

Two-level hybrid Schwarz preconditioners with preasymptotic piecewise-polynomial coarse spaces for the high-frequency Helmholtz equation

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October 8, 2026

Abstract

We consider two-level hybrid Schwarz domain-decomposition preconditioners for finite element discretizations of the high-frequency Helmholtz equation (with wavenumber k). To date, all existing k -explicit theory for these preconditioners considers coarse spaces in the *asymptotic regime*, i.e., where the Galerkin solution is k -uniformly quasi-optimal.

We consider the situation where the fine and coarse spaces consist, respectively, of piecewise polynomials of degree p on meshes of width h and H_c . We prove results for the coarse space in the *preasymptotic regime*, under the familiar mesh threshold that $(H_c k)^{2p} \rho(k)$ is sufficiently small, where $\rho(k)$ is the norm of the solution operator (such that $\rho(k) \sim k$ for the Helmholtz equation in free space). We prove that, in this regime, the fixed-point iteration with this preconditioner converges in a number of iterations that grows at most like $\log k$.

We give numerical experiments demonstrating this theory. Furthermore, these experiments indicate that our theoretical results are sharp, in that the number of iterations required for these preconditioners to converge with $(H_c k)^{2p} \rho(k) \gg 1$ grows rapidly with k .

1 Context and informal statement of the main result

Coarse grids are the key to parallel scalability of domain-decomposition (DD) methods for self-adjoint positive-definite problems (such as Laplace’s equation); see, e.g., [43, 44], [13, Chapter 4]. However, both the development of and the frequency-explicit theory of two-level DD methods for high-frequency wave problems, such as the high-frequency Helmholtz equation, are longstanding open problems (see, e.g., the recent computational studies [8, 12] and the references therein).

The six recent papers [28, 34, 35, 16, 20, 24] considered preconditioning finite-element discretizations of the Helmholtz equation with two-level Schwarz DD, and gave sufficient conditions for either the fixed-point method or GMRES (depending on the paper) to converge in a k -independent number of iterations (working in an appropriate k -weighted norm). All these papers consider coarse spaces in the *asymptotic* regime, i.e., where the Galerkin solution in the coarse space is k -uniformly quasi-optimal (with this terminology famously introduced in [29, 30])

The present paper considers fine and coarse spaces consisting of piecewise polynomials of degree p on meshes of width h and H_c , respectively. Piecewise-polynomial coarse spaces for high-frequency wave problems have previously been studied empirically in [25, 26, 7, 6, 5, 8, 42]. In contrast to the existing two-level analyses mentioned above, we consider the situation where the coarse space is in the *preasymptotic* regime; i.e., the Galerkin solution in the coarse space has k -uniformly bounded relative error for oscillatory data [30, 14, 19] but is not necessarily k -uniformly quasi-optimal. Specifically, we consider coarse spaces satisfying $(H_c k)^{2p} \rho(k) \ll 1$, where $\rho(k)$ is the norm of the Helmholtz solution operator (and such that $\rho(k) \sim k$ for the Helmholtz equation in free space); here and in the rest of the paper we use the notation that $A \ll B$ if $A \leq cB$ for a small constant c .

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In its simplest-possible form, the main result of the paper is the following.

Informal Theorem 1.1 (Informal statement of the main result (Theorem 2.7))

Consider the Helmholtz scattering problem in the exterior of a smooth, bounded Dirichlet or Neumann obstacle, with piecewise smooth coefficients (modelling a penetrable obstacle), and with an absorbing layer approximating the radiation condition. Suppose that $\rho(k)$ is polynomially bounded in k (as occurs for “most” k by [33]).

Consider finite-element discretisations of this problem in a space of mapped piecewise polynomials with meshwidth h and a hybrid Schwarz preconditioner with the following properties:

- the subdomains are of size $\sim k^{-1}$ with overlaps $\sim k^{-1}$,
- the subdomain problems have absorption on the subdomain boundaries that are not part of the obstacle boundary, and
- the piecewise-polynomial coarse space satisfies $(H_c k)^{2p} \rho(k) \ll 1$, and $h \ll H_c$.

Then the preconditioned fixed point method (and hence also GMRES) converges in a number of iterations $\lesssim \log k$.

In such a two-level preconditioner applied to the Helmholtz equation one would ideally like the coarse space to be small. However, the numerical experiments in §3 below indicate that the sufficient conditions on H_c in Informal Theorem 1.1/Theorem 2.7 are *optimal* in the sense that for piecewise polynomial coarse spaces with $(H_c k)^{2p} \rho(k) \gg 1$ the number of GMRES iterations grows quickly with k , even when h is chosen very small.

The underlying reason for the requirement $(H_c k)^{2p} \rho(k) \ll 1$ is that a one-level DD method with subdomains of size k^{-1} requires at least $\sim k$ iterations to resolve the propagation of the Helmholtz solution at length scales independent of k . This propagation must therefore be resolved by the coarse space, requiring the familiar condition $(H_c k)^{2p} \rho(k) \ll 1$ for accuracy of the h -version FEM solution in the coarse space.

2 Statement of the main result

2.1 Helmholtz problems

The general Helmholtz-type PDE to which our theory applies is given by (2.2) below. We arrive at this PDE motivated by scattering problems with the radiation condition approximated by an absorbing layer, such as a perfectly matched layer (PML) [3, 11] or a complex absorbing potential (CAP) [37].

To define the scattering problem, let $\Omega_- \subset \mathbb{R}^d$ be a bounded open obstacle with connected complement, $\Omega_+ := \mathbb{R}^d \setminus \Omega_-$. Furthermore, let A be a symmetric, positive-definite matrix with real coefficients that equals the identity outside a compact set and let n be a strictly positive, real-valued function that equals one outside a compact set. We consider the Helmholtz scattering problem: given $f \in L^2_{\text{comp}}(\Omega_+)$, find $u \in H^1_{\text{loc}}(\Omega_+)$ such that

$$-k^{-2} \operatorname{div}(A \nabla u) - nu = f \text{ in } \Omega_+, \quad (\mathcal{B}u)|_{\partial\Omega_+} = 0, \quad (k^{-1} \partial_r - i)u = o_{r \rightarrow \infty}(r^{\frac{1-d}{2}}) \quad (2.1)$$

where $\Gamma_s := \partial\Omega_+ = \Gamma_{\text{hard}} \sqcup \Gamma_{\text{soft}}$ with Γ_{hard} and Γ_{soft} smooth, closed hypersurfaces and $(\mathcal{B}u)|_{\Gamma_{\text{soft}}} = u$ in the sound-soft case and $(\mathcal{B}u)|_{\Gamma_{\text{hard}}} = \nu \cdot (A \nabla u)$, with ν the normal to Γ_s in the sound-hard case.

We then approximate the Sommerfeld radiation condition in (2.1) using an absorbing layer. Let $\Omega_{\text{tr}} \subset \mathbb{R}^d$ be open, bounded, connected and contain $\overline{\Omega_-} \cup \operatorname{supp}(A - I) \cup \operatorname{supp}(n - 1)$. We truncate the problem (2.1) to the computational domain $\Omega := \Omega_+ \cap \Omega_{\text{tr}}$ and apply the finite-element method to the problem: given $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

$$Pu := -k^{-2} \operatorname{div}(\tilde{A} \nabla u) - i k^{-1} \frac{1}{2} (\tilde{b} \cdot \nabla u + \operatorname{div}(\tilde{b}u)) - \tilde{n}u = f \text{ in } \Omega, \quad (\tilde{\mathcal{B}}u)|_{\partial\Omega_+} = 0, \quad u|_{\partial\Omega_{\text{tr}}} = 0, \quad (2.2)$$

where $\tilde{\mathcal{B}}u|_{\Gamma_{\text{soft}}} = u$, $\tilde{\mathcal{B}}u|_{\Gamma_{\text{hard}}} = \nu \cdot (\tilde{A} k^{-1} \nabla u + \frac{i}{2} \tilde{b}u)$ and we assume that $\tilde{A}$ is symmetric. (Note that if $\tilde{A}$, $\tilde{b}$, and $\tilde{n}$ are real then (2.2) is self adjoint – this is the reason we have written the Helmholtz

PDE in this form). When approximating solutions to (2.1) using a perfectly matched layer (PML), $\tilde{A}$, $\tilde{b}$ and $\tilde{n}$ are respectively equal to A , 0, and n near $\text{supp}(A - I) \cup (n - 1) \cup \Omega_-$, differing only because of the addition of the PML.

We denote the sesquilinear form associated to (2.2) by $a = a(k) : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbb{C}$; i.e.,

$$a(u, v) := \langle \tilde{A}k^{-1}\nabla u, k^{-1}\nabla v \rangle_{\Omega} - \frac{i}{2} \langle \tilde{b} \cdot k^{-1}\nabla u, v \rangle_{\Omega} + \frac{i}{2} \langle \tilde{b}u, k^{-1}\nabla v \rangle_{\Omega} - \langle \tilde{n}u, v \rangle_{\Omega}, \quad (2.3)$$

and

$$\mathcal{Z} := \{u \in H^1(\Omega) : u|_{\partial\Omega_{\text{tr}}} = 0, u|_{\Gamma_{\text{soft}}} = 0\} \quad (2.4)$$

with norm

$$\|u\|_{\mathcal{Z}}^2 := \|u\|_{H_k^1(\Omega)}^2 := \|k^{-1}\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2.$$

For a sesquilinear form $a : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbb{C}$, we denote by

$$\rho(a) := \left(\inf_{0 \neq u \in \mathcal{Z}} \sup_{0 \neq v \in \mathcal{Z}} \frac{|a(u, v)|}{\|u\|_{\mathcal{Z}} \|v\|_{\mathcal{Z}}} \right)^{-1}$$

the norm of the solution operator (a.k.a. the continuous inf-sup constant).

Definition 2.1 (Helmholtz problem with absorption and truncation) *We say that the sesquilinear form a in (2.3) is a piecewise C^{p+1} Helmholtz problem with absorption and C^r truncation if the following holds.*

(i) *There is $\theta_0 \in \mathbb{R}$ such that*

$$\inf_{k > k_0} \Re(e^{i\theta_0} \tilde{A}^{ij}(x) \xi_i \xi_j) \geq c|\xi|^2, \quad x \in \Omega, \xi \in \mathbb{R}^d. \quad (2.5)$$

(ii) $\partial\Omega_- \in C^{p,1}$, $\partial\Omega_{\text{tr}} \in C^{r-1,1}$, and there is $\Gamma_p \Subset \Omega$ a $C^{p,1}$, smooth hypersurface such that $\tilde{A}$, $\tilde{b}$, and $\tilde{n}$ are all $\overline{C^{p-1,1}}(\Omega \setminus \Gamma_p)$.

(iii) *There is a neighborhood, U of $\partial\Omega_{\text{tr}}$ such that for all $k_0 > 0$ there are $c > 0$ and $\theta_1, \theta_2 \in \mathbb{R}$ such that for $k \geq k_0$,*

$$\inf_{k > k_0} \Re(e^{i\theta_1} (\tilde{A}^{ij}(x) \xi_i \xi_j + \tilde{b}^i(x) \xi_i - \tilde{n}(x))) \geq c(|\xi|^2 + 1), \quad x \in U, \xi \in \mathbb{R}^d \quad (2.6)$$

and

$$\left(\text{supp} \Im(e^{i\theta_2} \tilde{A}) \cup \text{supp} \Im(e^{i\theta_2} \tilde{b}) \cup \text{supp} \Im(e^{i\theta_2} \tilde{n}) \right) \cap \left(\overline{(\Gamma_p \cup \partial\Omega_-)} \setminus \overline{U} \right) = \emptyset. \quad (2.7)$$

We immediately highlight that, by multiplying P by a constant, one of θ_0, θ_1 , and θ_2 can be set to zero. Point (i) of Definition 2.1 is that P is strongly elliptic in Ω (in the sense of, e.g., [36, Page 122]).

In Point (ii) of Definition 2.1, we use the notation that a function $u \in \overline{C^{r,1}}(V)$ if $u \in C^{r,1}(V)$ and

$$\|\partial^\alpha u\|_{L^\infty(V)} < \infty, \quad |\alpha| \leq r + 1$$

(i.e., u is $C^{r,1}$ “up to the boundary” of V).

In Point (iii) of Definition 2.1, the condition (2.6) corresponds to P being “absorbing” in U , and, with $\theta_2 = 0$, the condition (2.7) says that any absorption in the coefficients away from U is also away from the boundary of the scatterer $\partial\Omega_-$ and the interfaces Γ_p (i.e., the Helmholtz problem is self adjoint near the parts of $\partial\Omega_-$ and Γ_p that are away from U).

Note that the domain of the unique, closed, unbounded operator on L^2 generated by a piecewise C^1 Helmholtz problem with C^1 truncation is given by

$$\left\{ u \in H^1(\Omega) \cap H^2(\Omega \setminus \Gamma_p) : \begin{aligned} &\nu \cdot (\tilde{A}k^{-1}\nabla u + \frac{i}{2}\tilde{b}u)|_{\Gamma_{\text{hard}}} = 0, u|_{\Gamma_{\text{soft}} \cup \Gamma_{\text{tr}}} = 0, \\ &\nu \cdot (\tilde{A}_+ k^{-1}\nabla u_+ - \tilde{A}_- k^{-1}\nabla u_- + \frac{i}{2}(\tilde{b}_+ - \tilde{b}_-)u)|_{\Gamma_p} = 0 \end{aligned} \right\},$$

where the notation $\tilde{A}_\pm$, $\tilde{b}_\pm$, and $u_\pm$ denotes restriction from either side of Γ_p .

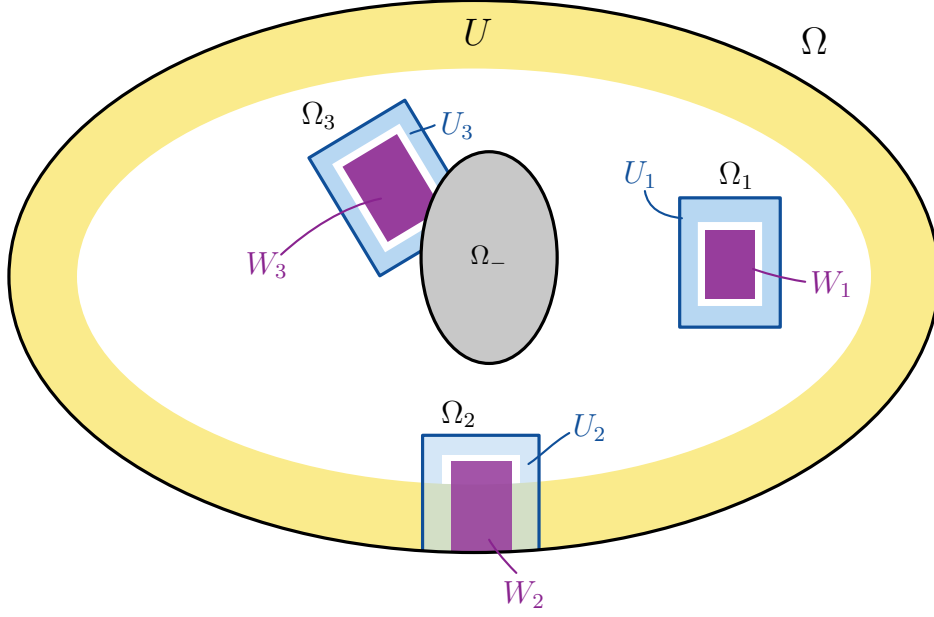


Figure 2.1: The diagram shows an example of a domain Ω with an obstacle Ω_- and its absorbing layer, U . Also shown are three subdomains, Ω_j , $j = 1, 2, 3$, with their absorbing layers, U_j and regions of agreement with the global problem, W_j .

2.2 Components of the two-level hybrid Schwarz preconditioner

The definition of the two-level hybrid Schwarz preconditioner involves the following components, which are defined precisely in the following subsections.

1. A Helmholtz problem with absorption and truncation, a (in the sense of Definition 2.1 above).
2. Two open covers of Ω , $\{\Omega_\ell\}_{\ell=1}^N$ and $\{W_\ell\}_{\ell=1}^N$ with

$$W_\ell \subset \Omega_\ell \quad \text{and} \quad \partial W_\ell \cap \partial \Omega_\ell \subset \partial \Omega. \quad (2.8)$$

(Here, W_ℓ will be where the Helmholtz problem on the ℓ th subdomain equals the underlying Helmholtz problem; i.e., for Ω_ℓ not touching $\partial \Omega$, W_ℓ is inside Ω_ℓ).

3. Open sets $\{U_\ell\}_{\ell=1}^N$ with $U_\ell \subset \Omega_\ell$ and

$$\partial \Omega_\ell \setminus \partial \Omega \subset \overline{U}_\ell. \quad (2.9)$$

(Here, U_ℓ will be the absorbing part of the Helmholtz problem on the ℓ th subdomain.)

4. Subdomain problems on Ω_ℓ for (a, W_ℓ, U_ℓ) , $\{a_\ell\}_{\ell=1}^N$.
5. A partition of unity $\{\chi_\ell\}_{\ell=1}^N$ subordinate to W_ℓ .
6. Coarse and fine spaces, $\mathcal{V}_{H_c}$ and $\mathcal{V}_h$ with $\mathcal{V}_{H_c} \subset \mathcal{V}_h$.
7. Maps $\mathcal{I}_\ell : H_{\partial \Omega}^1(\Omega_\ell) \cap \overline{C^0}(\Omega_\ell) \rightarrow \mathcal{V}_{\ell,h}$ where

$$H_{\partial \Omega}^1(\Omega_\ell) := \{u \in H^1(\Omega_\ell) : u|_{\partial \Omega_\ell \setminus \partial \Omega} = 0, u|_{\Gamma_{\text{soft}}} = 0, u|_{\Gamma_{\text{tr}}} = 0\}, \quad \mathcal{V}_{\ell,h} := \mathcal{V}_h \cap H_{\partial \Omega}^1(\Omega_\ell) \quad (2.10)$$

(the standard choice of $\mathcal{I}_\ell$ is nodal interpolation on Lagrange points).

The geometric set up described here is illustrated in Figure 2.1.

2.3 The subdomain problems

We now define the subdomain problems. This definition requires somewhat complicated assumptions about the absorption at the boundaries of the subdomains (similar to those in the definition of the underlying Helmholtz problem; i.e., Definition 2.1). However, we highlight that these conditions are all met if $\tilde{A}$, $\tilde{b}$, and $\tilde{n}$ are all real away from the absorption approximating the radiation condition (i.e., one starts from a problem with no absorption) and the absorption in the subdomain problems is a complex absorbing potential.

Definition 2.2 (Subdomain problems) *Given a piecewise C^{p+1} Helmholtz problem with absorption and C^r truncation, a (2.3), let $c_\ell > 0$, $W_\ell \subset \Omega_\ell \subset \Omega$ and $U_\ell \subset \Omega_\ell$ open such that*

$$\overline{W}_\ell \cap (\partial\Omega_\ell \setminus \partial\Omega) = \emptyset, \quad \partial\Omega_\ell \setminus \partial\Omega \subset \overline{U}_\ell.$$

Let

$$\mathcal{Z}_\ell := \{u \in H^1(\Omega) : u|_{\Gamma_{\text{soft}}} = 0, \text{supp } u \subset \overline{\Omega_\ell}\}.$$

We say that $a_\ell : \mathcal{Z}_\ell \times \mathcal{Z}_\ell \rightarrow \mathbb{C}$ is a piecewise C^{p+1} Helmholtz subdomain problem on Ω_ℓ with absorption and Lipschitz truncation for (a, W_ℓ, U_ℓ) on Ω_ℓ with constant c_ℓ if $\partial\Omega_\ell \in C^{0,1}$

$$a_\ell(u, v) = \langle A_\ell k^{-1} \nabla u, k^{-1} \nabla v \rangle_{\Omega_\ell} - \frac{i}{2} \langle b_\ell \cdot k^{-1} \nabla u, v \rangle_{\Omega_\ell} + \frac{i}{2} \langle b_\ell u, k^{-1} \nabla v \rangle - \langle n_\ell u, v \rangle_{\Omega_\ell},$$

with $A_\ell, b_\ell, n_\ell \in \overline{C^{p-1,1}}(\Omega_\ell \setminus \Gamma_p)$,

$$(\text{supp}(A_\ell - \tilde{A}) \cup \text{supp}(b_\ell - \tilde{b}) \cup \text{supp}(n_\ell - \tilde{n})) \cap W_\ell = \emptyset, \quad (2.11)$$

$$(2.5) \text{ and } (2.6) \text{ hold with } (\tilde{A}, \tilde{b}, \tilde{n}, c, U, \Omega) \text{ replaced by } (A_\ell, b_\ell, n_\ell, c_\ell, U_\ell, \Omega_\ell), \quad (2.12)$$

$$(\text{supp} \Im(e^{i\theta_2} A_\ell) \cup \text{supp} \Im(e^{i\theta_2} b_\ell)) \cap ((\partial\Omega_\ell \cup \Gamma_p) \setminus \overline{U}_\ell) = \emptyset, \quad (2.13)$$

where θ_2 is as in (2.7).

We sometimes write that a_ℓ is a subdomain problem on Ω_ℓ for a if there exist U_ℓ, W_ℓ, c_ℓ such that a_ℓ is a subdomain problem on Ω_ℓ for (a, W_ℓ, U_ℓ) with constant c_ℓ .

Regarding the three conditions (2.11), (2.12), (2.13): the conditions (2.11) and (2.12) imply that the region W_ℓ is where the subdomain problem agrees with the global Helmholtz problem and U_ℓ is the part of Ω_ℓ in which there is absorption. The condition (2.13) with $\theta_2 = 0$ requires that when Ω_ℓ touches Γ_p or $\partial\Omega_-$, the subdomain absorption at the boundary of the absorbing layer must be implemented with a CAP (involving complex $\tilde{n}$) rather than a PML.

To state the main result below, we need a quantitative notion of the scale of functions, sesquilinear forms, and Lipschitz domains.

Definition 2.3 (Scale of a function) *Let $\{D_\alpha\}_{\alpha \in \mathbb{N}^d} \subset (0, \infty)$. We say that the scale of $\chi \in \overline{C^{m-1,1}}(\Omega \setminus \Gamma_p)$ has with respect to $\{D_\alpha\}$ is*

$$\delta(\chi, \{D_\alpha\}, m) := \begin{cases} \left(\sup_{1 \leq |\alpha| \leq m} (D_\alpha^{-1} \|\partial^\alpha \chi\|_{L^\infty(\Omega \setminus \Gamma_p)})^{\frac{1}{|\alpha|}} \right)^{-1} & \text{if } \|\chi\|_{L^\infty} \leq D_0 \\ 0 & \text{else.} \end{cases}$$

Observe that if $\chi \in \overline{C^{m-1,1}}(\Omega \setminus \Gamma_p)$ has scale δ with respect to $\{D_\alpha\}$ then

$$\|\partial^\alpha \chi\|_{L^\infty(\Omega \setminus \Gamma_p)} \leq D_\alpha \delta^{-|\alpha|}.$$

Given the notion of the scale of a function, we define the scale of the sesquilinear form via the scale of its coefficients.

