|_{\partial\mathcal{P}} = 0$. Next, since $\|\nabla u_n\|_{L^2(\mathcal{P})} \to 0$, we have $\nabla u = 0$ so that u is a constant. Thus, since $u|_{\partial\mathcal{P})} = 0$, $u = 0$. But then $\|u_n\|_{L^2} \to 0$ and which is a contradiction since

$$1 = \|u_n\|^2_{H^1(\mathcal{P})} = \|\nabla u_n\|^2_{L^2(\mathcal{P})} + \|u_n\|^2_{L^2(\mathcal{P})} \to 0.$$

□

Lemma A.3. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a curvilinear polygon with maximal angle α. Then for all* $\max(-\frac{\pi}{\alpha}+\frac{1}{2}, -\frac{1}{2}) < s < \min(\frac{\pi}{\alpha}+\frac{1}{2}, \frac{3}{2})$, $(1+\Lambda)^{-1} : H^{s-1}(\partial\mathcal{P}) \to H^s(\partial\mathcal{P})$

Proof. We consider the $\frac{1}{2} \leq s < \min(\frac{\pi}{\alpha}+\frac{1}{2}, \frac{3}{2})$. The other values of s follow by duality.

Step 1: Invertibility in $H^{\frac{1}{2}}$.

First observe that for $g \in H^{1/2}(\partial\mathcal{P})$,

$$\langle (1+\Lambda^{\mathcal{P}})g, g\rangle_{L^2(\partial\mathcal{P})} = \|\nabla\mathcal{H}^{\mathcal{P}} g\|^2_{L^2(\mathcal{P})} + \|g\|^2_{L^2(\partial\mathcal{P})},$$

where

$$-\Delta \mathcal{H}^{\mathcal{P}} g = 0 \text{ in } \mathcal{P}, \qquad \mathcal{H} g|_{\partial\mathcal{P}} = g.$$

In particular, by (A.3) and the fact that the trace is bounded from $H^1(\mathcal{P}) \to H^{\frac{1}{2}}(\partial\mathcal{P})$,

$$\|g\|^2_{H^{\frac{1}{2}}(\partial\mathcal{P})} \leq C\|\mathcal{H}^{\mathcal{P}} g\|^2_{H^1(\mathcal{P})} \leq C(\|\nabla \mathcal{H}^{\mathcal{P}} g\|^2_{L^2(\mathcal{P})} + \|g\|^2_{L^2(\mathcal{P})}) = \langle (1+\Lambda^{\mathcal{P}})g, g\rangle.$$

Hence,

$$\|g\|_{H^{\frac{1}{2}}(\partial\mathcal{P})} \leq C\|(1+\Lambda^{\mathcal{P}})g\|_{H^{-\frac{1}{2}}(\partial\mathcal{P})}$$

and, since $\Lambda^{\mathcal{P}} : H^{\frac{1}{2}} \to H^{-\frac{1}{2}}$ is self-adjoint, this implies $(1+\Lambda^{\mathcal{P}}) : H^{\frac{1}{2}}(\partial\mathcal{P}) \to H^{-\frac{1}{2}}(\partial\mathcal{P})$ is invertible.

Step 2: Elliptic regularity for the DtN map at higher regularity

Let $\frac{1}{2} \leq s < \frac{\pi}{\alpha} + \frac{1}{2}$. Then, by [Dau88, Theorem 23.3], for all $w \in H^1(\mathcal{P})$ satisfying

$$\langle \nabla w, \nabla v\rangle = \langle f, v\rangle \qquad \text{for all } v \in H^1(\mathcal{P}), \tag{A.4}$$

with $f \in \dot{H}^{-1} \cap H^{s-\frac{3}{2}}(\mathcal{P})$, we have $w \in H^{s+\frac{1}{2}}(\mathcal{P})$ and

$$\|w\|_{H^{s+\frac{1}{2}}(\mathcal{P})} \leq C(\|f\|_{H^{s-\frac{3}{2}}(\mathcal{P})} + \|w\|_{H^1(\mathcal{P})}).$$

Consider now the space $\underline{H}^s(\mathcal{P}) := \{u \in H^s(\mathcal{P}) : \int u = 0\}$. Then, the same arguments leading to (A.2) but applying the Poincaré inequality in $\underline{H}^1$ shows that for all $f \in \underline{\dot{H}}^{-1} \cap H^{s-\frac{3}{2}}(\mathcal{P})$, there is $w \in \underline{H}^s(\mathcal{P})$ satisfying (A.4) and

$$\|w\|_{H^{s+\frac{1}{2}}(\mathcal{P})} \leq C\|f\|_{H^{s-\frac{3}{2}}(\mathcal{P})}.$$

Now, let $1 < s < \min(\frac{\pi}{\alpha} + \frac{1}{2}, \frac{3}{2})$. Then, by [Gri11, Theorem 1.5.2.8], there is $E_\nu : H^{s-1}(\partial\mathcal{P}) \to H^{s+\frac{1}{2}}(\mathcal{P})$ such that

$$(\partial_\nu E_\nu)|_{\partial\mathcal{P}} = I, \qquad E_\nu|_{\partial\mathcal{P}} = 0$$