Definition 2.4 (Scale of a sesquilinear form) *Let $m \geq 0$ and $\{D_\alpha\}_{\alpha \in \mathbb{N}^d} \subset (0, \infty)$. We say that the scale of a sesquilinear form a given by*

$$a(u, v) = \langle A k^{-1} \nabla u, k^{-1} \nabla v \rangle - \frac{i}{2} \langle b \cdot k^{-1} \nabla u, v \rangle + \frac{i}{2} \langle b u, k^{-1} \nabla v \rangle - \langle n u, v \rangle$$

is

$$\delta(a, \{D_\alpha\}, m+1) := \min \left\{ \min_{i,j} \delta(A^{ij}, \{D_\alpha\}, m), \min_i \delta(b^i, \{D_\alpha\}, m), \delta(n, \{D_\alpha\}, m) \right\}.$$

We now define precisely the notion of the scale of a domain. Roughly speaking, a Lipschitz domain has scale δ if after zooming in to scale δ , one obtains a Lipschitz domain with both the Lipschitz constant and the number of domains in the covering controlled independently of δ .

Definition 2.5 (Scale of a Lipschitz domain) *Let $D > 0$ and define for $r > 0$*

$$V_r^1 := \{x \in \mathbb{R}^d : |x| < r\}, \quad V_r^2 := \{(x_1, x') \in (0, r) \times \mathbb{R}^{d-1} : |x'| < r\}.$$

We say that $\Omega \in C^{0,1}$ has scale δ with respect to D if there are $\{U_i^j\}_{i=1}^{N_j} \subset \delta^{-1}\Omega$, $j = 1, 2$, with $N_j \leq D$

$$\overline{U_i^1} \cap (\delta^{-1}\partial\Omega) = \emptyset,$$

and $\{F_i^j\}_{i=1}^{N_j}$, $F_i^j : \overline{V_2^j} \rightarrow \overline{U_i^j}$ bi-Lipschitz maps such that

$$\begin{aligned} \max_{i,j} \{ \|DF_i^j\|_{L^\infty}, \|(DF_i^j)^{-1}\|_{L^\infty} \} &\leq D, \\ F_i^2(\{x_1 > 0\} \cap \overline{V_2^2}) &\subset \delta^{-1}\Omega, \quad F_i^2(\{x_1 = 0\} \cap \overline{V_2^2}) = \partial(\delta^{-1}\Omega) \cap \overline{U_i^2}, \\ \delta^{-1}\Omega &\subset \bigcup_{j=1,2} \bigcup_{i=1}^{N_j} F_i^j(V_1^j). \end{aligned}$$

We then write

$$\delta(\Omega, D) := \sup \{ \delta > 0 : \Omega \text{ has scale } \delta \text{ with respect to } D \}.$$

2.4 The finite-element space

We consider finite element spaces consisting of mapped piecewise polynomials on curved elements. Specifically, for $\mathcal{T}$ a simplicial, affine conforming triangulation of Ω with element maps $\{\mathcal{F}_T\}_{T \in \mathcal{T}}$ and reference element $\hat{T}$ (in the sense of, e.g., [21, §A.1]), we denote the space of mapped polynomials of degree p on the triangulation $\mathcal{T}$ by

$$V^p(\mathcal{T}, \{\mathcal{F}_T\}_{T \in \mathcal{T}}) := \{u \in H^1(\Omega) : u|_T \circ \mathcal{F}_T \in \mathbb{P}^p(\hat{T})\}.$$

We then let

$$V_{\mathcal{T}}^p := V(\mathcal{T}, \{\mathcal{F}_T\}_{T \in \mathcal{T}}) \cap \mathcal{Z}.$$

For a triangulation $\mathcal{T}$ and $U \subset \Omega$, we denote the maximal width of an element by

$$h(\mathcal{T}, U) := \sup_{T \in \mathcal{T}, T \cap U \neq \emptyset} \text{diam}(T),$$

and the *local uniformity constant of $\mathcal{T}$ at scale ϵ* by

$$\mathcal{U}(\mathcal{T}, \epsilon) := \sup_{x \in \Omega} \sup_{\substack{T_1, T_2 \in \mathcal{T} \\ T_1 \cap B(x, \epsilon) \neq \emptyset \\ T_2 \cap B(x, \epsilon) \neq \emptyset}} \frac{\text{diam}(T_1)}{\text{diam}(T_2)}.$$

2.5 Definition of the two-level hybrid Schwarz preconditioner

Denote the respective inclusion maps by $\iota : \mathcal{V}_h \rightarrow \mathcal{Z}$, $\iota_0 : \mathcal{V}_{H_c} \rightarrow \mathcal{Z}$, $\iota_\ell : \mathcal{V}_{\ell,h} \rightarrow H_{\partial\Omega}^1(\Omega_\ell)$, $\iota_{\ell,h} : \mathcal{V}_{\ell,h} \rightarrow \mathcal{V}_h$, and $\iota_{0,h} : \mathcal{V}_{H_c} \rightarrow \mathcal{V}_h$. At the level of functions, the hybrid Schwarz left-preconditioner for P is $B_L^{-1}(P) : (\mathcal{V}_h)^* \rightarrow \mathcal{V}_h$

$$B_L^{-1} = B_L^{-1}(P) := \iota_{0,h} A_0^{-1} \iota_{0,h}^* + \left(\sum_{\ell=1}^N \iota_{\ell,h} \mathcal{I}_\ell \chi_\ell \iota_\ell A_\ell^{-1} \iota_\ell^* \right) (I - A \iota_{0,h} A_0^{-1} \iota_{0,h}^*), \quad (2.14)$$

where $\mathcal{I}_\ell : H_{\partial\Omega}^1(\Omega_\ell) \cap \overline{C^0}(\Omega_\ell) \rightarrow \mathcal{V}_{\ell,h}$ and

$$A_0 := \iota_0^* P \iota_0, \quad A := \iota^* P \iota, \quad A_\ell := \iota_\ell^* P_\ell \iota_\ell,$$

where P is the operator defined by a and P_ℓ that defined by a_ℓ . Note that $\iota_\bullet^*$ is the adjoint of $\iota_\bullet$; recall that this can be identified with the corresponding L^2 orthogonal projector.

Fix a basis $\{\phi_i\}_{i=1}^L$ of $\mathcal{V}_h$ with mass matrix $M_{ij} = \langle \phi_j, \phi_i \rangle$, as well as bases $\{\phi_i^c\}_{i=1}^{L_c}$ of $\mathcal{V}_{H_c}$, and $\{\phi_{i,\ell}\}_{i=1}^{L_\ell}$ of $\mathcal{V}_{\ell,h}$ respectively. Then set

$$\begin{aligned} (R_\ell)_ij^* &:= \sum_m M_{im}^{-1} \langle \phi_{j,\ell}, \phi_m \rangle, & (R_0)_ij^* &:= \sum_m M_{im}^{-1} \langle \phi_j^c, \phi_m \rangle, \\ (R_\ell^{\chi_\ell})_{ij}^* &:= \sum_m M_{im}^{-1} \langle \mathcal{I}_\ell \chi_\ell \phi_{j,\ell}, \phi_m \rangle, \\ (A_0)_{ij} &:= a(\phi_j^c, \phi_i^c), & (A_\ell)_{ij} &:= a_\ell(\phi_{j,\ell}, \phi_{i,\ell}), & A_{ij} &:= a(\phi_j, \phi_i). \end{aligned}$$

At the level of matrices, the two-level hybrid-Schwarz left preconditioner is then given by

$$B_L^{-1} = B_L^{-1}(A) := R_0^* A_0^{-1} R_0 + \left(\sum_{\ell=1}^N (R_\ell^{\chi_\ell})^* A_\ell^{-1} R_\ell \right) (I - A R_0^* A_0^{-1} R_0) \quad (2.15)$$

and the right preconditioner given by

$$B_R^{-1} = B_R^{-1}(A) := R_0^* A_0^{-1} R_0 + (I - R_0^* A_0^{-1} R_0 A) \left(\sum_{\ell=1}^N R_\ell^* A_\ell^{-1} R_\ell^{\chi_\ell} \right).$$

To state our theorem, we need a few more definitions.

Definition 2.6 Let $\{\Omega_\ell\}_{\ell=1}^N$ be subsets of Ω . We define the k^{-1} -overlap growth of $\{\Omega_\ell\}_{\ell=1}^N$, by

$$\Lambda(\{\Omega_\ell\}, k) := \max_{\ell=1, \dots, N} \sup_{R \geq 0} \frac{1}{\log(2+R)} \log \left(\#\{j : kd(\Omega_j, \Omega_\ell) \leq R\} \right).$$

That is, $\Lambda(\{\Omega_\ell\}, k)$ is the maximal growth of the number of domains that intersect the ball of radius R/k around Ω_ℓ . Define the *overlap number* by

$$\tilde{\Lambda}(\{\Omega_\ell\}) := \max\{\#\mathcal{N}(\ell) : \ell = 1, \dots, N\} < \infty, \quad \text{where } \mathcal{N}(\ell) := \{\ell' : \Omega_\ell \cap \Omega_{\ell'} \neq \emptyset\} < \infty. \quad (2.16)$$

Then $\Lambda(\{\Omega_\ell\}, k) \geq \log \tilde{\Lambda}(\{\Omega_\ell\}) / \log 2$ (by taking $R = 0$ in the definition); i.e., $\Lambda(\{\Omega_\ell\}, k)$ controls $\tilde{\Lambda}(\{\Omega_\ell\})$. Conversely, for any fixed D , $c > 0$ if $\tilde{\Lambda}(\{\Omega_\ell\}) < \infty$ and $\min_\ell \delta(\Omega_\ell, D) \geq ck^{-1}$, then $\Lambda(\{\Omega_\ell\}, k) < \infty$ (i.e., $\tilde{\Lambda}(\{\Omega_\ell\})$ controls $\Lambda(\{\Omega_\ell\}, k)$ under a scale assumption).

Finally, given a basis $\{\phi_i\}_{i=1}^L$, let the real symmetric positive-definite matrix $D_{k,\{\phi\}}$ be such that,

$$\left\| \sum_{i=1}^L \mathbf{V}_i \phi_i \right\|_{H_k^1(\Omega)}^2 = \langle D_{k,\{\phi\}} \mathbf{V}, D_{k,\{\phi\}} \mathbf{V} \rangle_2, \quad (2.17)$$

where $\langle \cdot, \cdot \rangle_2$ denotes the Euclidean inner product. With these definitions, one can check that, for any $K \geq 0$,

$$\|(I - B_L^{-1} A)^K\|_{\mathcal{Z} \rightarrow \mathcal{Z}} = \|D_{k,\{\phi\}} (I - B_L^{-1} A)^K D_{k,\{\phi\}}^{-1}\|_{\ell^2 \rightarrow \ell^2}$$

and

$$\|(I - A B_R^{-1})^K\|_{\mathcal{Z}^* \rightarrow \mathcal{Z}^*} = \|D_{k,\{\phi\}}^{-1} (I - A B_R^{-1})^K D_{k,\{\phi\}}\|_{\ell^2 \rightarrow \ell^2}.$$

2.6 The main result

We are now in a position to state our main result. The structure of this result is, roughly speaking,

1. fix some constants and a C^∞ Helmholtz problem,
2. state that there exist two other constants c and C ,
3. list the components of the two-level hybrid Schwarz preconditioner, stating how the geometry of the triangulations and subdomains, the regularity of the cutoffs, etc. depend on the constants in Points 1 and 2 (see (2.19), (2.20), (2.22), and then (2.23) below), and

4. give a bound on the norm of powers of the difference between the preconditioned matrix and the identity ((2.27) below).

Theorem 2.7 *Let $M > 0$, $0 < s < \frac{1}{2}$, $\Upsilon > 0$, $k_0 > 0$, $p \geq 1$, $c_0 > 0$, $\{D_\alpha\}_{\alpha \in \mathbb{N}^d} \subset (0, \infty)$, and $a = a(k)$ be a piecewise C^∞ Helmholtz problem with absorption and C^∞ truncation. There are $c, C > 0$ such that for all $k > k_0$ with*

$$\rho := \rho(a(k)) \leq \Upsilon k^M, \quad (2.18)$$

all simplicial, affine conforming C^{p+1} triangulations $\mathcal{T}_c$ and $\mathcal{T}_f$ of Ω with constant Υ satisfying $V_{\mathcal{T}_c}^p \subset V_{\mathcal{T}_f}^p$,

$$(H_c k)^{2p} \rho + H_c k \leq c, \quad H_c := h(\mathcal{T}_c, \Omega), \quad \mathcal{U}(\mathcal{T}_c, k^{-1}) + \mathcal{U}(\mathcal{T}_f, k^{-1}) \leq \Upsilon, \quad (2.19)$$

all coverings $\{\widetilde{W}_\ell\}_{\ell=1}^N$, $\{W_\ell\}_{\ell=1}^N$ and $\{\Omega_\ell\}_{\ell=1}^N$ satisfying (2.8), $\widetilde{W}_\ell \subset W_\ell$,

$$\Lambda(\{\Omega_\ell\}, k) \leq \Upsilon, \quad \partial\Omega_\ell \subset \cup_{T \in \mathcal{T}_f} \partial T, \quad \partial\Omega \cup \Gamma_p \subset \cup_{T \in \mathcal{T}_c} \partial T, \quad (2.20)$$

sets $\{U_\ell\}_{\ell=1}^N$ satisfying (2.9), piecewise C^∞ subdomain problems on Ω_ℓ with Lipschitz truncation for $(a(k), W_\ell, U_\ell)$ with constant c_0 , $\{a_\ell(k)\}_{\ell=1}^N$, and partitions of unity $\{\chi_\ell\}_{\ell=1}^N \subset \overline{C^{p,1}}(\Omega \setminus \Gamma_p)$ subordinate to $\{\widetilde{W}_\ell\}_{\ell=1}^N$ and satisfying

$$\nu \cdot \tilde{A} \nabla \chi_\ell = 0 \text{ on } \Gamma_{\text{hard}} \cap \partial\Omega_\ell, \quad \nu \cdot (\tilde{A}_+ \nabla(\chi_\ell)_+ - \tilde{A}_- \nabla(\chi_\ell)_-) = 0 \text{ on } \Gamma_p \cap \Omega_\ell, \quad (2.21)$$

and

$$\Upsilon \min \left\{ \delta(a_\ell, \{D_\alpha\}, p+1), \delta(\chi_\ell, \{D_\alpha\}, p+1) \right\} \geq \delta_\ell \geq c_0 k^{-1}, \quad \delta_\ell := \delta(\Omega_\ell, D_0), \quad (2.22)$$

$$\{x \in \Omega_\ell : d(x, \partial\Omega_\ell \setminus \partial\Omega) < \Upsilon^{-1} \delta_\ell\} \subset U_\ell, \quad d(W_\ell^c, \widetilde{W}_\ell) \geq \Upsilon^{-1} \delta_\ell, \quad (2.23)$$

$$\rho_\ell := \rho(a_\ell(k)) \leq \Upsilon (\delta_\ell k)^M, \quad (2.24)$$

with

$$\sup_\ell (h_\ell k)^p \rho_\ell + (h_\ell k)^s \leq c, \quad h_\ell := h(\mathcal{T}_f, \Omega_\ell), \quad h := \sup_\ell h_\ell \quad (2.25)$$

all $\mathcal{I}_\ell : \overline{C^0}(\Omega_\ell) \rightarrow \mathcal{V}_{\ell,h}$ and $\{\mu_\ell\}_{\ell=1}^N \subset (0, \infty)$ satisfying

$$\begin{aligned} \|\mathcal{I}_\ell v_{\ell,h}\|_{H_k^1(\Omega_\ell)} &\leq \Upsilon \|v_{\ell,h}\|_{H_k^1(\Omega_\ell)}, \quad \|(I - \mathcal{I}_\ell) \chi_\ell v_{\ell,h}\|_{H_k^1(\Omega_\ell)} \leq \Upsilon \mu_\ell \|v_{\ell,h}\|_{H_k^1(\Omega_\ell)} \quad \text{for } v_{\ell,h} \in \mathcal{V}_{\ell,h}, \\ \|(I - \mathcal{I}_\ell) u\|_{H_k^1(\Omega_\ell)}^2 &\leq \Upsilon (h_\ell k)^{2p} \sum_{\substack{T \in \mathcal{T}_f \\ T \cap \Omega_\ell \neq \emptyset}} \|u\|_{H_k^{p+1}(T)}^2, \quad u \in H_k^{p+1}(\Omega_\ell) \cap \overline{C^0}(\Omega_\ell), \end{aligned} \quad (2.26)$$

and all $K \geq 1$, the preconditioner $B_L^{-1}(A)$ defined with $\mathcal{V}_{H_c} = \mathcal{V}_{\mathcal{T}_c}^p$ and $\mathcal{V}_h = \mathcal{V}_{\mathcal{T}_f}^p$, using the basis $\{\phi\}_{i=1}^L$ of $\mathcal{V}_h$, satisfies

$$\begin{aligned} &\|D_{k,\{\phi\}}(I - B_L^{-1}A)^K D_{k,\{\phi\}}^{-1}\|_{\ell^2 \rightarrow \ell^2} + \|D_{k,\{\phi\}}^{-1}(I - AB_R^{-1})^K D_{k,\{\phi\}}\|_{\ell^2 \rightarrow \ell^2} \\ &\leq C^K \epsilon^{K-1} \left(\epsilon + \max_\ell \left\{ \rho_\ell (k \delta_\ell)^{-1} \rho (H_c k)^p + \rho_\ell (H_c k)^p + \langle k \delta_\ell \rangle^{-M} \right\} \right), \end{aligned} \quad (2.27)$$

where

$$\begin{aligned} \epsilon &:= \max_\ell \left\{ \rho_\ell (k \delta_\ell)^{-1} \rho (H_c k)^{2p} + \rho_\ell (H_c k)^{2p} + \langle k \delta_\ell \rangle^{-M} (H_c k)^p \right. \\ &\quad \left. + (h/H_c)^p + \mu_\ell + ((h_\ell k)^p \rho_\ell + (h_\ell k)^s) (\rho_\ell (k \delta_\ell)^{-1} + 1) \right\}. \end{aligned} \quad (2.28)$$

To the best of our knowledge, Theorem 2.7 is the first result on the convergence of two-level DD methods that does not require the coarse space to be in the asymptotic regime (i.e., the first result for which the Galerkin solution in the coarse space is not necessarily k -uniformly quasi-optimal).

2.7 Discussion of Theorem 2.7

Subdomain geometry. The definition of δ_ℓ in condition (2.22) implies that the domains Ω_ℓ must have diameters that are $\sim \delta_\ell$ and, in addition, that the absorbing layer have width at least $c\delta_\ell$. In particular, the minimal domain diameter is $\sim k^{-1}$ and we require generous overlap.

Making ϵ (2.28) small. To understand when ϵ is small, we need information about ρ, μ_ℓ , and ρ_ℓ .

In this discussion, for simplicity we assume that $\rho \geq ck$; this occurs if the Helmholtz problem is not absorbing everywhere. We also consider the standard choice of $\mathcal{I}_\ell$; i.e., nodal interpolation on Lagrange points. In this case (and in our setting of mapped piecewise polynomials on curved elements) the superapproximation property in the first line of (2.26) holds with

$$\mu_\ell = h_\ell \delta_\ell^{-1} \quad (2.29)$$

by [20, Lemma 4.6], and the approximation property in the second line of (2.26) holds by [2, §D.2]. Equation (2.29) together with the facts that $\delta_\ell \geq ck^{-1}$ (by (2.22)) and $h_\ell k \ll 1$ (by (2.25)) imply that the contribution of μ_ℓ to ϵ (2.28) is small.

We highlight that the most practically-interesting choice of subdomain size is arguably $\delta_\ell \sim k^{-1}$ (since one wants to minimise the number of degrees of freedom per subdomain for parallel scalability). However, it is important that the behaviour of the hybrid Schwarz preconditioner not be very sensitive to the number of wavelengths in a subdomain; i.e., we need to check that the result of Theorem 2.7 does not degenerate if $k\delta_\ell$ becomes large.

We now split the discussion into three cases: (i) $k\delta_\ell \leq c$ for c small, (ii) $k\delta_\ell \geq C$ with C large, and (iii) $c \leq k\delta_\ell \leq C$.

Case 1: domains that are small multiples of k^{-1} : $k\delta_\ell \leq c$. By the Poincaré–Friedrichs inequality, there is $c > 0$ such that if $\text{diam}(\Omega_\ell) \leq ck^{-1}$, then $\rho_\ell \leq 1$ (thus, in particular, the bound (2.24) holds).

Case 2: non-trapping domains that are much larger than k^{-1} : $k\delta_\ell \geq C$. We say that the subdomain problem a_ℓ is non-trapping if $\rho_\ell \leq C(k\delta_\ell)$. There are many situations where one can prove that a_ℓ is non-trapping in this sense. The simplest (although somewhat trivial) case is when $\Omega_\ell \subset U$. In this case, it is easy to see that $\rho_\ell \leq C$ for $k\delta_\ell \gg 1$. A more non-trivial situation occurs when $\Omega_\ell \cap \Gamma_p = \emptyset$. Set

$$p_\ell(x, \xi) := A_\ell^{ij}(x) \xi_i \xi_j + b_\ell^i(x) \xi_i - n_\ell(x),$$

and suppose there are $c, C, T > 0$ such that $\pm \Im p_\ell \geq -C(k\delta_\ell)^{-1}$ on $|p_\ell| \leq c$, and for all solutions to

$$\dot{x} = \partial_\xi \Re p_\ell(x, \xi), \quad \dot{\xi} = -\partial_x \Re p_\ell(x, \xi), \quad (x(0), \xi(0)) = (x_0, \xi_0), \quad \text{with } x_0 \in \Omega_\ell, |p_\ell(x_0, \xi_0)| \leq c$$

(extended by reflection at $\partial\Omega_-$, or, more precisely, as generalised broken bicharacteristics; see e.g. [2, Appendix C], [27, §24.3]) there is $0 \leq t \leq T$ such that $x(\pm t) \in U_\ell$. Then, for $k\delta_\ell \gg 1$, $\rho_\ell \leq Ck\delta_\ell$ (see e.g. [9, 22]). This is, in fact, the origin of the name non-trapping since every trajectory escapes into the absorbing layer. This example can be extended to subdomains intersecting Γ_p if one replaces generalised broken bicharacteristics by an analogue that may split into a reflected and transmitted part at Γ_p (see e.g. [31, 38] for the relevant dynamics).