Suppose that $g \in H^{\frac{1}{2}}(\partial\mathcal{P})$ with $\Lambda^{\mathcal{P}} g =: \tilde{f} \in H^{s-1}(\partial\mathcal{P})$. Note that

$$\langle \tilde{f}, 1\rangle_{\partial\mathcal{P}} = \langle \Lambda^{\mathcal{P}} g, 1\rangle = \langle \nabla \mathcal{H}^{\mathcal{P}} g, \nabla 1\rangle = 0.$$

Therefore, there is $w \in H^1(\mathcal{P})$ such that

$$\langle \nabla w, \nabla v\rangle_{L^2(\mathcal{P})} = \langle \nabla E_\nu \tilde{f}, \nabla v\rangle_{\mathcal{P}} - \langle \tilde{f}, v\rangle_{\partial\mathcal{P}}, \qquad \text{for all } v \in H^1(\mathcal{P}),$$

and

$$\|w\|_{H^{s+\frac{1}{2}}(\mathcal{P})} \leq C(\|\nabla E_\nu \tilde{f}\|_{H^{s-\frac{1}{2}}(\mathcal{P})} + \|\tilde{f}\|_{H^{s-1}(\partial\mathcal{P})}) \leq C\|\tilde{f}\|_{H^{s-1}(\partial\mathcal{P})}.$$

Now, observe that $u := \mathcal{H}^{\mathcal{P}} g + w$ satisfies

$$\langle \nabla u, \nabla v\rangle = \langle \nabla E_\nu \tilde{f}, \nabla v\rangle_{\mathcal{P}} \qquad \text{for all } v \in H^1(\mathcal{P}).$$

Therefore,

$$\begin{aligned}\|u\|_{H^{s+\frac{1}{2}}(\mathcal{P})} &\leq C(\|\nabla E_\nu \tilde{f}\|_{H^{s-\frac{1}{2}}} + \|u\|_{H^1(\mathcal{P})}) \\ &\leq C\|\tilde{f}\|_{H^{s-1}(\partial\mathcal{P})} + \|\mathcal{H}^{\mathcal{P}} g\|_{H^1(\mathcal{P})} \leq C(\|\tilde{f}\|_{H^{s-1}(\mathcal{P})} + \|g\|_{H^{\frac{1}{2}}(\partial\mathcal{P})}).\end{aligned}$$

Hence,

$$\|g\|_{H^s(\partial\mathcal{P})} \leq C\|\mathcal{H}^{\mathcal{P}}g\|_{H^{s+\frac{1}{2}}(\mathcal{P})} \leq C(\|u\|_{H^{s+\frac{1}{2}}(\partial\mathcal{P})} + \|w\|_{H^{s+\frac{1}{2}}(\partial\mathcal{P})}) \leq C(\|\tilde{f}\|_{H^{s-1}(\partial\mathcal{P})} + \|g\|_{H^{\frac{1}{2}}(\partial\mathcal{P})}). \tag{A.5}$$

Next, suppose that $(1+\Lambda^{\mathcal{P}})g = f$. Then, setting $\tilde{f} = \Lambda^{\mathcal{P}}g = f - g$ in (A.5), we obtain

$$\|g\|_{H^s(\partial\mathcal{P})} \leq C(\|f\|_{H^{s-1}(\partial\mathcal{P})} + \|g\|_{H^{s-1}(\partial\mathcal{P})} + \|g\|_{H^{\frac{1}{2}}(\partial\mathcal{P})}) \leq C\|f\|_{H^{s-1}(\partial\mathcal{P})}, \tag{A.6}$$

where we have used that, since $s < \frac{3}{2}$, by Step 1

$$\|g\|_{H^{s-1}(\partial\mathcal{P})} + \|g\|_{H^{\frac{1}{2}}(\partial\mathcal{P})} \leq \|(1+\Lambda^{\mathcal{P}})g\|_{H^{-\frac{1}{2}}(\partial\mathcal{P})} \leq \|f\|_{H^{-\frac{1}{2}}(\partial\mathcal{P})},$$

to absorb the $H^{1/2}$ and H^{s-1} errors. □

The following is now a direct corollary of interpolation together with Lemmas A.1 and A.3

Corollary A.4. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a curvilinear polygon with maximal angle α. Then for $m \in \mathbb{R}$ with $0 \leq m < \min(\frac{2\pi}{\alpha} + 1, 3)$, there is $C > 0$ such that*

$$\|(1+\Lambda^{\mathcal{P}})^m\|_{H^{\frac{m}{2}} \to H^{-\frac{m}{2}}} + \|(1+\Lambda^{\mathcal{P}})^{-m}\|_{H^{-\frac{m}{2}} \to H^{\frac{m}{2}}} \leq C.$$

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