We point out that, when $\delta_\ell \ll 1$, the coefficients of the Helmholtz problem are effectively constant on each subdomain and all scatterer boundaries $\partial\Omega_-$ and interfaces Γ_p are effectively hyperplanes. In this case, if the only change made to the (global) coefficients to create the subdomain problems is to add absorption, then a_ℓ will be non-trapping in the dynamical sense described above.

Case 3: middle-sized domains: $c \leq k\delta_\ell \leq C$. There are $c, C > 0$ depending on a, Υ , and $\{D_\alpha\}$, such that, except for $c \leq k\delta_\ell \leq C$, either the analysis in case 1 or the analysis in case 2 applies. To handle this intermediate regime, observe that the Fredholm alternative shows that for a fixed subdomain problem, ρ_ℓ may only fail to be uniformly bounded in k due to the presence of finitely many eigenvalues. If, for example, the absorption on the subdomains is a CAP (and k is large

enough if there is a PML on the global problem), then there are no such eigenvalues and $\rho_\ell \leq c$. Indeed, in this case, the imaginary part of the PDE on each subdomain has a sign, and uniqueness follows from the unique continuation principle.

In all of the above cases $\rho_\ell \leq Ck\delta_\ell$; hence, recalling that $\rho \geq ck$ and $\delta_\ell \leq c$, we see that ϵ (2.28) is controlled by

$$\epsilon \lesssim \rho(H_ck)^{2p} + (H_ck)^p + (h/H_c)^p + (hk)^s + \max_\ell (h_\ell k)^p k \delta_\ell.$$

Therefore, to make $\epsilon \ll 1$, we require $\rho(H_ck)^{2p} \ll 1$, $h \ll H_c$, and $(h_\ell k)^p (k\delta_\ell) \ll 1$. When $k\delta_\ell \sim 1$, this last requirement is automatic and this is the result summarised in Informal Theorem 1.1.

The condition $\rho(H_ck)^{2p} \ll 1$ is observed empirically to be necessary for the Galerkin solution in the coarse space to have controlled relative error and, in fact, the matrix A_0 may fail to be invertible above this threshold [4]. In Section 3, we give numerical evidence in the setup of Theorem 2.7 that indicates that the threshold $\rho(H_ck)^{2p} \ll 1$ is required to have the number of iterations – of both the fixed-point method and GMRES – bounded uniformly in k .

Consequences of having ϵ small for iteration counts. In all situations where $\epsilon \ll 1$, one can achieve a residual of size $\gamma > 0$ after

$$\sim \frac{\log \gamma - \log(1 + (H_ck)^p \rho)}{\log \epsilon} \quad (2.30)$$

iterations, for both the fixed-point method and GMRES in the $D_{k,\{\phi\}}$ weighted inner product (2.17). Recall that unweighted GMRES takes an extra $C \log(hk)^{-1}$ iterations than weighted GMRES [23, Corollary 5.8]. That is, the fixed-point method, weighted GMRES, and unweighted GMRES all converge in no more than $\sim \log k$ iterations provided that $(H_ck)^{2p} \rho \ll 1$ and h is not superalgebraically small in k . From (2.30), we also see that Theorem 2.7 recovers the result from [20] that if $(H_ck)^p \rho \leq C$ then convergence of the fixed-point method/weighted GMRES occurs in a k -independent number of iterations.

The assumptions on the underlying Helmholtz problem. To simplify the analysis of the coarse problem, Theorem 2.7 assumes that the global truncation boundary is smooth and that absorption of the form (2.6) holds. These assumptions, in particular, rule out truncation of the global Helmholtz problem with Cartesian PMLs. To cover Cartesian PMLs, one would need to repeat the preasymptotic h -FEM error analysis in [19] but explicitly take into account the better behaviour of the Helmholtz solution operator in the PML region (using, e.g., ideas from [17]).

In addition, Theorem 2.7 assumes that all coefficients and scatterer boundaries are smooth so that various localisation procedures produce errors that are super-algebraically small in k (see the proof of Lemma 5.1). This assumption could be relaxed to C^p -type regularity assumptions, but we would then need to strengthen the polynomial growth assumption on $\rho(k)$ (2.18); specifically, for each p one would allow a fixed p -dependent amount of polynomial growth in $\rho(k)$.

The assumptions on the cutoffs. Two features of the cutoffs χ_ℓ are important for the proof of Theorem 2.7. First, they need to have sufficiently high $(C^{p,1})$ regularity (encoded in Theorem 2.7 via (2.22)) so that they are bounded multipliers on H_k^{p+1} and hence approximation and super-approximation estimates hold in piecewise polynomial spaces of degree p . Second for the sum of local solves to plausibly be close to the true Helmholtz solution (which is in the domain of the operator P), it is crucial that χ_ℓ map the domain of P to itself. This is guaranteed by the conditions (2.21).

2.8 Discussion of the ideas of the proof

To understand the proof of Theorem 2.7, observe that the error in the preconditioned matrix is

$$I - B_L^{-1}A = \left(I - \sum_{\ell=1}^N \iota_{\ell,h} \mathcal{I}_\ell \chi_\ell \iota_\ell A_\ell^{-1} \iota_{\ell,h}^* A \right) (I - \iota_{0,h} A_0^{-1} \iota_{0,h}^* A) \quad (2.31)$$

$$\approx \left(\sum_{\ell=1}^N \chi_{\ell} (I - \iota_{\ell,h} A_{\ell}^{-1} \iota_{\ell,h}^* A) \right) (I - \iota_{0,h} A_0^{-1} \iota_{0,h}^* A),$$

where the error in passing to the second line is controlled using super-approximation results in $\mathcal{V}_{\ell,h}$. As advocated in [20, §1.2.3], one should think of the error in the coarse solve, $(I - \iota_{0,h} A_0^{-1} \iota_{0,h}^* A)$, as removing the low frequencies and the error in the subdomain solve $\sum_{\ell} (\chi_{\ell} (I - \iota_{\ell,h} A_{\ell}^{-1} \iota_{\ell,h}^* A))$ as removing the high-frequencies (since high frequencies do not propagate under the action of the Helmholtz solution operator, and thus can be well-approximated by a sum of local solves), so that their combination contracts the entire norm. The analysis in [20] rigorously implemented this idea in the case when the Galerkin solution in the coarse space is k -uniformly quasi-optimal. In contrast, here we consider piecewise-polynomial subspaces in the preasymptotic regime, i.e., for which the Galerkin solution in the coarse space is not k -uniformly quasi-optimal; see (2.34) below.

For the purposes of this discussion, we denote by $\Psi : H_k^1 \rightarrow H_k^{p+1}$ an operator that returns the “frequency k ”-part of a function u — this operator is constructed using the functional calculus of an associated self-adjoint operator, and is the key component of preasymptotic error analyses [19] (see Section 4 for the full definition of Ψ).

Coarse solve as frequency $\lesssim k$ removal.

Denote by $Q_{H_c}^0 : \mathcal{V}_h \rightarrow \mathcal{V}_{H_c}$ the solution to

$$a(Q_{H_c}^0 v_h, v_{H_c}) = a(v_h, v_{H_c}) \quad \text{for all } v_{H_c} \in \mathcal{V}_{H_c}; \quad (2.32)$$

i.e.

$$Q_{H_c}^0 = A_0^{-1} \iota_{0,h}^* A. \quad (2.33)$$

Then, by [19] there is $c > 0$ such that if $(H_c k)^{2p} \rho \leq c$, then $Q_{H_c}^0$ is well-defined; more precisely A_0 is invertible and

$$\|(I - Q_{H_c}^0)u\|_{H_k^1} \leq C(1 + (H_c k)^p \rho) \|(I - \Pi_{H_c})u\|_{H_k^1}. \quad (2.34)$$

Furthermore, we show in Lemma 5.1 (see also [21, Theorem 6.27]) that, with $\Pi_{H_c} : H_k^1 \rightarrow \mathcal{V}_{H_c}$ the H_k^1 orthogonal projector,

$$\begin{aligned} \|(I - \Psi)(I - Q_{H_c}^0)u\|_{H_k^1} &\leq C(1 + (H_c k)^{2p} \rho) \|(I - \Pi_{H_c})u\|_{H_k^1}, \\ \|\Psi(I - Q_{H_c}^0)u\|_{H_k^{p+1}} &\leq C(H_c k)^p \rho \|(I - \Pi_{H_c})u\|_{H_k^1}, \\ \|(I - Q_{H_c}^0)u\|_{H_k^{1-p}} &\leq C(H_c k)^p \rho \|(I - \Pi_{H_c})u\|_{H_k^1}. \end{aligned} \quad (2.35)$$

Since

$$\|(I - \Pi_{H_c})\Psi u\|_{H_k^1} \leq (H_c k)^p \|\Psi u\|_{H_k^1}, \quad (2.36)$$

the bounds (2.35) imply that, when $(H_c k)^{2p} \rho \ll 1$, the coarse solve effectively removes the low frequencies of u ; to see this, observe that if $u = \Psi u$, then the full norm contracts under the coarse solve. However, the high frequencies are not well approximated and so, when $1 \ll (H_c k)^p \rho$ and $(H_c k)^{2p} \rho \ll 1$, they produce a possibly large component in $(I - Q_{H_c}^0)$.

To quantitatively capture the difference between the coarse solve acting on high and low frequencies, we use the norm

$$\|u\|_X^2 := \|\Psi u\|_{H_k^{p+1}}^2 + (H_c k)^{-2p} \|(I - \Psi)u\|_{H_k^1}^2, \quad (2.37)$$

i.e., where high-frequencies are weighted by $(H_c k)^{-2p}$. Then

$$\|(I - Q_{H_c}^0)u\|_X \leq C(1 + (H_c k)^{2p} \rho) (H_c k)^{-p} \|(I - \Pi_{H_c})u\|_{H_k^1} \quad (2.38)$$

and

$$\|(I - \Pi_{H_c})u\|_{H_k^1} \leq C(H_c k)^p \|u\|_X,$$

so that

$$\|(I - Q_{H_c}^0)u\|_X \leq C(1 + (H_c k)^{2p} \rho) \|u\|_X;$$

compare to (2.34). That is, the coarse solve has bounded relative error in the norm $\|\cdot\|_X$ when $(H_c k)^{2p} \rho \leq c$.

Orthogonal decomposition on $\mathcal{V}_{H_c}$.

A key step in the proof is the simple observation that, by (2.34), $(I - Q_{H_c}^0)\Pi_{H_c} = 0$; i.e., the coarse solve is the identity in the coarse space. By (2.31) and (2.33), $I - B_L^{-1}A = E(I - Q_{H_c}^0)$ for some operator $E : H_k^1 \rightarrow H_k^1$. Therefore $I - B_L^{-1}A = (I - B_L^{-1}A)(I - \Pi_{H_c})$, so that

$$(I - B_L^{-1}A)^K = (I - B_L^{-1}A)((I - \Pi_{H_c})(I - B_L^{-1}A))^{K-1},$$

and thus it is enough to show that $(I - \Pi_{H_c})(I - B_L^{-1}A) : H_k^1 \rightarrow H_k^1$ is small and $(I - B_L^{-1}A) : H_k^1 \rightarrow H_k^1$ is controlled.

By (2.35) and (2.36),

$$\begin{aligned} \|(I - \Pi_{H_c})(I - Q_{H_c}^0)\Psi u\|_{H_k^1} &\leq \|(I - \Pi_{H_c})\Psi(I - Q_{H_c}^0)\Psi u\|_{H_k^1} + \|(I - \Pi_{H_c})(I - \Psi)(I - Q_{H_c}^0)\Psi u\|_{H_k^1} \\ &\leq C(1 + (H_c k)^{2p} \rho) \|(I - \Pi_{H_c})\Psi u\|_{H_k^1} \\ &\leq C(H_c k)^p (1 + (H_c k)^{2p} \rho) \|u\|_{H_k^1} \end{aligned} \quad (2.39)$$

and

$$\|(I - \Pi_{H_c})\Psi(I - Q_{H_c}^0)(I - \Psi)u\|_{H_k^1} \leq C(H_c k)^{2p} \rho \|u\|_{H_k^1}. \quad (2.40)$$

The two bounds (2.39) and (2.40) show that using the coarse solve as a preconditioner in $\mathcal{V}_h$ results in an error that is acceptable (i.e., small after approximation in $\mathcal{V}_{H_c}$) except at frequencies $\gg k$. Because these frequencies do not propagate under the Helmholtz solution operator, it is then reasonable to expect that the addition of a subdomain solve as in (2.31) can correct these errors. The analysis in the next step shows that this is indeed the case.

Analysis of the subdomain problems.

To finish the sketch proof of Theorem 2.7 it is therefore sufficient to prove that

$$\left\| (I - \Pi_{H_c}) \sum_{\ell} \chi_{\ell} (I - \iota_{\ell,h} A_{\ell}^{-1} \iota_{\ell,h}^* A) (I - Q_{H_c}^0) u \right\|_{H_k^1} \ll (H_c k)^p \|(I - Q_{H_c}^0)u\|_X \quad (2.41)$$

since, by (2.38),

$$\begin{aligned} (H_c k)^p \|(I - Q_{H_c}^0)u\|_X \\ \leq C(H_c k)^p (1 + (H_c k)^{2p} \rho) (H_c k)^{-p} \|(I - \Pi_{H_c})u\|_{H_k^1} \leq C(1 + (H_c k)^{2p} \rho) \|u\|_{H_k^1}. \end{aligned}$$

The proof of (2.41) is somewhat more subtle than the analysis of the coarse space, and is divided into two parts. We denote by $Q_h^{\ell} : \mathcal{V}_h \rightarrow \mathcal{V}_{\ell,h}$ the solution of

$$a_{\ell}(Q_h^{\ell} u_h, v_{\ell,h}) = a(u_h, E_{\ell} v_{\ell,h}) \quad \text{for all } v_{\ell,h} \in \mathcal{V}_{\ell,h}, \quad (2.42)$$

where $E_{\ell} : H_{\partial\Omega}^1(\Omega_{\ell}) \rightarrow \mathcal{Z}$ denotes extension by 0; that is, $Q_h^{\ell} = A_{\ell}^{-1} \iota_{\ell,h}^* A$, so that (2.41) involves $(I - Q_h^{\ell})(I - Q_{H_c}^0)$.

We then write

$$(I - Q_h^{\ell}) = (I - Q^{\ell}) + (Q^{\ell} - Q_h^{\ell}),$$

where

$$Q^{\ell} := P_{\ell}^{-1} E_{\ell}^* P. \quad (2.43)$$

That is, Q^{ℓ} is the continuous solution for which Q_h^{ℓ} is the Galerkin solution.

We start by showing that $\chi_{\ell}(I - Q^{\ell})(I - Q_{H_c}^0)$ is in H_k^{p+1} , and hence well approximated in $\mathcal{V}_{H_c}$. This relies on the following features of the setup

1. The quadratic forms a_{ℓ} are equal to a on $\text{supp} \chi_{\ell}$ and hence $\chi_{\ell}(I - P_{\ell}^{-1} E_{\ell}^* P)$ is smoothing. More precisely, away from $\partial\Omega_{\ell} \setminus \partial\Omega$, this allows us to write $\chi_{\ell}(I - Q^{\ell}) = \chi_{\ell} P_{\ell}^{-1} [\chi_{\ell}^{(1)}, P]$ for some $\chi_{\ell}^{(1)}$ with $\text{supp}(1 - \chi_{\ell}^{(1)}) \cap \text{supp} \chi_{\ell} = \emptyset$.

2. There is absorption in a neighborhood of the boundaries $\partial\Omega_\ell \setminus \partial\Omega$.
3. $P(I - Q_{H_c}^0)$ is orthogonal to $\mathcal{V}_{H_c}$ and hence, for terms near $\partial\Omega_\ell \setminus \partial\Omega$, the forcing term is small.

It then remains to understand $\chi_\ell(Q^\ell - Q_h^\ell)$. For this, we use the local-quasi-optimality results of [1] together with our analysis of Q^ℓ to see that it is enough to estimate $Q^\ell - Q_h^\ell$ in some very low regularity Sobolev space. This is then done using the preasymptotic Galerkin estimates from [19] applied to the subdomain problem.

3 Numerical experiments

In this section, we present numerical experiments for the Helmholtz equation *either* in free space *or* in the exterior of a trapping obstacle. We consider polynomials of degree 2 and 4, choose $\delta_\ell \sim k^{-1}$ (for which Theorem 2.7 gives the fastest convergence of the two-level method), and present results demonstrating that

- (A) under the assumptions of Theorem 2.7, one obtains exponential convergence of both GMRES and the iterative solver uniformly in k (i.e., consistent with Theorem 2.7), and
- (B) if $(H_c k)^{2p} \rho(k) \gg 1$, then the iterative method diverges and the number of iterations required for GMRES to converge grows quickly with k . This failure of convergence happens regardless of how small one chooses h .

More precisely, under Point 2 we present numerical experiments with

$$(B.1) \quad \left(\frac{hk}{C_f}\right)^{2p} \rho(k) = 1 \text{ and } 1 \ll \left(\frac{H_c k}{C_c}\right)^{2p} \rho(k) \ll \rho(k),$$

$$(B.2) \quad hk = C_f \text{ and } H_c k = C_c, \text{ and}$$

$$(B.3) \quad \left(\frac{hk}{C_f}\right)^{2p} \rho(k) = 1 \text{ and } H_c k = C_c.$$

In all these three cases the iterative method diverges and the number of iterations required for GMRES to converge grows quickly with k .

3.1 Numerical setup

All the numerical experiments were implemented in FreeFEM, with the code available at <https://github.com/FreeFem/FreeFem-sources/tree/develop/examples/ffddm/Helmholtz-2d-2lvl-nested.edp>.

Truncation and the sesquilinear form. We truncate to the ball of radius $R_{\text{tr}} = 0.9$, i.e., $\Omega_{\text{tr}} := B(0, R_{\text{tr}})$ with an absorbing layer starting at $R_{R_{\text{PML}}, -} = 0.7$. Since FreeFEM does not implement curved meshes, we commit a variational crime by approximating $\partial\Omega_{\text{tr}}$ using linear piecewise polynomials (see [10] for an k -explicit analysis of the FEM error in this situation).

The Sommerfeld radiation condition is approximated using a radial PML, leading to the sesquilinear form $a : H_0^1(\Omega_{\text{tr}}) \times H_0^1(\Omega_{\text{tr}}) \rightarrow \mathbb{C}$ given by (2.3) with $\tilde{A}, \tilde{b}$, and $\tilde{n}$ defined as follows.

We use the PML scaling function

$$g = \begin{cases} 10 \exp\left(-\frac{1}{3(r - R_{R_{\text{PML}}, -})}\right) & r > R_{R_{\text{PML}}, -} \\ 0 & r \leq R_{R_{\text{PML}}, -} \end{cases}.$$

Then, defining

$$\alpha(r) := 1 + ig'(r) \quad \text{and} \quad \beta(r) := 1 + ig(r)/r.$$

and the matrices D and H using polar coordinates (r, φ) by

$$D := \begin{pmatrix} \beta(r)\alpha(r)^{-1} & 0 \\ 0 & \alpha(r)\beta(r)^{-1} \end{pmatrix} \quad \text{and} \quad H := \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix},$$

we set $\tilde{b} = 0$,

$$\tilde{A} := \begin{cases} A & \text{in } \Omega, \\ HDH^T & \text{in } (B_{R_{\text{PML}},-})^c \end{cases} \quad \text{and} \quad \tilde{n} := \begin{cases} n & \text{in } \Omega \cap B_{R_{\text{PML}},-}, \\ \alpha(r)\beta(r)^{d-1} & \text{in } (B_{R_{\text{PML}},-})^c. \end{cases} \quad (3.1)$$

We use the source

$$f = \chi(|x|)J_0(k|x|), \quad \text{with} \quad \chi = \begin{cases} \frac{1}{1+\exp\left(\frac{1}{0.1-x}-\frac{1}{x-0.05}\right)} & 0.05 < x < 0.1 \\ 1 & x < 0.05 \\ 0 & x > 0.1 \end{cases}. \quad (3.2)$$

Subdomain problems. The subdomains Ω_ℓ are obtained using the METIS algorithm. They are chosen to be of size $\sim k^{-1}$ and have generous overlap. We deviate slightly from the setup of Theorem 2.7 by using piecewise linear partition-of-unity functions χ_ℓ and an absorbing profile that is not smooth. The subdomain problems are then given by $a_\ell : \mathcal{Z}_\ell \times \mathcal{Z}_\ell \rightarrow \mathbb{C}$,

$$a_\ell(u, v) = a(u, v) - i\langle q_\ell u, v \rangle,$$

where

$$0 \leq q_\ell = \begin{cases} 10 \exp\left(-\frac{L_\ell}{0.6(d(x, \partial\Omega_\ell \setminus \partial\Omega) - L_\ell)}\right) & d(x, \partial\Omega_\ell \setminus \partial\Omega) < L_\ell \\ 0 & \text{else,} \end{cases}$$

and $L_\ell \sim k^{-1}$ is half of the subdomain overlap width.

Piecewise polynomial spaces The mesh is generated using a standard Delaunay algorithm as implemented in FreeFEM and the basis functions $\{\phi\}_{i=1}^K$ from (2.17) are the standard representation of piecewise polynomials using Lagrange nodes expressed in barycentric coordinates.

3.2 Experiments in free space.

In this case $\rho \sim Ck$; in the experiments below we take $\rho = k$.

P2 elements. Figures 3.1, 3.2, 3.3, and 3.4 show numerical results for $p = 2$ in the scenarios in Points A, B.1, B.2, and B.3, respectively. In Figure 3.1 we see that both the fixed-point iteration and GMRES converge in a k -uniform number of iterations (in accordance with Theorem 2.7), but in the other three experiments we see the number of iterations for both methods increasing as k increases.

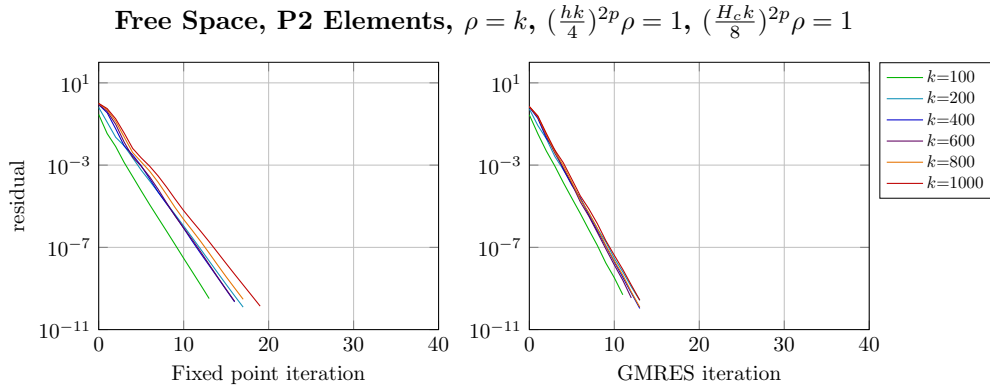


Figure 3.1: Experiments for free space in the set up of Point A above (i.e., covered by Theorem 2.7).

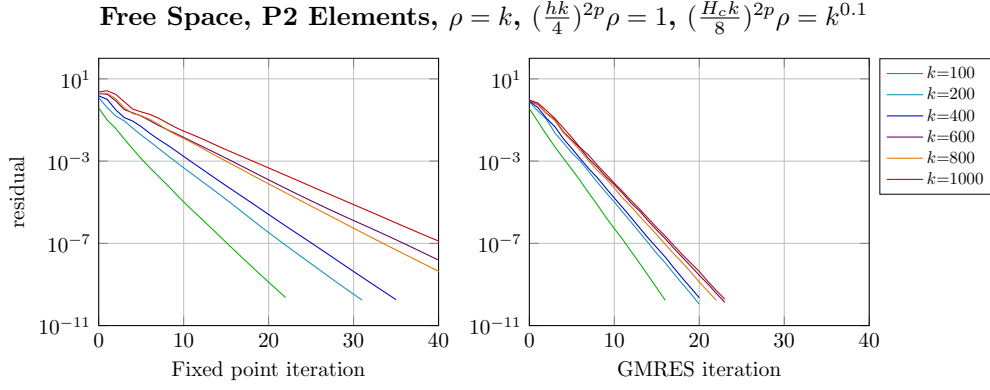


Figure 3.2: Experiments for free space with $p = 2$ in the set up of Point B.1 above (i.e., a scenario not covered by Theorem 2.7).

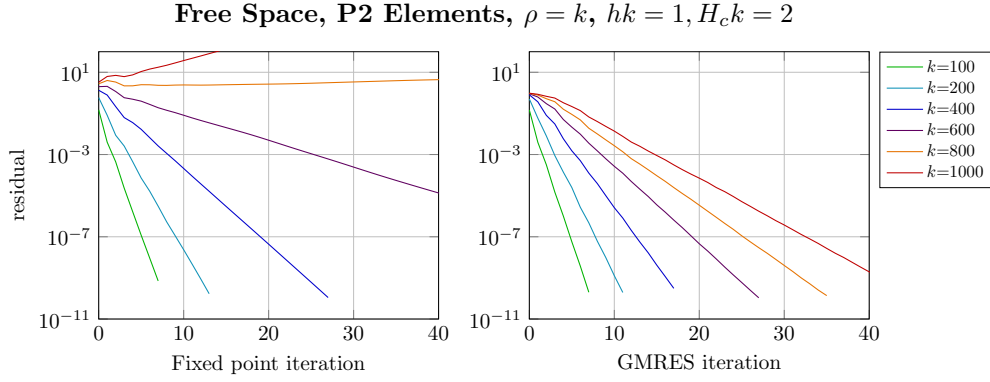


Figure 3.3: Experiments for free space with $p = 2$ in the set up of Point B.2 above (i.e., a scenario not covered by Theorem 2.7).

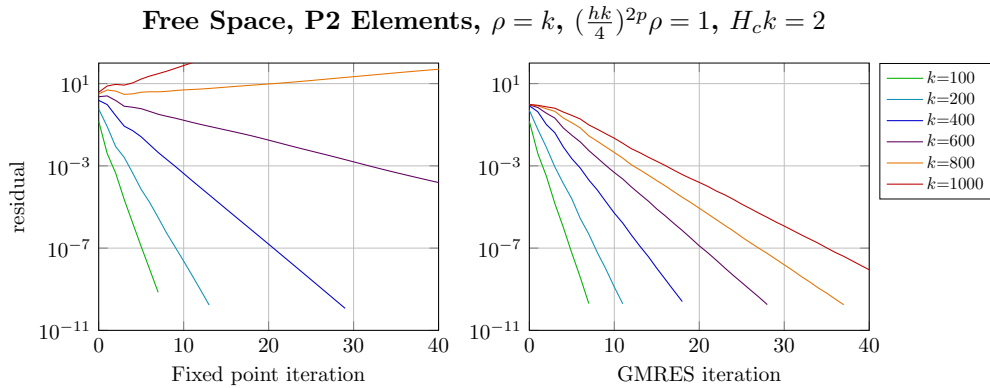


Figure 3.4: Experiments for free space with $p = 2$ in the set up of Point B.3 above (i.e., a scenario not covered by Theorem 2.7).

P4 elements. Figures 3.5, 3.6, 3.7, and 3.8 show numerical results for $p = 4$ in the scenarios in Points A, B.1, B.2, and B.3 respectively. In Figure 3.1 we see that both the fixed-point iteration and GMRES converge in a k -uniform number of iterations (in accordance with Theorem 2.7), but in the other two experiments we see the number of iterations for both iterations increasing as k increases.

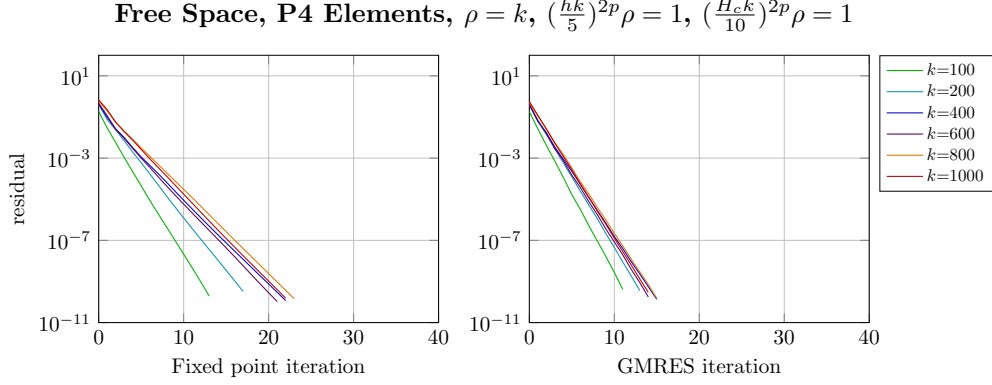


Figure 3.5: Experiments for free space with $p = 4$ in the set up of Point A above (i.e., a scenario covered by Theorem 2.7).

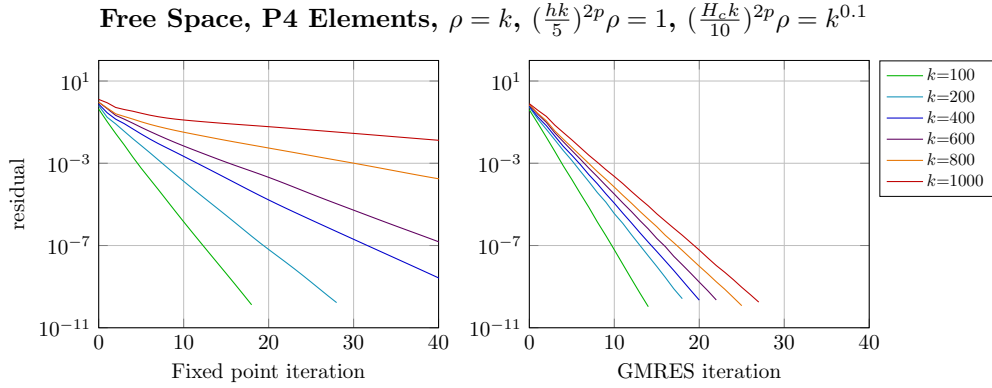


Figure 3.6: Experiments for free space with $p = 4$ in the set up of Point B.1 above (i.e., a scenario not covered by Theorem 2.7).

3.3 Experiments in the exterior of a trapping obstacle

Here we present experiments in the following domain, pictured in Figure 3.9,

$$\Omega = \Omega_{\text{tr}} \setminus \Omega_-, \quad \Omega_- := \cup_{x_0 \in \{\pm 1\}, y_0 \in \{\pm 1\}} 0.3(x_0, y_0) + \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}(-0.15, 0.15)^2. \quad (3.3)$$

In this case, one can show that $\rho(k) \gtrsim k^2$ when

$$k = \frac{2\pi}{0.6\sqrt{2} - 0.3} \sqrt{n^2 + m^2} \text{ with } m = 0, n \in \mathbb{Z} \text{ or } n/m \in \{0, \pm 1, \pm 2\}. \quad (3.4)$$

In the experiments we take $\rho(k) = k^2$. The data f is given by (3.2) (with the source point in the centre of the four squares).

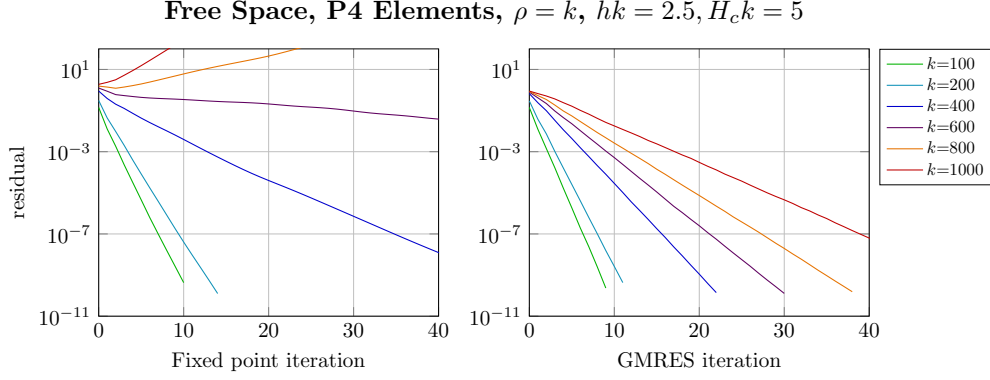


Figure 3.7: Experiments for free space with $p = 4$ in the set up of Point B.2 above (i.e., a scenario not covered by Theorem 2.7).

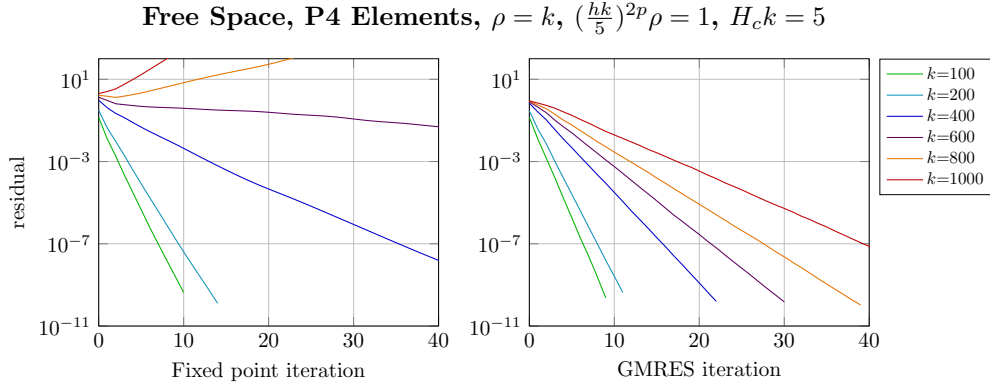


Figure 3.8: Experiments for free space with $p = 4$ in the set up of Point B.3 above (i.e., a scenario not covered by Theorem 2.7).

We present results under Point A for both $p = 2$ and $p = 4$. For the results under Point B, for brevity we present results under Point B.1 for $p = 2$ only, and results under Points B.2 and B.3 for $p = 4$ only.

P2 elements. Figures 3.10 and 3.11 show numerical results for $p = 2$ in the scenarios in Points A and B.1. In Figure 3.10 we see that both the fixed-point iteration and GMRES converge in a k -uniform number of iterations (in accordance with Theorem 2.7) but in the other three experiments we see the number of iterations for both methods increasing as k increases.

Remark 3.1 *In each of the Figures 3.10, 3.11, 3.12, 3.13, and 3.14, the fixed-point iteration behaves significantly worse at $k = 194.729$ than at any other frequency tested. This can be explained by the fact that, in addition to being in $\frac{2\pi}{0.6\sqrt{2}-0.3}\mathbb{N}$, 194.729 is very close to $\frac{2\pi}{0.6\sqrt{2}-0.3}\sqrt{2}\mathbb{N}$ and hence the solution operator is more badly behaved at this particular frequency than the others (see (3.4)).*

P4 elements. Figures 3.12, 3.13, 3.14 show numerical results for $p = 4$ in the scenarios in Points A, B.2, and B.3, respectively. In Figure 3.1 we see that both the fixed-point iteration and GMRES converge in a k -uniform number of iterations (in accordance with Theorem 2.7), but in the other two experiments we see the number of iterations for both iterations increasing as k increases.

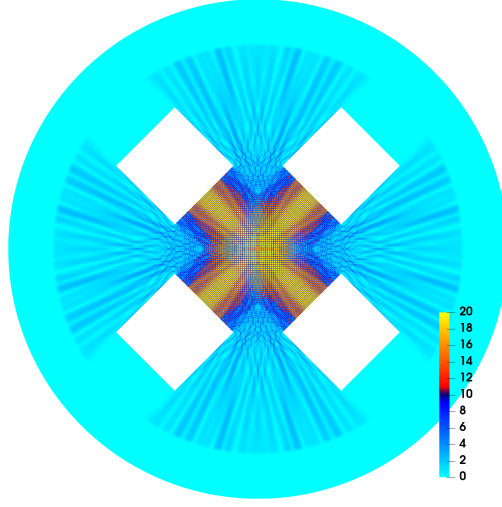


Figure 3.9: The absolute value of the total field in the exterior of the domain Ω_- defined in (3.3), at wavenumber $k = 595.641$, and where the data f is defined by (3.2).

Trapping Obstacle, P2 Elements, $\rho = k^2$, $(\frac{hk}{8})^{2p}\rho = 1$, $(\frac{H_ck}{16})^{2p}\rho = 1$, $k \in \frac{2\pi}{0.6\sqrt{2}-0.3}\mathbb{N}$

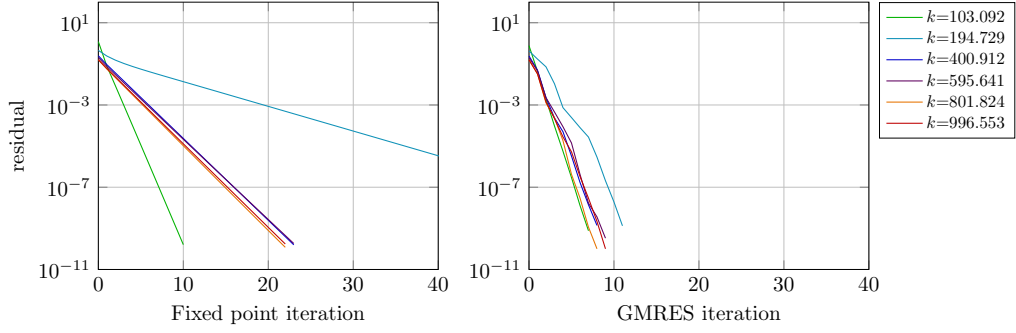


Figure 3.10: Experiments in the exterior of a trapping obstacle in the set up of Point A above (i.e., covered by Theorem 2.7).

Trapping Obstacle, P2 Elements, $\rho = k^2$, $(\frac{hk}{8})^{2p}\rho = 1$, $(\frac{H_ck}{4})^{2p}\rho = k$, $k \in \frac{2\pi}{0.6\sqrt{2}-0.3}\mathbb{N}$

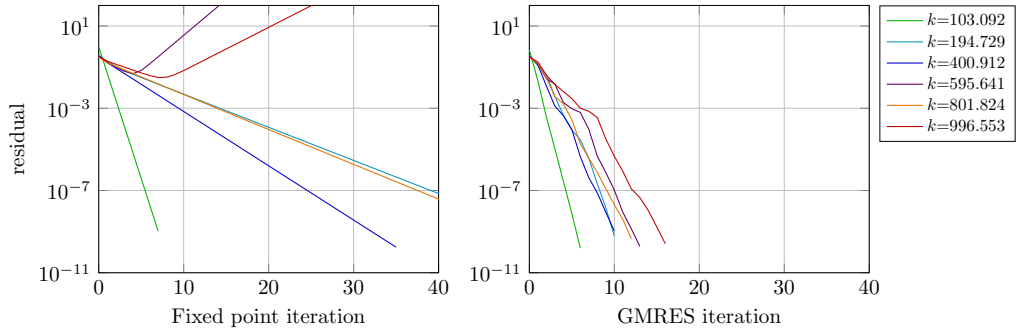


Figure 3.11: Experiments in the exterior of a trapping obstacle with $p = 2$ in the set up of Point B.1 above (i.e., a scenario not covered by Theorem 2.7).

Trapping Obstacle, P4 Elements, $\rho = k^2$, $(\frac{hk}{6})^{2p}\rho = 1$, $(\frac{H_c k}{12})^{2p}\rho = 1$, $k \in \frac{2\pi}{0.6\sqrt{2}-0.3}\mathbb{N}$

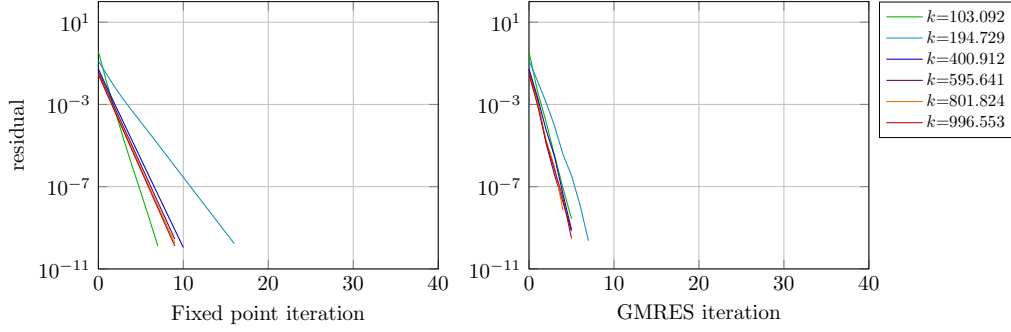


Figure 3.12: Experiments in the exterior of a trapping obstacle with $p = 4$ in the set up of Point A above (i.e., covered by Theorem 2.7).

Trapping Obstacle, P4 Elements, $\rho = k^2$, $hk = 2$, $H_c k = 4$, $k \in \frac{2\pi}{0.6\sqrt{2}-0.3}\mathbb{N}$

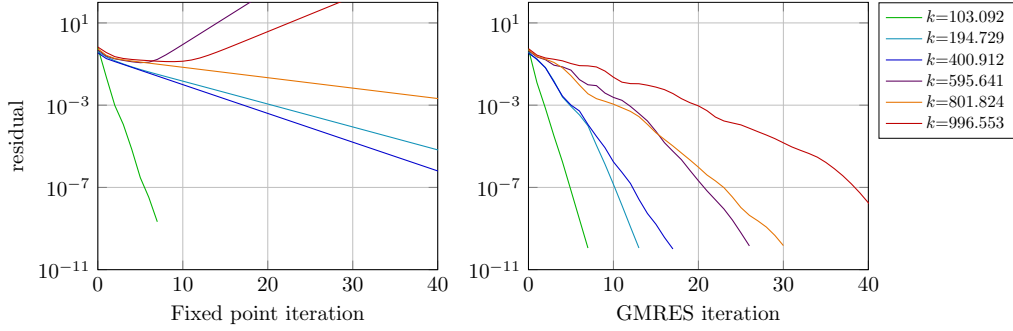


Figure 3.13: Experiments in the exterior of a trapping obstacle with $p = 4$ in the set up of Point B.2 above (i.e., covered by Theorem 2.7).

Trapping Obstacle, P4 Elements, $\rho = k^2$, $(\frac{hk}{6})^{2p}\rho = 1$, $H_c k = 4$, $k \in \frac{2\pi}{0.6\sqrt{2}-0.3}\mathbb{N}$

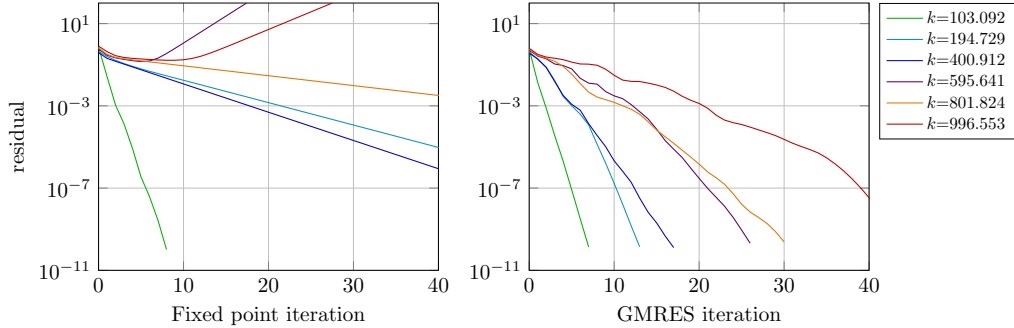


Figure 3.14: Experiments in the exterior of a trapping obstacle with $p = 4$ in the set up of Point B.3 above (i.e., covered by Theorem 2.7).

4 Functional-analytic set up

For convenience, and (up to multiplication by a $re^{-i\theta_0}$ for some $r > 0$) without loss of generality we will assume that $\theta_0 = 0$, hence by (2.5) $\cos \theta_2 \neq 0$, and there is a constant $c > 0$ such that for all $k > k_0$,

$$\Re a(u, u) \geq c \|u\|_{H_k^1(\Omega)}^2 - \frac{3}{2} \|u\|_{L^2(\Omega)}^2 \quad \text{for all } u \in \mathcal{Z}$$

(where the sesquilinear form a and space $\mathcal{Z}$ are defined by (2.3) and (2.4), respectively). We denote by $\mathcal{P} : \mathcal{Z} \rightarrow \mathcal{Z}^*$ the self-adjoint operator generated by the quadratic form

$$(\Re a)(u, v) := \frac{1}{2} (a(u, v) + \overline{a(v, u)}).$$

To state more conveniently the mapping properties of P , P^* , and $\mathcal{P}$ we define several new scales of Hilbert spaces. For $n = 0$, set $\mathcal{H}^0 = \mathcal{Z}^0 = L^2(\Omega)$ and for $n \geq 1$, we define the Hilbert spaces

$$\mathcal{H}^n := H_k^n(\Omega \setminus \Gamma_p) \cap H_k^1(\Omega),$$

$$\mathcal{Z}^n := \begin{cases} \mathcal{Z}, & n = 1 \\ \mathcal{H}^n \cap \left\{ u \in \mathcal{Z} : \begin{aligned} &(\nu \cdot \tilde{A} k^{-1} \nabla u + \frac{i}{2} \tilde{b} u)|_{\Gamma_{\text{hard}}} = 0, \\ &\nu \cdot (\tilde{A}_+ k^{-1} \nabla u_+ - \tilde{A}_- k^{-1} \nabla u_- + \frac{i}{2} (\tilde{b}_+ - \tilde{b}_-) u)|_{\Gamma_p} = 0 \end{aligned} \right\}, & n \geq 2. \end{cases}$$

For $n \leq -1$, we define $\mathcal{Z}^n = (\mathcal{Z}^{-n})^*$ and $\mathcal{H}^n = (\mathcal{H}^{-n})^*$. Then we set

$$\mathcal{W}^n := \mathcal{Z}^n \cap \mathcal{H}^n, \quad \mathcal{Y}^n := \mathcal{Z}^n + \mathcal{H}^n. \quad (4.1)$$

By the inclusion $\mathcal{Z}^n \subset \mathcal{H}^n$ for $n \geq 0$ and duality,

$$\mathcal{W}^n = \begin{cases} \mathcal{Z}^n & \text{if } n \geq 0, \\ \mathcal{H}^n & \text{if } n \leq 0, \end{cases} \quad \text{and} \quad \mathcal{Y}^n = \begin{cases} \mathcal{H}^n & \text{if } n \geq 0, \\ \mathcal{Z}^n & \text{if } n \leq 0. \end{cases}$$

Thus, since $\mathcal{Z}^{-n} = (\mathcal{Z}^n)^*$,

$$\mathcal{Y}^{-n} = (\mathcal{W}^n)^*, \quad \text{for all } n \in \mathbb{Z}.$$

One may think of $\mathcal{W}^n$ (respectively $\mathcal{Y}^n$) as the space “with” (respectively “without”) boundary conditions, and the application of P “removes” the boundary conditions.

For use later, we fix $\Psi_0, \Psi \in C_c^\infty(\mathbb{R}; [0, \infty))$ such that

$$\begin{aligned} \Re P + \Psi_0(\Re P) &\geq c > 0, & \Re P_\ell + \Psi_0(\mathcal{P}_\ell) &\geq c > 0 \\ \text{supp}(I - \Psi) \cap \text{supp} \Psi_0 &= \emptyset. \end{aligned} \quad (4.2)$$

We then set $S := \Psi_0(\Re P)$ and define

$$\|u\|_X^2 := \|\Psi(\mathcal{P})u\|_{\mathcal{W}^{p+1}}^2 + (H_c k)^{-2p} \|(I - \Psi(\mathcal{P}))u\|_{H_k^1}^2. \quad (4.3)$$

and similarly

$$\|u\|_{X(\Omega_\ell)}^2 := \|\Psi(\mathcal{P})u\|_{H_k^{p+1}(\Omega_\ell \setminus \Gamma_p)}^2 + (H_c k)^{-2p} \|(I - \Psi(\mathcal{P}))u\|_{H_k^1(\Omega_\ell)}^2. \quad (4.4)$$

5 Preasymptotic analysis of coarse problem

Lemma 5.1 *If $(H_c k)^{2p} \rho \leq c$ then $Q_{H_c}^0 : \mathcal{V}_h \rightarrow \mathcal{V}_{H_c}$ defined by (2.32) exists, is unique, and satisfies*

$$\|(I - Q_{H_c}^0)v_h\|_X \leq C(1 + (H_c k)^{2p} \rho)(H_c k)^{-p} \|(I - \Pi_{H_c})v_h\|_{\mathcal{Z}}. \quad (5.1)$$

Furthermore,

$$\|(I - Q_{H_c}^0)v_h\|_{\mathcal{W}^{1-p}} \leq C(H_c k)^{2p} \rho (H_c k)^{-p} \|(I - \Pi_{H_c})v_h\|_{\mathcal{Z}}. \quad (5.2)$$

Proof. By the definition of $\|\cdot\|_X$ (4.3),

$$\|(I - Q_{H_c}^0)v_h\|_X^2 = \|\Psi(\mathcal{P})(I - Q_{H_c}^0)v_h\|_{\mathcal{W}^{p+1}}^2 + (H_c k)^{-2p} \|(I - \Psi(\mathcal{P}))(I - Q_{H_c}^0)v_h\|_{\mathcal{Z}}^2.$$

To prove (5.1), we prove the two bounds

$$\|\Psi(\mathcal{P})(I - Q_{H_c}^0)v_h\|_{\mathcal{W}^{p+1}} \leq C(H_c k)^p \rho \|(I - \Pi_{H_c})v_h\|_{\mathcal{Z}}, \quad (5.3)$$

and

$$\|(I - \Psi(\mathcal{P}))(I - Q_{H_c}^0)v_h\|_{\mathcal{Z}} \leq C(1 + (H_c k)^{2p} \rho) \|(I - \Pi_{H_c})v_h\|_{\mathcal{Z}}. \quad (5.4)$$

We start with (5.3), i.e., the low-frequencies of FEM error. Here we use the bounds proved in [19] (recapped in [21, §6]). By [21, Equation 6.24],

$$\|\Psi(\mathcal{P})\|_{\mathcal{Y}^{1-p} \rightarrow \mathcal{W}^{p+1}} + \|\Psi_0(\mathcal{P})\|_{\mathcal{Y}^{1-p} \rightarrow \mathcal{W}^{p+1}} \leq C \quad (5.5)$$

and, by [21, Equations 6.33 and 6.43], if $(H_c k)^{2p} \rho \leq c$ then

$$\|(I - Q_{H_c}^0)v_h\|_{\mathcal{W}^{1-m}} \leq C((H_c k)^m + (H_c k)^p \rho) \|(I - \Pi_{H_c})v_h\|_{\mathcal{Z}}. \quad (5.6)$$

This last inequality with $m = p$ is (5.2) (since $\rho \geq c$), and then combined with the bound (5.5) it gives (5.3).

We now prove (5.4) by modifying arguments in [2]. Define $\Pi_S : \mathcal{Z} \rightarrow \mathcal{V}_{H_c}$ by

$$\langle (P + S)w_h, (I - \Pi_S)u \rangle = 0, \quad \text{for all } w_h \in \mathcal{V}_{H_c}. \quad (5.7)$$

Then, by the definition of $(P^*)^{-1}$, Galerkin orthogonality, the definition $P_S := P + S$, and the smoothing property of S (5.5),

$$\begin{aligned} & \langle (I - \Psi(\mathcal{P}))(I - Q_{H_c}^0)v_h, v \rangle \\ &= \langle P(I - Q_{H_c}^0)v_h, (P^*)^{-1}(I - \Psi(\mathcal{P}))v \rangle \\ &= \langle P((I - Q_{H_c}^0)v_h), (I - \Pi_S)(P^*)^{-1}(I - \Psi(\mathcal{P}))v \rangle \\ &= \langle P_S(I - \Pi_{H_c})v_h, (I - \Pi_S)(P^*)^{-1}(I - \Psi(\mathcal{P}))v \rangle - \langle S((I - Q_{H_c}^0)v_h), (I - \Pi_S)(P^*)^{-1}(I - \Psi(\mathcal{P}))v \rangle \\ &\leq C\|(I - \Pi_{H_c})v_h\|_{\mathcal{Z}}\|(P^*)^{-1}(I - \Psi(\mathcal{P}))\|_{\mathcal{Z}^* \rightarrow \mathcal{Z}}\|v\|_{\mathcal{Z}^*} \\ &\quad + C\|(I - Q_{H_c}^0)v_h\|_{\mathcal{Y}^{1-p}}\|(I - \Pi_S)(P^*)^{-1}(I - \Psi(\mathcal{P}))v\|_{\mathcal{Y}^{1-p}}. \end{aligned} \quad (5.8)$$

By (4.2), $P + S$ is k -uniformly coercive (see [21, Theorem 6.21]) and then a standard Aubin-Nitsche-type duality argument (see [2, Lemma 8.2/Corollary 8.3]) implies that

$$\begin{aligned} \|(I - \Pi_S)(P^*)^{-1}(I - \Psi(\mathcal{P}))v\|_{\mathcal{Y}^{1-p}} &\leq (H_c k)^p \|(I - \Pi_{H_c})(P^*)^{-1}(I - \Psi(\mathcal{P}))v\|_{\mathcal{Z}} \\ &\leq (H_c k)^p \|(P^*)^{-1}(I - \Psi(\mathcal{P}))\|_{\mathcal{Z}^* \rightarrow \mathcal{Z}}\|v\|_{\mathcal{Z}^*}. \end{aligned} \quad (5.9)$$

We claim that

$$\|(P^*)^{-1}(I - \Psi(\mathcal{P}))\|_{\mathcal{Z}^* \rightarrow \mathcal{Z}} \leq C; \quad (5.10)$$

combining this with (5.8) and (5.9) gives

$$\|(I - \Psi(\mathcal{P}))(I - Q_{H_c}^0)v_h\|_{\mathcal{Z}} \leq C\|(I - \Pi_{H_c})v_h\|_{\mathcal{Z}} + C(H_c k)^p \|(I - Q_{H_c}^0)v_h\|_{\mathcal{Y}^{1-p}},$$

so that by (5.6) with $m = p$, the bound (5.4) follows.

It therefore remains to prove the bound (5.10). Since $P_S^* := P^* + S$,

$$(P^* + S)(P^*)^{-1} = I + S(P^*)^{-1},$$

and thus

$$(P^*)^{-1} = (P_S^*)^{-1} + (P_S^*)^{-1}S(P^*)^{-1} \quad (5.11)$$

so that

$$(P^*)^{-1}(I - \Psi(\mathcal{P})) = (P_S^*)^{-1}(I - \Psi(\mathcal{P})) + (P_S^*)^{-1}S(P^*)^{-1}(I - \Psi(\mathcal{P})). \quad (5.12)$$

By (5.5), $\|I - \Psi(\mathcal{P})\|_{\mathcal{Z} \rightarrow \mathcal{Z}} \leq C$ so that, by self-adjointness, $\|I - \Psi(\mathcal{P})\|_{\mathcal{Z}^* \rightarrow \mathcal{Z}^*} \leq C$. Since $\|(P_S^*)^{-1}\|_{\mathcal{Z}^* \rightarrow \mathcal{Z}} \leq C$, the $\mathcal{Z}^* \rightarrow \mathcal{Z}$ norm of the first term in (5.12) is $\leq C$.

To deal with the second term in (5.12), we let $\varphi, \tilde{\varphi} \in C^\infty(\bar{\Omega})$ be such that $\text{supp} \varphi \cap \Gamma_{\text{tr}} = \emptyset$, $\text{supp}(1 - \varphi) \cap \text{supp} \tilde{\varphi} = \emptyset$ (i.e. $\tilde{\varphi} \prec \varphi$) and $1 - \tilde{\varphi}$ is supported in the PML region (note that this last condition requires that the support of φ extend into the PML region). We write

$$(P_S^*)^{-1} S(P^*)^{-1} (I - \Psi(\mathcal{P})) = (P_S^*)^{-1} S(P^*)^{-1} (I - \Psi(\mathcal{P})) \varphi + (P_S^*)^{-1} S(P^*)^{-1} (I - \Psi(\mathcal{P})) (1 - \varphi). \quad (5.13)$$

By [2, Lemma 8.6] and the fact that $\text{supp}(I - \Psi) \cap \text{supp} \Psi_0 = \emptyset$,

$$\|(P^*)^{-1} (I - \Psi(\mathcal{P})) \varphi\|_{\mathcal{Z}^* \rightarrow \mathcal{Z}} \leq C, \quad (5.14)$$

so that the $\mathcal{Z}^* \rightarrow \mathcal{Z}$ norm of the first term on the right-hand side of (5.13) is bounded (Here, we use (2.7) to see that, near $(\Gamma_p \cup \partial\Omega_-) \setminus \bar{U}$, $\Im(e^{i\theta_2} P) = 0$ so that $P = (1 - i \tan \theta_2) \mathcal{P}$ and hence the proof of [2, Lemma 8.6], involving [2, Lemma 6.9], applies.) Furthermore,

$$\begin{aligned} & (P_S^*)^{-1} S(P^*)^{-1} (I - \Psi(\mathcal{P})) (1 - \varphi) \\ &= (P_S^*)^{-1} S(P^*)^{-1} (1 - \tilde{\varphi}) (I - \Psi(\mathcal{P})) (1 - \varphi) + (P_S^*)^{-1} S(P^*)^{-1} \tilde{\varphi} (I - \Psi(\mathcal{P})) (1 - \varphi) \\ &= (P_S^*)^{-1} S(P^*)^{-1} (1 - \tilde{\varphi}) (I - \Psi(\mathcal{P})) (1 - \varphi) + O(k^{-\infty})_{(H_k^N)^* \rightarrow H_k^N} \end{aligned}$$

by spatial pseudolocality of $\Psi(\mathcal{P})$ [2, Theorem 6.2]. Finally, by [18, Lemma 4.4] (see also Lemma 6.1 below with Ω_ℓ replaced by Ω), $\|(P^*)^{-1} (1 - \tilde{\varphi})\|_{\mathcal{Z}^* \rightarrow \mathcal{Z}} \leq C$ since $1 - \tilde{\varphi}$ is supported in the absorbing layer. We have therefore proved (5.10), and the proof is complete. $\blacksquare$

6 FEM analysis of the subdomain problems

For $1 \leq s < \frac{3}{2}$, let

$$\mathcal{Z}_\ell^s := H_{\partial\Omega}^1(\Omega_\ell) \cap H_k^s(\Omega_\ell), \quad \|u\|_{\mathcal{Z}_\ell^s} := \|u\|_{H_k^s(\Omega_\ell)},$$

with $\mathcal{Z}_\ell^0 = L^2$ and $\mathcal{Z}_\ell^s$ for $s \in (0, 1)$ defined by interpolation. We then set $\mathcal{Z}_\ell^s := (\mathcal{Z}_\ell^{-s})^*$ for $-\frac{3}{2} < s < 0$.

Lemma 6.1 (Solution operator bounds in the absorbing region) *Let $c > 0$ and $\{D_\alpha\}_{\alpha \in \mathbb{N}^d}$ and $-\frac{1}{2} < s < \frac{1}{2}$. Then there exists $C > 0$ such that for all $\varphi \in \overline{C}^\infty(\Omega_\ell)$ with $\text{supp} \varphi \subset U_\ell$ and*

$$\delta(\varphi, \{D_\alpha\}, 2) \geq c\delta_\ell,$$

we have

$$\|P_\ell^{-1} \varphi\|_{\mathcal{Z}_\ell^{s-1} \rightarrow \mathcal{Z}_\ell^{s+1}} + \|\varphi P_\ell^{-1}\|_{\mathcal{Z}_\ell^{s-1} \rightarrow \mathcal{Z}_\ell^{s+1}} \leq C. \quad (6.1)$$

Moreover for all $M > 0$, there is $C > 0$ such that for all $\varphi_2 \in \overline{C}^\infty(\Omega_\ell)$ with

$$d(\text{supp} \varphi, \text{supp} \varphi_2) \geq c\delta_\ell, \quad \delta(\varphi_2, \{D_\alpha\}, 2) \geq c\delta_\ell,$$

and

$$\|\varphi_2 P_\ell^{-1} \varphi\|_{\mathcal{Z}_\ell^{s-1} \rightarrow \mathcal{Z}_\ell^{s+1}} + \|\varphi P_\ell^{-1} \varphi_2\|_{\mathcal{Z}_\ell^{s-1} \rightarrow \mathcal{Z}_\ell^{s+1}} \leq C \langle k\delta_\ell \rangle^{-M}. \quad (6.2)$$

Proof. To prove Lemma 6.1, we set $y := \delta_\ell^{-1} x$, rescaling the subdomain problem on Ω_ℓ to one with scale ~ 1 and φ to a cutoff with scale ~ 1 . With $\tilde{k} = k\delta_\ell$, The operator P_ℓ then becomes

$$\tilde{P}_\ell = -\tilde{k}^{-2} \text{div}_y (A_\ell(y\delta_\ell) \nabla_y u) - i\tilde{k}^{-1} b_\ell(y\delta_\ell) \cdot \nabla_y - n_\ell(y\delta_\ell)$$

Since there is $C > 0$ such that

$$C^{-1} \|u\|_{H_k^s(\Omega_\ell)} \leq \delta_\ell^{d/2} \|u\|_{H_k^s(\delta_\ell^{-1} \Omega_\ell)} \leq C \|u\|_{H_k^s(\Omega_\ell)},$$

it is enough to prove estimates in $\tilde{\mathcal{Z}}_\ell^s$ spaces; i.e. rescaled $\mathcal{Z}_\ell^s$ space with the H_k^s norm. In the proof of this lemma, we will now abuse notation by writing Ω_ℓ for $\delta_\ell^{-1} \Omega_\ell$ and U_ℓ for $\delta_\ell^{-1} U_\ell$.

By Lemma A.1, for all $u \in H_{\partial\Omega}^1(\Omega_\ell)$ with $\text{supp } u \subset \delta_\ell^{-1}U_\ell$,

$$\|u\|_{\tilde{Z}_\ell^1} \leq C\|\tilde{P}_\ell u\|_{\tilde{Z}_\ell^{-1}}. \quad (6.3)$$

Then, for $\tilde{\varphi} \in \overline{C^\infty}(\Omega_\ell)$ with $\text{supp } \tilde{\varphi} \subset \overline{U_\ell}$ and $\text{supp}(1 - \tilde{\varphi}) \cap \text{supp } \nabla \varphi = \emptyset$, and $\text{supp } \tilde{\varphi} \cap \partial\Omega_\ell \setminus \partial\Omega = \emptyset$,

$$\tilde{P}_\ell \varphi u = [\tilde{P}_\ell, \varphi] \tilde{\varphi} u + \varphi \tilde{P}_\ell u$$

and hence, using (6.3)

$$\|\varphi u\|_{\tilde{Z}_\ell^1} \leq C\tilde{k}^{-1}\|\tilde{\varphi} u\|_{\tilde{Z}_\ell^1} + C\|\varphi \tilde{P}_\ell u\|_{\tilde{Z}_\ell^{-1}}$$

Iterating this argument, and letting $\varphi_+ \in \overline{C^\infty}(U_\ell)$ with $\text{supp}(1 - \varphi_+) \cap \text{supp } \varphi = \emptyset$ and $\text{supp } \varphi_+ \cap \text{supp } \varphi_1 = \emptyset$ we obtain

$$\|\varphi u\|_{\tilde{Z}_\ell^1} \leq C\tilde{k}^{-N}\|\varphi_+ u\|_{\tilde{Z}_\ell^1} + C\|\varphi_+ \tilde{P}_\ell u\|_{\tilde{Z}_\ell^{-1}},$$

and hence

$$\|\varphi \tilde{P}_\ell^{-1}\|_{\tilde{Z}_\ell^{-1} \rightarrow \tilde{Z}_\ell^1} \leq C(1 + \tilde{k}^{-N}\rho_\ell) \leq C, \quad (6.4)$$

and

$$\|\varphi \tilde{P}_\ell^{-1} \varphi_1\|_{\tilde{Z}_\ell^{-1} \rightarrow \tilde{Z}_\ell^1} \leq C\tilde{k}^{-N}\rho_\ell. \quad (6.5)$$

Now, by elliptic regularity on a Lipschitz domain (see, e.g., [39, Theorem 3], [32]), for $0 \leq s < \frac{1}{2}$,

$$\|\varphi \tilde{P}_\ell^{-1}\|_{\tilde{Z}_\ell^{s-1} \rightarrow \tilde{Z}_\ell^{s+1}} \leq C(1 + \|\varphi \tilde{P}_\ell^{-1}\|_{\tilde{Z}_\ell^{-1} \rightarrow \tilde{Z}_\ell^1}) \leq C. \quad (6.6)$$

and

$$\|\varphi \tilde{P}_\ell^{-1} \varphi_1\|_{\tilde{Z}_\ell^{s-1} \rightarrow \tilde{Z}_\ell^{s+1}} \leq C(\|\varphi \tilde{P}_\ell^{-1} \varphi_1\|_{\tilde{Z}_\ell^{-1} \rightarrow \tilde{Z}_\ell^1}) \leq C\tilde{k}^{-N}. \quad (6.7)$$

Now, we show that

$$\|\tilde{P}_\ell^{-1} \varphi\|_{\tilde{Z}_\ell^{s-1} \rightarrow \tilde{Z}_\ell^{s+1}} \leq C; \quad (6.8)$$

i.e., the result (6.1) for $0 \leq s < 1/2$. Indeed, observe that for φ as above, and $\tilde{\varphi} \in \overline{C^\infty}(\Omega_\ell)$ with $\text{supp } \tilde{\varphi} \subset \overline{U_\ell}$ and $\text{supp}(1 - \tilde{\varphi}) \cap \text{supp } \nabla \varphi = \emptyset$, and $\text{supp } \tilde{\varphi} \cap \partial\Omega_\ell \setminus \partial\Omega = \emptyset$,

$$\tilde{P}_\ell^{-1} \varphi = \varphi \tilde{P}_\ell^{-1} + [\tilde{P}_\ell^{-1}, \varphi] = \varphi \tilde{P}_\ell^{-1} - \tilde{P}_\ell^{-1} \tilde{\varphi} [\tilde{P}_\ell, \varphi] \tilde{\varphi} \tilde{P}_\ell^{-1}$$

Hence, by (6.4),

$$\|\tilde{P}_\ell^{-1} \varphi\|_{\tilde{Z}_\ell^{-1} \rightarrow \tilde{Z}_\ell^1} \leq C + C\|\tilde{P}_\ell^{-1} \tilde{\varphi}\|_{\tilde{Z}_\ell^{-1} \rightarrow \tilde{Z}_\ell^1} \tilde{k}^{-1}.$$

By iteration and the polynomial bound on ρ_ℓ ,

$$\|\tilde{P}_\ell^{-1} \varphi\|_{\tilde{Z}_\ell^{-1} \rightarrow \tilde{Z}_\ell^1} \leq C,$$

and hence (6.8) holds by elliptic regularity. The fact that

$$\|\varphi_1 \tilde{P}_\ell^{-1} \varphi\|_{\tilde{Z}_\ell^{-1} \rightarrow \tilde{Z}_\ell^1} \leq C\tilde{k}^{-N}$$

follows in the same way.

The same arguments applied to $\tilde{P}_\ell^*$ imply that for $0 \leq s < \frac{1}{2}$

$$\begin{aligned} \|\varphi(\tilde{P}_\ell^*)^{-1}\|_{\tilde{Z}_\ell^{s-1} \rightarrow \tilde{Z}_\ell^{s+1}} + \|(\tilde{P}_\ell^*)^{-1} \varphi\|_{\tilde{Z}_\ell^{s-1} \rightarrow \tilde{Z}_\ell^{s+1}} &\leq C, \\ \|\varphi(\tilde{P}_\ell^*)^{-1} \varphi_1\|_{\tilde{Z}_\ell^{s-1} \rightarrow \tilde{Z}_\ell^{s+1}} + \|\varphi_1(\tilde{P}_\ell^*)^{-1} \varphi\|_{\tilde{Z}_\ell^{s-1} \rightarrow \tilde{Z}_\ell^{s+1}} &\leq C\tilde{k}^{-N}. \end{aligned} \quad (6.9)$$

and hence the result (6.1) for $-1/2 < s < 0$ follows from (6.9) and duality. $\blacksquare$

Lemma 6.2 (Solution operator bounds at high frequency) *With Φ as in (4.2), there exists $C > 0$ such that*

$$\|(I - \Psi(\mathcal{P}_\ell))P_\ell^{-1}\|_{\mathcal{Z}_\ell^* \rightarrow \mathcal{Z}_\ell} \leq C(\rho_\ell(k\delta_\ell)^{-1} + 1). \quad (6.10)$$

Proof. By (4.2),

$$\begin{aligned}\Re\langle P_\ell(I - \Psi(\mathcal{P}_\ell))u, (I - \Psi(\mathcal{P}_\ell))u \rangle &= \langle \mathcal{P}_\ell(I - \Psi(\mathcal{P}_\ell)), (I - \Psi(\mathcal{P}_\ell))u \rangle \\ &= \langle (\mathcal{P}_\ell + \Psi_0(\Psi))(I - \Psi(\mathcal{P}_\ell)), (I - \Psi(\mathcal{P}_\ell))u \rangle \geq c\|(I - \Psi(\mathcal{P}_\ell))u\|_{\mathcal{Z}}^2,\end{aligned}$$

so that

$$\|(I - \Psi(\mathcal{P}_\ell))u\|_{\mathcal{Z}_\ell} \leq C\|P_\ell(I - \Psi(\mathcal{P}_\ell))u\|_{\mathcal{Z}_\ell^*}.$$

Thus, since

$$P_\ell(I - \Psi(\mathcal{P}_\ell))P_\ell^{-1} = -[P_\ell, \Psi(\mathcal{P}_\ell)]P_\ell^{-1} + (I - \Psi(\mathcal{P}_\ell)),$$

we have that

$$\begin{aligned}\|(I - \Psi(\mathcal{P}_\ell))P_\ell^{-1}\|_{\mathcal{Z}_\ell^* \rightarrow \mathcal{Z}_\ell} &\leq \|[P_\ell, \Psi(\mathcal{P}_\ell)]P_\ell^{-1}\|_{\mathcal{Z}_\ell^* \rightarrow \mathcal{Z}_\ell^*} + \|(I - \Psi(\mathcal{P}_\ell))\|_{\mathcal{Z}_\ell^* \rightarrow \mathcal{Z}_\ell^*} \\ &\leq \|[P_\ell, \Psi(\mathcal{P}_\ell)]P_\ell^{-1}\|_{\mathcal{Z}_\ell^* \rightarrow \mathcal{Z}_\ell^*} + C.\end{aligned}\tag{6.11}$$

Now, by (2.13) there exist $\varphi_{\text{in}} + \varphi_{\text{abs}} + \varphi_{\text{out}} = 1$ on Ω_ℓ such that

$$\text{supp}\varphi_{\text{abs}} \subset U_\ell, \quad \text{supp}\varphi_{\text{in}} \cap (\partial\Omega_\ell \cup \Gamma_p) = \emptyset, \quad \text{supp}\varphi_{\text{out}} \cap (\text{supp}\Im e^{i\theta_2} A_\ell \cup \text{supp}\Im e^{i\theta_2} b_\ell) = \emptyset,\tag{6.12}$$

$$\nu \cdot A_\ell \nabla \varphi_{\text{out}}|_{\partial\Omega_\ell} = 0, \quad \nu \cdot (A_+ \nabla(\varphi_{\text{out}})_+ - A_- \nabla(\varphi_{\text{out}})_-)|_{\Gamma_p} = 0\tag{6.13}$$

(i.e., φ_{out} can touch the boundary/interfaces only when $\Im e^{i\theta_2} A_\ell$ and $\Im e^{i\theta_2} b_\ell$ are zero). Thus

$$[P_\ell, \Psi(\mathcal{P}_\ell)]P_\ell^{-1} = [P_\ell, \Psi(\mathcal{P}_\ell)]\varphi_{\text{abs}}P_\ell^{-1} + [P_\ell, \Psi(\mathcal{P}_\ell)]\varphi_{\text{in}}P_\ell^{-1} + [P_\ell, \Psi(\mathcal{P}_\ell)]\varphi_{\text{out}}P_\ell^{-1}$$

By Lemma 6.1 and the facts that $\Psi(\mathcal{P}_\ell) : \mathcal{Z}_\ell \rightarrow \mathcal{Z}_\ell$, $\Psi(\mathcal{P}_\ell) : \mathcal{Z}_\ell^* \rightarrow \mathcal{Z}_\ell^*$ and $P_\ell : \mathcal{Z}_\ell \rightarrow \mathcal{Z}_\ell^*$ are all uniformly bounded,

$$\|[P_\ell, \Psi(\mathcal{P}_\ell)]\varphi_{\text{abs}}P_\ell^{-1}\|_{\mathcal{Z}_\ell^* \rightarrow \mathcal{Z}_\ell^*} \leq C\|[P_\ell, \Psi(\mathcal{P}_\ell)]\|_{\mathcal{Z}_\ell \rightarrow \mathcal{Z}_\ell^*} \leq C.$$

Next, write

$$\begin{aligned}[P_\ell, \Psi(\mathcal{P}_\ell)]\varphi_{\text{in}} &= [P_\ell\varphi_{\text{in}}, \Psi(\mathcal{P}_\ell)] + P_\ell[\Psi(\mathcal{P}_\ell), \varphi_{\text{in}}], \\ [P_\ell, \Psi(\mathcal{P}_\ell)]\varphi_{\text{out}} &= [P_\ell\varphi_{\text{out}}, \Psi(\mathcal{P}_\ell)] + P_\ell[\Psi(\mathcal{P}_\ell), \varphi_{\text{out}}].\end{aligned}\tag{6.14}$$

By the Helffer–Sjöstrand formula (see, e.g., [45, Theorem 14.8]),

$$\Psi(\mathcal{P}_\ell) = \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\Psi}(z) (\mathcal{P}_\ell - z)^{-1} dm_{\mathbb{C}}(z),$$

where $\tilde{\Psi}(z)$ is an almost-analytic extension of Ψ (see, e.g., [45, Theorem 3.6]),

$$[A, \Psi(\mathcal{P}_\ell)] = \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\Psi}(z) [A, (\mathcal{P}_\ell - z)^{-1}] dm_{\mathbb{C}}(z) = \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\Psi}(z) (\mathcal{P}_\ell - z)^{-1} [\mathcal{P}_\ell, A] (\mathcal{P}_\ell - z)^{-1} dm_{\mathbb{C}}(z).$$

Therefore, to show that

$$\|[A, \Psi(\mathcal{P}_\ell)]\|_{\mathcal{Z}_\ell \rightarrow \mathcal{Z}_\ell} \leq C(k\delta_\ell)^{-1}, \quad \|[B, \Psi(\mathcal{P}_\ell)]\|_{\mathcal{Z}_\ell \rightarrow \mathcal{Z}_\ell^*} \leq C(k\delta_\ell)^{-1},$$

it is enough to show that

$$\|[A, \mathcal{P}_\ell]\|_{\mathcal{D}_\ell^3 \rightarrow \mathcal{Z}_\ell^*} \leq C(k\delta_\ell)^{-1}, \quad \|[B, \mathcal{P}_\ell]\|_{\mathcal{D}_\ell^3 \rightarrow \mathcal{D}_\ell^{-3}} \leq C(k\delta_\ell)^{-1},\tag{6.15}$$

where, letting $M > 0$ large enough so that $(\mathcal{P}_\ell + M) > c > 0$,

$$\mathcal{D}_\ell^s := (\mathcal{P}_\ell + M)^{-s/2} L^2.$$

(Note that $\mathcal{D}_\ell^1 = \mathcal{Z}$ and $\mathcal{D}_\ell^{-1} = \mathcal{Z}^*$ and recall that $(\mathcal{P} - z)^{-1} : \mathcal{D}_\ell^s \rightarrow \mathcal{D}_\ell^{s+2}$ for $z \in \mathbb{C} \setminus \mathbb{R}$; see, e.g., [2, Proposition 5.21].) We now prove (6.15) with $A = \varphi_{\text{in/out}}$ and $B = P_\ell\varphi_{\text{in/out}}$; the result then follows from combining (6.15) with (6.14) and (6.11).

For $A = \varphi_{\text{in/out}}$, observe that $A : \mathcal{Z} \rightarrow \mathcal{Z}$ and hence for $u, v \in \mathcal{Z}$,

$$|\langle [A, \mathcal{P}_\ell]u, v \rangle| = |(\Re a_\ell)(Au, v) - (\Re a_\ell)(u, Av)| \leq C\tilde{k}^{-1}\|u\|_{\mathcal{Z}}\|v\|_{\mathcal{Z}},$$

hence $\|[A, \mathcal{P}_\ell]\|_{\mathcal{Z}_\ell \rightarrow \mathcal{Z}_\ell^*} \leq C\tilde{k}^{-1}$ as required.

The computation for B is more complicated. For $B = P_\ell \varphi_{\text{in}}$, the operators are supported away from all boundaries; hence the claim follows from the fact that P_ℓ and $\mathcal{P}_\ell$ are semiclassical differential operators with coefficients having derivatives of size δ_ℓ^{-1} .

For $B = P_\ell \varphi_{\text{out}}$, observe that since $\varphi_{\text{out}} : \mathcal{Z}_\ell \rightarrow \mathcal{Z}_\ell$ and $\varphi_{\text{out}} : \mathcal{D}_\ell^3 \rightarrow \mathcal{D}_\ell^3$ (by the conditions (6.13)), the support condition (6.12) on φ_{out} implies that $P_\ell \varphi_{\text{out}} : \mathcal{D}_\ell^3 \rightarrow \mathcal{Z}$.

Furthermore, the support condition on φ_{out} combined with the assumption (2.13) implies that $P = (1 - i \tan \theta_2) \mathcal{P}_\ell + V_\ell$, where $V_\ell := -i \sec \theta_2 \Im n_\ell$, on $\text{supp} \varphi_{\text{out}}$. (Recall that $\cos \theta_2 \neq 0$ since (2.5) holds and we have assumed, without loss of generality, that $\theta_0 = 0$.) Finally, $V_\ell : \mathcal{Z}_\ell \rightarrow \mathcal{Z}_\ell$ and hence

$$\begin{aligned} \langle [P_\ell \varphi_{\text{out}}, \mathcal{P}_\ell]u, v \rangle &= \langle P_\ell \varphi_{\text{out}} \mathcal{P}_\ell u, v \rangle - \langle \mathcal{P}_\ell P_\ell \varphi_{\text{out}} u, v \rangle \\ &= a_\ell(\varphi_{\text{out}} \mathcal{P}_\ell u, v) - (\Re a_\ell)(P_\ell \varphi_{\text{out}} u, v) \\ &= (1 - i \tan \theta_2) \Re a_\ell(\varphi_{\text{out}} \mathcal{P}_\ell u, v) - (\Re a_\ell)((1 - i \tan \theta_2) \mathcal{P}_\ell + V_\ell) \varphi_{\text{out}} u, v \\ &\quad + \langle V_\ell \varphi_{\text{out}} \mathcal{P}_\ell u, v \rangle \\ &= (1 - i \tan \theta_2) \Re a_\ell([\varphi_{\text{out}}, \mathcal{P}_\ell]u, v) - (\Re a_\ell)(V_\ell \varphi_{\text{out}} u, v) + (\Re a_\ell)(u, \varphi_{\text{out}} \overline{V_\ell} v) \\ &= (1 - i \tan \theta_2) \langle \mathcal{P}_\ell [\varphi_{\text{out}}, \mathcal{P}_\ell]u, v \rangle - (\Re a_\ell)(V_\ell \varphi_{\text{out}} u, v) + (\Re a_\ell)(u, \varphi_{\text{out}} \overline{V_\ell} v). \end{aligned}$$

Now,

$$\|\mathcal{P}_\ell [\varphi_{\text{out}}, \mathcal{P}_\ell]\|_{\mathcal{D}_\ell^3 \rightarrow \mathcal{D}_\ell^{-3}} \leq C\|[\varphi_{\text{out}}, \mathcal{P}_\ell]\|_{\mathcal{Z}_\ell \rightarrow \mathcal{Z}_\ell^*} \leq C(k\delta_\ell)^{-1},$$

and

$$|(\Re a_\ell)(V_\ell \varphi_{\text{out}} u, v) - (\Re a_\ell)(u, \varphi_{\text{out}} \overline{V_\ell} v)| \leq C(k\delta_\ell)^{-1}\|u\|_{\mathcal{Z}}\|v\|_{\mathcal{Z}}.$$

Hence,

$$\|[P_\ell \varphi_{\text{out}}, \mathcal{P}_\ell]\|_{\mathcal{D}_\ell^3 \rightarrow \mathcal{D}_\ell^{-3}} \leq C(k\delta_\ell)^{-1};$$

the proof of (6.15), and hence also the result, is therefore complete. $\blacksquare$

Lemma 6.3 (Quasi-optimality of the FEM solution for the subdomain problems) *Let $0 < s < \frac{1}{2}$. Then there is $c > 0$ such that if*

$$\rho_\ell(h_\ell k)^p + (h_\ell k)^s \leq c,$$

then for all $u \in H_{\partial\Omega}^1(\Omega_\ell)$ there is a unique $u_h \in \mathcal{V}_{\ell,h}$ satisfying

$$a_\ell(u_h, w_h) = a_\ell(u, w_h) \quad \text{for all } w_h \in \mathcal{V}_{\ell,h}, \quad (6.16)$$

and

$$\|u - u_h\|_{(H_k^{p-1}(\Omega_\ell \setminus \Gamma_p) \cap H_\partial^1(\Omega_\ell))^*} \leq C((h_\ell k)^p \rho_\ell + (h_\ell k)^s) \inf_{w_h \in \mathcal{V}_{\ell,h}} \|u - w_h\|_{H_k^1(\Omega_\ell)}. \quad (6.17)$$

Proof. The proof follows the Schatz duality argument [40, 41] (i.e., the Aubin–Nitsche lemma combined with a Gårding inequality). Since $\mathcal{V}_{\ell,h}$ is finite dimensional it is sufficient to prove the bound (6.17) under the assumption that u_h exists. Let φ be as in Lemma 6.1 with $\text{supp}(1 - \varphi) \cap \partial\Omega_\ell \setminus \partial\Omega = \emptyset$. Observe first that, for all $w_h = w_h^1 + w_h^2 \in \mathcal{V}_{\ell,h}$,

$$\begin{aligned} |\langle u - u_h, v \rangle| &= |\langle P_\ell(u - u_h), (P_\ell^*)^{-1}v - w_h \rangle| \\ &= |\langle P_\ell(u - u_h), \varphi(P_\ell^*)^{-1}v - w_h^1 \rangle + \langle P_\ell(u - u_h), (1 - \varphi)(P_\ell^*)^{-1}v - w_h^2 \rangle| \\ &\leq C\|u - u_h\|_{H_k^1}((h_\ell k)^s \|v\|_{L^2} + (h_\ell k)^p \rho_\ell \|v\|_{H_k^{p-1}(\Omega_\ell \setminus \Gamma_p)}) \end{aligned}$$

by Lemma 6.1 and the polynomial approximation result (2.26). Hence,

$$\|u - u_h\|_{(H_k^{p-1}(\Omega_\ell \setminus \Gamma_p))^*} \leq C((h_\ell k)^s + (h_\ell k)^p \rho_\ell) \|u - u_h\|_{H_k^1}.$$

Now, by the Gårding inequality satisfied by a_ℓ , Galerkin orthogonality (6.16), and interpolation, for all $\epsilon > 0$ there is $C_\epsilon > 0$ such that for all $w_h \in \mathcal{V}_{\ell,h}$

$$\begin{aligned} \|u - u_h\|_{H_k^1}^2 &\leq C|\langle P_\ell(u - u_h), u - u_h \rangle| + C\|u - u_h\|_{L^2}^2 \\ &\leq C_\epsilon\|u - w_h\|_{H_k^1}^2 + \epsilon\|u - u_h\|_{H_k^1}^2 + \epsilon\|u - u_h\|_{H_k^1}^2 + C_\epsilon\|u - u_h\|_{(H_k^{p-1}(\Omega_\ell \setminus \Gamma_p))^*}^2 \\ &\leq C_\epsilon\|u - w_h\|_{H_k^1}^2 + \left(2\epsilon + C_\epsilon((h_\ell k)^s + (h_\ell k)^p \rho_\ell)^2\right)\|u - u_h\|_{H_k^1}^2. \end{aligned}$$

The result then follows by taking ϵ small and $(h_\ell k)^s + (h_\ell k)^p \rho_\ell$ small. $\blacksquare$

7 Superapproximation and subdomain overlaps

Recall that, by the quasi-interpolation results of [2, Appendix D] (which cover curved meshes; see also, e.g., [15, Theorem 22.6(i)] for analogous results for simplicial meshes), for $\ell = 1, \dots, N$, there exists a bounded linear operator

$$\tilde{\mathcal{I}}_\ell : H_{\partial\Omega}^1(\Omega_\ell) \rightarrow \mathcal{V}_{\ell,h}, \quad \chi_\ell \tilde{\mathcal{I}}_\ell : H_k^1(\Omega_\ell) \rightarrow \mathcal{V}_{\ell,h} \quad (7.1)$$

such that

$$\tilde{\mathcal{I}}_\ell v_{\ell,h} = v_{\ell,h} \text{ for all } v_{\ell,h} \in \mathcal{V}_{\ell,h}, \quad (7.2)$$

$$\chi_\ell \tilde{\mathcal{I}}_\ell (1 - \chi_\ell^{(1)}) = 0, \quad (7.3)$$

and

$$\sum_{T \cap \text{supp} \chi_\ell \neq \emptyset} \|(I - \tilde{\mathcal{I}}_\ell)u\|_{H_k^{p+1}(T)}^2 \leq C\|u\|_{H_k^{p+1}(\Omega_\ell)}^2. \quad (7.4)$$

The following lemma, which appears new and may be of independent interest, shows that if a function is H_k^{p+1} close to a finite element function, then one still has a type of superapproximation.

Lemma 7.1 *There exists $C > 0$ such that for all $v_h \in \mathcal{V}_h$ and $w \in H_k^1(\Omega_\ell)$ and $\chi_\ell^{(1)} w \in H_k^{p+1}(\Omega_\ell)$,*

$$\|(I - \mathcal{I}_\ell)\chi_\ell v_h\|_{H_k^1(\Omega_\ell)} \leq C\left((h_\ell k)^p \|\chi_\ell^{(1)} w\|_{H_k^{p+1}} + \mu_\ell \|\chi_\ell^{(1)}(v_h - w)\|_{H_k^1(\Omega_\ell)}\right) \quad (7.5)$$

(where $\mathcal{I}_\ell$ and μ_ℓ are as in (2.26)). In particular, there is $C > 0$ such that for all $v_h \in \mathcal{V}_h$,

$$\|(I - \mathcal{I}_\ell)\chi_\ell v_h\|_{H_k^1(\Omega_\ell)} \leq C\left((h_\ell k)^p + (H_c k)^p \mu_\ell\right)\|v_h\|_{X(\Omega_\ell)}. \quad (7.6)$$

Proof. By (7.2) and (7.3),

$$\begin{aligned} (I - \mathcal{I}_\ell)\chi_\ell v_h &= (I - \mathcal{I}_\ell)\chi_\ell w + (I - \mathcal{I}_\ell)\chi_\ell(v_h - w) \\ &= (I - \mathcal{I}_\ell)\chi_\ell w + (I - \mathcal{I}_\ell)\chi_\ell(v_h - \tilde{\mathcal{I}}_\ell w + \tilde{\mathcal{I}}_\ell w - w) \\ &= (I - \mathcal{I}_\ell)\chi_\ell w + (I - \mathcal{I}_\ell)\chi_\ell \tilde{\mathcal{I}}_\ell(v_h - w) - (I - \mathcal{I}_\ell)\chi_\ell(I - \tilde{\mathcal{I}}_\ell)w \\ &= (I - \mathcal{I}_\ell)\chi_\ell w + (I - \mathcal{I}_\ell)\chi_\ell \tilde{\mathcal{I}}_\ell \chi_\ell^{(1)}(v_h - w) - (I - \mathcal{I}_\ell)\chi_\ell(I - \tilde{\mathcal{I}}_\ell)\chi_\ell^{(1)}w. \end{aligned}$$

Now, by (2.26),

$$\|(I - \mathcal{I}_\ell)\chi_\ell w\|_{H_k^1(\Omega_\ell)} \leq C(h_\ell k)^p \|\chi_\ell w\|_{H_k^{p+1}(\Omega_\ell)} \leq C(h_\ell k)^p \|\chi_\ell^{(1)} w\|_{H_k^{p+1}(\Omega_\ell)}.$$

By (2.26), (7.1), and (4.4),

$$\|(I - \mathcal{I}_\ell)\chi_\ell \tilde{\mathcal{I}}_\ell \chi_\ell^{(1)}(v_h - w)\|_{H_k^1(\Omega_\ell)} \leq \mu_\ell \|\tilde{\mathcal{I}}_\ell \chi_\ell^{(1)}(v_h - w)\|_{H_k^1(\Omega_\ell)} \leq C\mu_\ell \|\chi_\ell^{(1)}(v_h - w)\|_{H_k^1(\Omega_\ell)}.$$

By (2.26) and (7.4),

$$\|(I - \mathcal{I}_\ell)\chi_\ell(I - \tilde{\mathcal{I}}_\ell)w\|_{H_k^1}^2 \leq C(h_\ell k)^{2p} \sum_{T \cap \text{supp} \chi_\ell \neq \emptyset} \|(I - \tilde{\mathcal{I}}_\ell)w\|_{H_k^{p+1}(T)}^2 \leq C(h_\ell k)^{2p} \|\chi_\ell^{(1)} w\|_{H_k^{p+1}(\Omega_\ell)}^2.$$

This completes the proof of (7.5).

To prove (7.6), we take $w = \Psi(\mathcal{P})v_h$ in (7.5) and use the definition of $\|\cdot\|_{X(\Omega_\ell)}$ (see (4.4)) to obtain

$$\begin{aligned} \|(I - \mathcal{I}_\ell)\chi_\ell v_h\|_{H_k^1(\Omega_\ell)} &\leq C\left((h_\ell k)^p \|\Psi(\mathcal{P})v_h\|_{H_k^{p+1}(\Omega_\ell)} + \mu_\ell(H_c k)^p (H_c k)^{-p} \|(I - \Psi(\mathcal{P}))v_h\|_{H_k^1(\Omega_\ell)}\right) \\ &\leq C((h_\ell k)^p + \mu_\ell(H_c k)^p) \|v_h\|_{X(\Omega_\ell)}. \end{aligned}$$

■

7.1 Cutoff functions associated to the overlapping subdomains

Given χ_ℓ , let $\chi_\ell^{(1)} \in \overline{C^\infty}(\Omega_\ell \setminus \Gamma_p)$ be such that

$$\nu \cdot A \nabla \chi_\ell^{(1)} = 0 \text{ on } \Gamma_{\text{hard}} \quad \text{and} \quad \nu \cdot (A_+ \nabla (\chi_\ell^{(1)})_+ - A_- \nabla (\chi_\ell^{(1)})_-) = 0 \text{ on } \Gamma_p,$$

(compare to (2.21)),

$$\delta_\ell \leq \delta(\chi_\ell^{(1)}, \{D_\alpha\}, p+1), \quad (7.7)$$

$$\chi_\ell^{(1)} P_\ell = \chi_\ell^{(1)} P, \quad d(\chi_\ell, \text{supp}(1 - \chi_\ell^{(1)})) \geq c\delta_\ell, \quad \text{and} \quad d(\text{supp}\chi_\ell^{(1)}, \Omega_\ell \setminus W_\ell) \geq c\delta_\ell;$$

we abbreviate the penultimate condition to $\chi_\ell \prec \chi_\ell^{(1)}$.

We then define three sets of even larger cut-offs, supported in Ω_ℓ , but such that the “transition” from zero to one happens in the absorbing region. That is, let $\chi_\ell^{(2)}, \chi_\ell^{(3)}, \chi_\ell^{(4)} \in \overline{C^\infty}(\Omega_\ell \setminus \Gamma_p; [0, 1])$ be such that $\text{supp}\chi_\ell^{(4)} \subset \Omega_\ell \cup \partial\Omega$,

$$\chi_\ell^{(1)} \prec \chi_\ell^{(2)} \prec \chi_\ell^{(3)} \prec \chi_\ell^{(4)}, \quad d(\text{supp}(1 - \chi_\ell^{(3)}), \Omega_\ell \setminus U_\ell) \geq c\delta_\ell, \quad \text{and} \quad d(\chi_\ell^{(2)}, \Omega_\ell \setminus W_\ell) \geq c\delta_\ell, \quad (7.8)$$

and satisfying

$$\delta_\ell \leq \delta(\chi_\ell^{(j)}, \{D_\alpha\}, p+1), \quad j = 2, 3, 4.$$

We emphasise that only χ_ℓ appears in the definition of the preconditioner (8.3); the other cutoffs are only used in the analysis.

7.2 Approximate orthogonality of spaces on subdomains

Lemma 7.2 *For all $u \in H^1(\Omega)$,*

$$\sum_{\ell=1}^N \|u\|_{X(\Omega_\ell)}^2 \leq \tilde{\Lambda}(\{\Omega_\ell\}) \|u\|_{X(\Omega)}^2. \quad (7.9)$$

To prove Lemma 7.2, we recall the following result.

Lemma 7.3 (Consequences of the definition of $\tilde{\Lambda}(\{\Omega_\ell\})$ (2.16) [25, Lemma 4.2], [20, Lemma 3.3]) *Let $r \in \{0, 1, \dots\}$ and $v \in H^r(\Omega)$. Then*

$$\sum_{\ell=1}^N \|v\|_{H_k^r(\Omega_\ell \setminus \Gamma_p)}^2 \leq \tilde{\Lambda}(\{\Omega_\ell\}) \|v\|_{H_k^r(\Omega \setminus \Gamma_p)}^2. \quad (7.10)$$

Furthermore, given $v_\ell \in \mathcal{V}_\ell$,

$$\left\| \sum_{\ell=1}^N v_\ell \right\|_{H_k^r(\Omega \setminus \Gamma_p)}^2 \leq 2\tilde{\Lambda}(\{\Omega_\ell\}) \sum_{\ell=1}^N \|v_\ell\|_{H_k^r(\Omega_\ell \setminus \Gamma_p)}^2. \quad (7.11)$$

Proof of Lemma 7.2. By (7.10) with $w = (I - \Psi)u$,

$$\sum_{\ell=1}^N (H_c k)^{-2p} \|(I - \Psi(\mathcal{P}))u\|_{H_k^1(\Omega_\ell)}^2 \leq \tilde{\Lambda} (H_c k)^{-2p} \|(I - \Psi(\mathcal{P}))u\|_{H_k^1(\Omega)}^2$$

Similarly,

$$\sum_{\ell=1}^N \|\Psi(\mathcal{P})u\|_{H_k^{p+1}(\Omega_\ell \setminus \Gamma_p)}^2 \leq \tilde{\Lambda} \|\Psi(\mathcal{P})u\|_{H_k^{p+1}(\Omega \setminus \Gamma_p)}^2,$$

and then (7.9) follows from the combination of the last two displayed inequalities. $\blacksquare$

We also need an analogue of Lemma 7.3 in negative Sobolev spaces.

Lemma 7.4 *Let $1 \leq m \leq p+1$, $\psi_\ell \in \overline{C^\infty}(\Omega \setminus \Gamma_p) \cap \overline{C^0(\Omega)}$ with $\text{supp} \psi_\ell \subset \Omega_\ell$,*

$$\delta(\psi_\ell, \{D_\alpha\}, p+1) \geq \delta_\ell, \quad (7.12)$$

Then for all $w \in \mathcal{W}^{-m}$,

$$\sum_{\ell=1}^N \|\psi_\ell w\|_{\mathcal{W}^{-m}}^2 \leq C \tilde{\Lambda} \|w\|_{\mathcal{W}^{-m}}^2.$$

Proof. By (7.12), there is $C > 0$ such that for all ℓ ,

$$\begin{aligned} \|\psi_\ell w\|_{\mathcal{W}^{-m}} &\leq C \|w\|_{(\dot{H}_k^m(\Omega_\ell \setminus \Gamma_p) \cap H_k^1(\Omega_\ell))^*}, \\ \text{where } \dot{H}_k^m(\Omega_\ell \setminus \Gamma_p) &:= \{u \in H_k^m(\Omega \setminus \Gamma_p) : \text{supp} u \subset \overline{\Omega_\ell}\} \end{aligned} \quad (7.13)$$

Let $\varphi_\ell \in \overline{C^\infty}(\Omega_\ell \setminus \Gamma_p) \cap H_k^1(\Omega_\ell)$ with $\text{supp} \varphi_\ell \cap \partial\Omega_\ell \setminus \partial\Omega = \emptyset$, $\|\varphi_\ell\|_{\dot{H}_k^m(\Omega_\ell \setminus \Gamma_p)} = 1$, define

$$\theta_\ell := \|w\|_{(\dot{H}_k^m(\Omega_\ell \setminus \Gamma_p) \cap H_k^1(\Omega_\ell))^*},$$

and set

$$\varphi := \sum_{\ell} \theta_\ell \varphi_\ell.$$

Then, by Lemma 7.3

$$\|\varphi\|_{\mathcal{Y}^m}^2 \leq \tilde{\Lambda} \sum_{\ell=1}^N \theta_\ell^2,$$

and hence

$$\|w\|_{\mathcal{W}^{-m}}^2 \geq \frac{|\langle w, \varphi \rangle|^2}{\|\varphi\|_{\mathcal{Y}^m}^2} \geq \tilde{\Lambda}^{-1} \frac{\left| \sum_{\ell} \theta_\ell \langle w, \varphi_\ell \rangle \right|^2}{\sum_{\ell=1}^N \theta_\ell^2} = \tilde{\Lambda}^{-1} \frac{\left| \sum_{\ell} \|w\|_{(\dot{H}_k^m(\Omega_\ell \setminus \Gamma_p) \cap H_k^1(\Omega_\ell))^*} \langle w, \varphi_\ell \rangle \right|^2}{\sum_{\ell=1}^N \|w\|_{(\dot{H}_k^m(\Omega_\ell \setminus \Gamma_p) \cap H_k^1(\Omega_\ell))^*}^2}.$$

Taking the supremum over $\{\varphi_\ell\}_{\ell=1}^N$ then implies

$$\sum_{\ell} \|w\|_{(\dot{H}_k^m(\Omega_\ell \setminus \Gamma_p) \cap H_k^1(\Omega_\ell))^*}^2 \leq \tilde{\Lambda} \|w\|_{\mathcal{W}^{-m}}^2,$$

and hence the lemma follows from (7.13) $\blacksquare$

8 Proof of Theorem 2.7

We prove the result for left preconditioning. Since all the assumptions are invariant under taking the adjoint, the result for right preconditioning follows (see, e.g., [20, Equation 1.8]).

8.1 Definition of the preconditioner

Recall from (2.32) that $Q_{H_c}^0 : \mathcal{V}_h \rightarrow \mathcal{V}_{H_c}$ is defined by

$$a(Q_{H_c}^0 u_h, v_{H_c}) = a(u_h, v_{H_c}) \quad \text{for all } v_{H_c} \in \mathcal{V}_{H_c}, \quad (8.1)$$

and recall from (2.42) that $Q_h^\ell : \mathcal{V}_h \rightarrow \mathcal{V}_{\ell,h}$ is defined by

$$a_\ell(Q_h^\ell u_h, v_{\ell,h}) = a(u_h, E_\ell v_{\ell,h}) \quad \text{for all } v_{\ell,h} \in \mathcal{V}_{\ell,h} \quad (8.2)$$

where $E_\ell : H_{\partial\Omega}^1(\Omega_\ell) \rightarrow \mathcal{Z}$ denotes extension by zero. The preconditioner B_L^{-1} defined by (2.14) then satisfies $B_L^{-1}A = Q\iota$, where $Q : \mathcal{Z} \rightarrow \mathcal{V}_h$ is given by

$$Q := Q_{H_c}^0 + \sum_{\ell=1}^N \mathcal{I}_\ell(\chi_\ell Q_h^\ell (I - Q_{H_c}^0)). \quad (8.3)$$

8.2 Subdomain problems at the continuous level

Recall that $P : \mathcal{Z} \rightarrow \mathcal{Z}^*$ is the operator defined by a and $P_\ell : \mathcal{V}_\ell \rightarrow \mathcal{V}_\ell^*$ that defined by a_ℓ . The subdomain problems defined by (8.2) can then be written as

$$\langle P_\ell Q_h^\ell u_h, v_{\ell,h} \rangle = \langle Pu_h, E_\ell v_{\ell,h} \rangle \quad \text{for all } v_{\ell,h} \in \mathcal{V}_{\ell,h}. \quad (8.4)$$

Recall from (2.43) that the “continuous-level counterparts” of $Q_h^\ell : \mathcal{V}_h \rightarrow \mathcal{V}_{\ell,h}$, $Q^\ell : \mathcal{V}_h \rightarrow \mathcal{V}_\ell$, are defined by $Q^\ell := P_\ell^{-1} E_\ell^* P$.

8.3 Start of the proof of Theorem 2.7

We first observe that, by Lemma 5.1 together with the definition of Q (8.3),

$$Q\Pi_{H_c} = \Pi_{H_c},$$

where Π_{H_c} denotes the $\mathcal{Z}$ -orthogonal projector onto $\mathcal{V}_{H_c}$. Hence, we may decompose $\mathcal{V}_h = \Pi_{H_c}\mathcal{V}_h \oplus (I - \Pi_{H_c})\mathcal{V}_h$ and, written with respect to this decomposition,

$$(Q - I)\iota = \begin{pmatrix} 0 & \Pi_{H_c}(Q - I)\iota(I - \Pi_{H_c}) \\ 0 & (I - \Pi_{H_c})(Q - I)\iota(I - \Pi_{H_c}) \end{pmatrix}$$

Hence, to prove Theorem 2.7 it is enough to show that

$$\begin{aligned} & \| (I - \Pi_{H_c})(Q - I)\iota \|_{\mathcal{Z} \rightarrow \mathcal{Z}} \\ & \leq C\Lambda \left[\max_\ell \{ \rho_\ell (k\delta_\ell)^{-1} \} \rho(H_c k)^{2p} + \max_\ell \rho_\ell (H_c k)^{2p} + \max_\ell \langle k\delta_\ell \rangle^{-M} (H_c k)^p \right. \\ & \quad \left. + (h/H_c)^p + \max_\ell \mu_\ell + \max_\ell ((h_\ell k)^p \rho_\ell + (h_\ell k)^s) \left(\rho_\ell (h_\ell k)^p + \rho_\ell (k\delta_\ell)^{-1} + 1 \right) \right] =: \epsilon, \\ & \| (Q - I)\iota \|_{\mathcal{Z} \rightarrow \mathcal{Z}} \leq \epsilon + C\Lambda \left(\max_\ell (\rho_\ell (k\delta_\ell)^{-1}) \rho(H_c k)^p + \max_\ell \rho_\ell (H_c k)^p + \max_\ell \langle k\delta_\ell \rangle^{-M} \right). \end{aligned} \quad (8.5)$$

Indeed, once we have (8.5), then

$$\| [(Q - I)\iota]^K u \|_{\mathcal{Z}} \leq C^K \epsilon^{K-1} \| (Q - I)\iota \|_{\mathcal{Z} \rightarrow \mathcal{Z}} \| u \|_{\mathcal{Z}}.$$

Therefore, it remains to prove (8.5). We start by decomposing $Q - I$. Using the facts that $\{\chi_\ell\}_{\ell=1}^N$ is a partition of unity, $\mathcal{R}_\ell^* \chi_\ell \mathcal{R}_\ell = \chi_\ell$, and $\chi_\ell^{(1)} \equiv 1$ on $\text{supp} \chi_\ell$ (so that $\chi_\ell = \chi_\ell \chi_\ell^{(1)}$), for all $u_h \in \mathcal{V}_h$,

$$u_h = \sum_{\ell=1}^N \mathcal{R}_\ell^* \chi_\ell \mathcal{R}_\ell u_h = \sum_{\ell=1}^N \mathcal{R}_\ell^* \left(\mathcal{I}_\ell(\chi_\ell \mathcal{R}_\ell u_h) + (I - \mathcal{I}_\ell) \chi_\ell \mathcal{R}_\ell u_h \right).$$

By (8.3) and the last displayed equation (with $u_h = (I - Q_{H_c}^0)v_h$), for $v_h \in \mathcal{V}_h$,

$$\begin{aligned}
(Q - I)v_h &= -(I - Q_{H_c}^0)v_h + \sum_{\ell=1}^N \mathcal{R}_\ell^* \mathcal{I}_\ell (\chi_\ell Q_h^\ell (I - Q_{H_c}^0)v_h) \\
&= \sum_{\ell=1}^N \mathcal{R}_\ell^* \left[-\mathcal{I}_\ell (\chi_\ell \mathcal{R}_\ell (I - Q_{H_c}^0)v_h) + \mathcal{I}_\ell (\chi_\ell Q_h^\ell (I - Q_{H_c}^0)v_h) \right. \\
&\quad \left. - (I - \mathcal{I}_\ell) \chi_\ell \mathcal{R}_\ell (I - Q_{H_c}^0)v_h \right] \\
&= \sum_{\ell=1}^N \mathcal{R}_\ell^* \left[\mathcal{I}_\ell (\chi_\ell (Q_h^\ell - \mathcal{R}_\ell) (I - Q_{H_c}^0)v_h) - (I - \mathcal{I}_\ell) \chi_\ell \mathcal{R}_\ell (I - Q_{H_c}^0)v_h \right] \\
&= \sum_{\ell=1}^N \mathcal{R}_\ell^* \left[(\chi_\ell (Q_h^\ell - \mathcal{R}_\ell) (I - Q_{H_c}^0)v_h) - (\chi_\ell (Q_h^\ell - Q_h^\ell) (I - Q_{H_c}^0)v_h) \right. \\
&\quad \left. - (I - \mathcal{I}_\ell) (\chi_\ell (Q_h^\ell - \mathcal{R}_\ell) (I - Q_{H_c}^0)v_h) - (I - \mathcal{I}_\ell) \chi_\ell \mathcal{R}_\ell (I - Q_{H_c}^0)v_h \right]. \tag{8.6}
\end{aligned}$$

8.4 Study of the local problems at the continuous level

In this subsection, we prove two results operator $Q^\ell(I - Q_{H_c}^0)$. The first – Lemma 8.1 – splits $Q^\ell(I - Q_{H_c}^0)$ into “low” and “high” frequency components. The second – Lemma 8.2 – studies the difference between $Q^\ell(I - Q_{H_c}^0)$ and $I - Q_{H_c}^0$.

Lemma 8.1 *Given $M > 0$ there exists $C > 0$ such that for all ℓ , $\mathcal{L}_\ell : \mathcal{V}_h \rightarrow H^{p+1}(\Omega_\ell) \cap H_{\partial\Omega}^1(\Omega_\ell)$ and $\mathcal{H}_\ell : \mathcal{V}_h \rightarrow H_{\partial\Omega}^1(\Omega_\ell)$ such that for all $v_h \in \mathcal{V}_h$,*

$$Q^\ell(I - Q_{H_c}^0)v_h = \mathcal{L}_\ell v_h + \mathcal{H}_\ell v_h,$$

with

$$\begin{aligned}
\|\mathcal{L}_\ell v_h\|_{W^{p+1}(\Omega_\ell)} &\leq C \rho_\ell (H_c k)^p \sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \|(I - Q_{H_c}^0)v_h\|_{X(\Omega_j)} \\
\|\mathcal{H}_\ell v_h\|_{H_k^1(\Omega_\ell)} &\leq C (\rho_\ell (k\delta_\ell)^{-1} + 1) (H_c k)^p \sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \|(I - Q_{H_c}^0)v_h\|_{X(\Omega_j)} \tag{8.7}
\end{aligned}$$

Lemma 8.2 *Given $M > 0$ there exists $C > 0$ such that for all ℓ , $\mathcal{A}_\ell : \mathcal{V}_h \rightarrow \mathcal{Z}$ and $\mathcal{B}_\ell : \mathcal{V}_h \rightarrow \mathcal{Z}$ $\chi_\ell^{(1)} \mathcal{A}_\ell : \mathcal{V}_h \rightarrow \mathcal{W}^{p+1}$, $\chi_\ell^{(1)} \mathcal{B}_\ell : \mathcal{V}_h \rightarrow \mathcal{W}^{p+1}$ and for all $v_h \in \mathcal{V}_h$,*

$$(Q^\ell - I)(I - Q_{H_c}^0)v_h = \mathcal{A}_\ell v_h + \mathcal{B}_\ell v_h,$$

with

$$\begin{aligned}
&\|\chi_\ell^{(1)} \mathcal{A}_\ell v_h\|_{\mathcal{W}^{p+1}} \\
&\leq C \rho_\ell \left((k\delta_\ell)^{-1} \|\chi_\ell^{(2)} (I - Q_{H_c}^0)v_h\|_{\mathcal{W}^{1-p}} + (H_c k)^{2p} \sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \|(I - Q_{H_c}^0)v_h\|_{X(\Omega_j)} \right), \tag{8.8}
\end{aligned}$$

$$\|\chi_\ell^{(1)} \mathcal{B}_\ell v_h\|_{\mathcal{W}^{p+1}} \leq C \langle k\delta_\ell \rangle^{-M} (H_c k)^p \sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \|(I - Q_{H_c}^0)v_h\|_{X(\Omega_j)}. \tag{8.9}$$

A key ingredient of the proofs of both Lemmas 8.1 and 8.2 is the following result.

Lemma 8.3 *Given $M > 0$ there exists $C > 0$ such that*

$$\|P(I - Q_{H_c}^0)v_h\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*} \leq C(H_c k)^p \sum_j \langle d(\Omega_j, \Omega_\ell)k \rangle^{-M} \|(I - Q_{H_c}^0)v_h\|_{X(\Omega_j)}, \quad (8.10)$$

and

$$\|\chi_\ell^{(4)} P(I - Q_{H_c}^0)v_h\|_{Y^{-p-1}} \leq C(H_c k)^{2p} \sum_j \langle d(\Omega_j, \Omega_\ell)k \rangle^{-M} \|(I - Q_{H_c}^0)v_h\|_{X(\Omega_j)}, \quad (8.11)$$

Proof. In this proof, we use the fact that $\|u\|_{\mathcal{Z}}$ is equivalent to $\langle (\mathcal{P} + 2)u, u \rangle^{1/2}$. Let $\Pi_{\mathcal{Z}}^c : \mathcal{Z} \rightarrow \mathcal{V}_{H_c}$ denote the $\mathcal{Z}$ orthogonal projection using this equivalent norm. Thus, by Galerkin orthogonality, for all $v_h \in \mathcal{V}_h$,

$$\langle (\Pi_{\mathcal{Z}}^c)^* P(I - Q_{H_c}^0)v_h, w \rangle = 0 \quad \text{for all } w \in \mathcal{Z},$$

so that

$$P(I - Q_{H_c}^0)v_h = (I - (\Pi_{\mathcal{Z}}^c)^*)P(I - Q_{H_c}^0)v_h \quad \text{for all } v_h \in \mathcal{V}_h. \quad (8.12)$$

To prove (8.10), it is therefore sufficient to prove that

$$\|(I - (\Pi_{\mathcal{Z}}^c)^*)Pu\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*} \leq C(H_c k)^p \sum_j \langle d(\Omega_j, \Omega_\ell)k \rangle^{-N} \|u\|_{X(\Omega_j)}, \quad (8.13)$$

and to prove (8.11) it is sufficient to prove

$$\|\chi_\ell^{(4)}(I - (\Pi_{\mathcal{Z}}^c)^*)Pu\|_{Y^{-p-1}} \leq C(H_c k)^{2p} \sum_j \langle d(\Omega_j, \Omega_\ell)k \rangle^{-N} \|u\|_{X(\Omega_j)}. \quad (8.14)$$

Let $\tilde{\chi}_\ell \in C^\infty(\Omega)$ with $\text{supp}(1 - \tilde{\chi}_\ell) \cap \Omega_\ell = \emptyset$,

$$\sup\{d(x, \Omega_\ell) : x \in \text{supp}\tilde{\chi}_\ell\} < \frac{1}{2k},$$

and

$$\|\partial^\alpha \tilde{\chi}_\ell\|_{L^\infty} \leq C_\alpha k^{|\alpha|} \quad \text{for all } \alpha.$$

Since $\text{supp}(1 - \tilde{\chi}_\ell) \cap \Omega_\ell = \emptyset$,

$$\|(I - (\Pi_{\mathcal{Z}}^c)^*)Pu\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*} \leq \|\tilde{\chi}_\ell(I - (\Pi_{\mathcal{Z}}^c)^*)Pu\|_{\mathcal{Z}^*} \quad (8.15)$$

and

$$\|\chi_\ell^{(4)}(I - (\Pi_{\mathcal{Z}}^c)^*)Pu\|_{Y^{-1-p}} \leq \|\tilde{\chi}_\ell(I - (\Pi_{\mathcal{Z}}^c)^*)Pu\|_{Y^{-1-p}}. \quad (8.16)$$

Furthermore,

$$1_{\Omega_\ell}(I - (\Pi_{\mathcal{Z}}^c)^*)Pu = \sum_j 1_{\Omega_\ell} \tilde{\chi}_\ell(I - (\Pi_{\mathcal{Z}}^c)^*)\chi_j P\chi_j^{(1)}(\Psi u + (I - \Psi)u). \quad (8.17)$$

Set

$$\mathcal{F}_\ell := \{j : d(\Omega_\ell, \Omega_j) > k^{-1}\}, \quad \mathcal{N}_\ell := \{j : d(\Omega_\ell, \Omega_j) \leq k^{-1}\}$$

Then, by (8.17) and the fact that $P : \mathcal{W}^{p+1} \rightarrow Y^{p-1}$ and $P : \mathcal{Z} \rightarrow \mathcal{Z}^*$ with bounded norms,

$$\begin{aligned} & \|\tilde{\chi}_\ell(I - (\Pi_{\mathcal{Z}}^c)^*)Pu\|_{Y^{-s}} \\ & \leq \sum_{j \in \mathcal{F}_\ell} \|\tilde{\chi}_\ell(I - (\Pi_{\mathcal{Z}}^c)^*)\chi_j\|_{\mathcal{Z}^* \rightarrow Y^{-s}} \|\chi_j^{(1)}u\|_{\mathcal{Z}} \\ & \quad + \sum_{j \in \mathcal{N}_\ell} \left(\|\tilde{\chi}_\ell(I - (\Pi_{\mathcal{Z}}^c)^*)\chi_j\|_{Y^{p-1} \rightarrow Y^{-s}} \|\chi_j^{(1)}\Psi u\|_{\mathcal{W}^{p+1}} + \|\tilde{\chi}_\ell(I - (\Pi_{\mathcal{Z}}^c)^*)\chi_j\|_{\mathcal{Z}^* \rightarrow Y^{-s}} \|\chi_j^{(1)}(I - \Psi)u\|_{\mathcal{Z}} \right). \end{aligned} \quad (8.18)$$

Notice that since

$$\|u\|_{H_k^1} \leq \|\Psi(\mathcal{P})u\|_{H_k^1} + \|(I - \Psi(\mathcal{P}))u\|_{H_k^1} \leq \|u\|_X,$$

we have

$$\|\chi_j^{(1)}u\|_{\mathcal{Z}} \leq C\|u\|_{X(\Omega_j)}. \quad (8.19)$$

Since $\Pi_{\mathcal{Z}}^c : \mathcal{Z} \rightarrow \mathcal{V}_{H_c}$ is bounded,

$$\|I - (\Pi_{\mathcal{Z}}^c)^*\|_{\mathcal{Z}^* \rightarrow \mathcal{Y}^{-s}} = \|I - \Pi_{\mathcal{Z}}^c\|_{\mathcal{W}^s \rightarrow \mathcal{Z}} \leq C(H_ck)^{s-1}. \quad (8.20)$$

We now claim that

$$\|I - (\Pi_{\mathcal{Z}}^c)^*\|_{\mathcal{Y}^{p-1} \rightarrow \mathcal{Y}^{-s}} \leq C(H_ck)^{p+s-1}, \quad (8.21)$$

Indeed, $(\Pi_{\mathcal{Z}}^c)^* = (\mathcal{P} + 2)\Pi_{\mathcal{Z}}^c(\mathcal{P} + 2)^{-1}$ and hence

$$\|I - (\Pi_{\mathcal{Z}}^c)^*\|_{\mathcal{Y}^{p-1} \rightarrow \mathcal{Z}^*} = \|I - \Pi_{\mathcal{Z}}^c\|_{\mathcal{W}^{p+1} \rightarrow \mathcal{Z}} \leq C(H_ck)^p,$$

Also, by, e.g., [2, Lemma D.15], for $1 \leq s \leq p+1$,

$$\|I - (\Pi_{\mathcal{Z}}^c)^*\|_{\mathcal{Z}^* \rightarrow \mathcal{Y}^{-s}} = \|I - \Pi_{\mathcal{Z}}^c\|_{\mathcal{W}^s \rightarrow \mathcal{Z}} \leq C(H_ck)^{s-1};$$

the combination of these last two inequalities then implies (8.21).

Furthermore for any N there is $C > 0$ such that

$$\|\tilde{\chi}_\ell(I - (\Pi_{\mathcal{Z}}^c)^*)\chi_j\|_{\mathcal{Z}^* \rightarrow \mathcal{Y}^{-s}} \leq C(d(\Omega_j, \Omega_\ell)k)^{-N}(H_ck)^{p+s-1} \quad \text{for } j \in \mathcal{F}_\ell; \quad (8.22)$$

this follows from [2, Theorem 7.2] or more precisely its proof after letting $d(\text{supp}\chi, \text{supp}\psi) \geq dk^{-1}$ and assuming that $h(\mathcal{T})k \leq c$ and accounting for the interface Γ_p .

Combining (8.15) with (8.18), (8.19), (8.20), (8.21), and (8.22), we obtain that

$$\begin{aligned} \|(I - (\Pi_{\mathcal{Z}}^c)^*)Pu\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*} &\leq C \left(\sum_{j \in \mathcal{F}_\ell} (H_ck)^p (d(\Omega_j, \Omega_\ell)k)^{-N} \|u\|_{X(\Omega_j)} + \sum_{j \in \mathcal{N}_\ell} (H_ck)^p \|u\|_{X(\Omega_j)} \right) \\ &\leq C(H_ck)^p \sum_j \langle d(\Omega_j, \Omega_\ell)k \rangle^{-N} \|u\|_{X(\Omega_j)}, \end{aligned}$$

i.e., (8.13). The proof of the bound (8.14) is analogous, starting from (8.16) instead of (8.15). ■

Proof of Lemma 8.1. Observe that

$$\begin{aligned} Q^\ell(I - Q_{H_c}^0)v_h &= P_\ell^{-1}E_\ell^*P(I - Q_{H_c}^0)v_h \\ &= \underbrace{\chi_\ell^{(4)}\Psi(\mathcal{P}_\ell)P_\ell^{-1}E_\ell^*P(I - Q_{H_c}^0)v_h}_{=:\mathcal{L}_\ell} \\ &\quad + \underbrace{\chi_\ell^{(4)}(I - \Psi(\mathcal{P}_\ell))P_\ell^{-1}E_\ell^*P(I - Q_{H_c}^0) + (1 - \chi_\ell^{(4)})P_\ell^{-1}E_\ell^*P(I - Q_{H_c}^0)v_h}_{=:\mathcal{H}_\ell}. \end{aligned}$$

Now, since $\text{supp}\chi_\ell^{(4)} \Subset \Omega_\ell$,

$$\|\chi_\ell^{(4)}\Psi(\mathcal{P}_\ell)\|_{H_{\partial\Omega}^1(\Omega_\ell) \rightarrow \mathcal{W}^{p+1} \cap H_{\partial\Omega}^1(\Omega_\ell)} \leq C.$$

By Lemma 6.1,

$$\|(1 - \chi_\ell^{(4)})P_\ell^{-1}\|_{(H_{\partial\Omega}^1(\Omega_\ell))^* \rightarrow H_{\partial\Omega}^1(\Omega_\ell)} \leq C,$$

and by Lemma 6.2

$$\|\chi_\ell^{(4)}(I - \Psi(\mathcal{P}_\ell))\mathcal{P}_\ell^{-1}\|_{(H_{\partial\Omega}^1(\Omega_\ell))^* \rightarrow H_{\partial\Omega}^1(\Omega_\ell)} \leq C\rho_\ell(k\delta_\ell)^{-1}.$$

Thus

$$\begin{aligned} \|\mathcal{L}_\ell v_h\|_{\mathcal{W}^{p+1}} &\leq C\rho_\ell\|P(I - Q_{H_c}^0)v_h\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*}, \\ \|\mathcal{H}_\ell v_h\|_{H_k^1} &\leq C(1 + \rho_\ell(k\delta_\ell)^{-1})\|P(I - Q_{H_c}^0)v_h\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*}. \end{aligned}$$

The bounds (8.7) then follow from (8.10). $\blacksquare$

The rest of this subsection proves Lemma 8.2. By the definition of Q^ℓ (2.43), the fact that $\chi_\ell = \chi_\ell \chi_\ell^{(1)}$, and the fact that E_ℓ denotes extension by zero,

$$\begin{aligned}
& (Q^\ell - I)(I - Q_{H_c}^0)v_h \\
&= (P_\ell^{-1}E_\ell^*P - I)(I - Q_{H_c}^0)v_h \\
&= (P_\ell^{-1}E_\ell^*P - \chi_\ell^{(2)})(I - Q_{H_c}^0)v_h + (1 - \chi_\ell^{(2)})(I - Q_{H_c}^0)v_h \\
&= (P_\ell^{-1}E_\ell^*P - P_\ell^{-1}P_\ell\chi_\ell^{(2)})(I - Q_{H_c}^0)v_h + (1 - \chi_\ell^{(2)})(I - Q_{H_c}^0)v_h \\
&= (P_\ell^{-1}\chi_\ell^{(4)}P - P_\ell^{-1}P_\ell\chi_\ell^{(2)})(I - Q_{H_c}^0)v_h + P_\ell^{-1}(1 - \chi_\ell^{(3)})E_\ell^*(1 - \chi_\ell^{(4)})P(I - Q_{H_c}^0)v_h \\
&\quad + (1 - \chi_\ell^{(2)})(I - Q_{H_c}^0)v_h \\
&= \underbrace{P_\ell^{-1}[\chi_\ell^{(2)}, P](I - Q_{H_c}^0)v_h}_{=:\mathcal{A}_\ell^1} + \underbrace{P_\ell^{-1}(\chi_\ell^{(4)} - \chi_\ell^{(2)})P(I - Q_{H_c}^0)v_h}_{=:\mathcal{A}_\ell^2} + \underbrace{(1 - \chi_\ell^{(2)})(I - Q_{H_c}^0)v_h}_{=:\mathcal{A}_\ell^3} \\
&\quad + \underbrace{P_\ell^{-1}(1 - \chi_\ell^{(3)})E_\ell^*(1 - \chi_\ell^{(4)})P(I - Q_{H_c}^0)v_h}_{=:\mathcal{B}_\ell}.
\end{aligned}$$

We then set $\mathcal{A}_\ell := \mathcal{A}_\ell^1 + \mathcal{A}_\ell^2 + \mathcal{A}_\ell^3$. After noting $\nu \cdot A\nabla\chi_\ell^{(1)} = 0$ on Γ_{hard} and $\nu \cdot (A_+(\nabla\chi_\ell^{(1)})_+ - A_-(\nabla\chi_\ell^{(1)})_-) = 0$ on Γ_p implies $\chi_\ell^{(1)}\mathcal{A}_\ell v_h, \chi_\ell^{(1)}\mathcal{B}_\ell \in \mathcal{D}$, we have by Lemma 8.3, that to prove Lemma 8.2 it is sufficient to prove that

$$\sum_{j=1}^3 \|\chi_\ell^{(1)}\mathcal{A}_\ell^j v_h\|_{\mathcal{W}^{p+1}} \leq C\rho_\ell \left((k\delta_\ell)^{-1} \|\chi_\ell^{(3)}(I - Q_{H_c}^0)v_h\|_{\mathcal{W}^{1-p}} + \|\chi_\ell^{(4)}P(I - Q_{H_c}^0)v_h\|_{\mathcal{Y}^{-p-1}} \right). \quad (8.23)$$

and

$$\|\chi_\ell^{(1)}\mathcal{B}_\ell\|_{\mathcal{W}^{p+1}} \leq C(k\delta_\ell)^{-\infty} \|P(I - Q_{H_c}^0)v_h\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*} \quad (8.24)$$

Note first that $\chi_\ell^{(1)}\mathcal{A}_\ell^3 = 0$, so it is enough to consider $j = 1, 2$. Our proof of (8.23) uses the following commutator bound.

Lemma 8.4 *Suppose that χ_ℓ is as in Theorem 2.7 and $\chi_\ell^{(1)}$ is as in §7.1. Given $-p \leq s \leq p$ there exists $C > 0$ such that for all ℓ*

$$\|[P, \chi_\ell]\|_{\mathcal{W}^{s+1} \rightarrow \mathcal{Y}^s} \leq C(k\delta_\ell)^{-1}(1 + (k\delta_\ell)^{-1}).$$

Proof. By the assumed smoothness of χ_ℓ , $[P, \chi_\ell]u$ agrees with a differential operator when $u \in \mathcal{Z}_k^{s+1}$ for $0 \leq s \leq p$. Then, by direct calculation and the bounds (7.7) on χ_ℓ ,

$$\begin{aligned}
|[P, \chi_\ell]u| &\leq Ck^{-1} \left(\|\nabla\chi_\ell\|_{L^\infty} |k^{-1}\nabla u| + (\|\nabla\chi_\ell\|_{L^\infty} + k^{-1}\|\Delta\chi_\ell\|_{L^\infty})|u| \right) \\
&\leq C \left((k\delta_\ell)^{-1} |k^{-1}\nabla u| + ((k\delta_\ell)^{-1} + (k\delta_\ell)^{-2})|u| \right)
\end{aligned}$$

which proves the result for $s = 0$. Taking further derivatives proves the result for $1 \leq s \leq p$. The result for $-p \leq s \leq 0$ follows by duality, using that $[P, \chi_\ell]^* = [\chi_\ell, P^*]$ $\blacksquare$

Lemma 8.5 *Suppose that χ_ℓ is as in Theorem 2.7 and $\chi_\ell^{(1)}$ is as in §7.1. Suppose further that $\nu \cdot A\nabla\chi_\ell = 0$ on Γ_{hard} and $\nu \cdot (A_+\nabla(\chi_\ell)_+ - A_-\nabla(\chi_\ell)_-) = 0$ on Γ_p . Given $c_0 > 0$ there is $C > 0$ such that if $k\delta_\ell \geq c_0$ then*

$$\|\chi_\ell u\|_{\mathcal{W}^{p+1}} \leq C \left(\|\chi_\ell P u\|_{\mathcal{Y}^{p-1}} + \|\chi_\ell^{(1)} u\|_{\mathcal{W}^{1-p}} \right).$$

Proof. Since the coefficients of P are smooth, elliptic regularity implies that there exists $C > 0$ such that $(P + C)^{-1} = O(1)_{\mathcal{Y}^{s-2} \rightarrow \mathcal{W}^s}$ (see, e.g., [36, Theorems 4.18 and 4.20]). Since $\nu \cdot A\nabla\chi_\ell = 0$ on Γ_{hard} and $\nu \cdot (A_+\nabla(\chi_\ell)_+ - A_-\nabla(\chi_\ell)_-) = 0$ on Γ_p and $u \in \mathcal{D}$, we have $\chi_\ell u \in \mathcal{D}$ and hence

$$(P + C)\chi_\ell u = \chi_\ell(P + C)u + [P, \chi_\ell]u,$$

$$\chi_\ell u = (P + C)^{-1} \chi_\ell (P + C)u + (P + C)^{-1} [P, \chi_\ell]u.$$

By the mapping property of $(P + C)^{-1}$, the fact that $[P, \chi_\ell] = [P, \chi_\ell] \chi_\ell^{(1)}$, and Lemma 8.4,

$$\begin{aligned} \|\chi_\ell u\|_{\mathcal{W}^m} &\leq C \left(\|\chi_\ell (P + C)u\|_{\mathcal{Y}^{m-2}} + \|[P, \chi_\ell]u\|_{\mathcal{Y}^{m-2}} \right) \\ &\leq C \left(\|\chi_\ell (P + C)u\|_{\mathcal{Y}^{m-2}} + (k\delta_\ell)^{-1} (1 + (k\delta_\ell)^{-1}) \|\chi_\ell^{(1)} u\|_{\mathcal{W}^{m-1}} \right) \\ &\leq C \left(\|\chi_\ell P u\|_{\mathcal{Y}^{m-2}} + (1 + (k\delta_\ell)^{-1}) (1 + (k\delta_\ell)^{-1}) \|\chi_\ell^{(1)} u\|_{\mathcal{W}^{m-1}} \right). \end{aligned}$$

The result then follows by iterating, using a sequence of cutoff functions “between” χ_ℓ and $\chi_\ell^{(1)}$. $\blacksquare$

Lemma 8.5 applied with $u = P_\ell^{-1} [\chi_\ell^{(2)}, P] (I - Q_{H_c}^0) v_h$ implies that

$$\begin{aligned} \|\chi_\ell^{(1)} \mathcal{A}^1 v_h\|_{\mathcal{W}^{p+1}} &= \|\chi_\ell P_\ell^{-1} [\chi_\ell^{(2)}, P] (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{p+1}} \\ &\leq C \left(\|\chi_\ell^{(1)} P P_\ell^{-1} [\chi_\ell^{(2)}, P] (I - Q_{H_c}^0) v_h\|_{\mathcal{Y}^{p-1}} + \|\chi_\ell^{(1)} P_\ell^{-1} [\chi_\ell^{(2)}, P] (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}} \right). \end{aligned}$$

Now $\chi_\ell^{(1)} P = \chi_\ell^{(1)} P_\ell$, $\chi_\ell^{(1)} [\chi_\ell^{(1)}, P] = 0$, and $[\chi_\ell^{(2)}, P] = [\chi_\ell^{(2)}, P] \chi_\ell^{(3)}$, so that

$$\begin{aligned} \|\chi_\ell^{(1)} \mathcal{A}^1 v_h\|_{\mathcal{W}^{p+1}} &\leq C \|\chi_\ell^{(2)} P_\ell^{-1} [\chi_\ell^{(2)}, P] (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}} \\ &\leq C \|\chi_\ell^{(2)} P_\ell^{-1}\|_{\mathcal{Y}^{-1-p} \rightarrow \mathcal{W}^{1-p}} \|[\chi_\ell^{(2)}, P]\|_{\mathcal{W}^{1-p} \rightarrow \mathcal{Y}^{-1-p}} \|\chi_\ell^{(3)} (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}} \\ &\leq C \rho_\ell (k\delta_\ell)^{-1} \|\chi_\ell^{(3)} (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}}. \end{aligned}$$

Similarly, by Lemma 8.5 (applied with $u = P_\ell^{-1} (\chi_\ell^{(4)} - \chi_\ell^{(2)}) P (I - Q_{H_c}^0) v_h$) and the facts that $\chi_\ell P = \chi_\ell P_\ell$ and that $\chi_\ell (\chi_\ell^{(3)} - \chi_\ell^{(1)}) = 0$,

$$\begin{aligned} \|\chi_\ell^{(1)} \mathcal{A}^2 v_h\|_{\mathcal{W}^{p+1}} &= \|\chi_\ell^{(1)} P_\ell^{-1} (\chi_\ell^{(4)} - \chi_\ell^{(2)}) P (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{p+1}} \\ &\leq C \|\chi_\ell^{(2)} P_\ell^{-1} (\chi_\ell^{(4)} - \chi_\ell^{(2)}) P (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}} \\ &= C \|\chi_\ell^{(2)} P_\ell^{-1} (1 - \chi_\ell^{(2)}) \chi_\ell^{(4)} P (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}}. \end{aligned}$$

Since $\chi_\ell^{(1)} P_\ell^{-1} (\chi_\ell^{(4)} - \chi_\ell^{(2)}) : \mathcal{Y}^{-p-1} \rightarrow \mathcal{W}^{p+1}$,

$$\|\chi_\ell^{(1)} \mathcal{A}^2 v_h\|_{\mathcal{W}^{p+1}} \leq C \rho_\ell \|\chi_\ell^{(4)} P (I - Q_{H_c}^0) v_h\|_{\mathcal{Y}^{-1-p}},$$

which completes the proof of the bound (8.23).

We now prove (8.24). By definition

$$\begin{aligned} \|\chi_\ell^{(1)} \mathcal{B}_\ell\|_{\mathcal{W}^{p+1}} &= \|\chi_\ell^{(1)} P_\ell^{-1} (1 - \chi_\ell^{(3)}) E_\ell^* (1 - \chi_\ell^{(4)}) P (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{p+1}} \\ &\leq \|\chi_\ell^{(1)} P_\ell^{-1} (1 - \chi_\ell^{(3)}) E_\ell^*\|_{(H_{\partial\Omega}^1(\Omega_\ell))^* \rightarrow \mathcal{W}^{p+1}} \|(1 - \chi_\ell^{(4)}) P (I - Q_{H_c}^0) v_h\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*} \\ &= \|E_\ell (1 - \chi_\ell^{(3)}) (P_\ell^*)^{-1} \chi_\ell^{(1)}\|_{\mathcal{Y}^{-p-1} \rightarrow H_{\partial\Omega}^1(\Omega_\ell)} \|(1 - \chi_\ell^{(4)}) P (I - Q_{H_c}^0) v_h\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*}. \end{aligned}$$

Since $(P_\ell^*)^{-1} \chi_\ell^{(1)} v \in H_{\partial\Omega}^1(\Omega_\ell)$, so is $(1 - \chi_\ell^{(3)}) (P_\ell^*)^{-1} \chi_\ell^{(1)} v$. Thus

$$E_\ell (1 - \chi_\ell^{(3)}) (P_\ell^*)^{-1} \chi_\ell^{(1)} \in H_{\partial\Omega}^1(\Omega_\ell)$$

with

$$\|E_\ell (1 - \chi_\ell^{(3)}) (P_\ell^*)^{-1} \chi_\ell^{(1)}\|_{\mathcal{Y}^{-p-1} \rightarrow H_k^1(\Omega_\ell)} = \|(1 - \chi_\ell^{(3)}) (P_\ell^*)^{-1} \chi_\ell^{(1)}\|_{\mathcal{Y}^{-p-1} \rightarrow H_k^1(\Omega_\ell)}.$$

Thus,

$$\|\chi_\ell^{(1)} \mathcal{B}_\ell\|_{\mathcal{W}^{p+1}} \leq \|(1 - \chi_\ell^{(3)}) (P_\ell^*)^{-1} \chi_\ell^{(1)}\|_{\mathcal{Y}^{-p-1} \rightarrow H_{\partial\Omega}^1(\Omega_\ell)} \|P (I - Q_{H_c}^0) v_h\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*}$$

$$= \|\chi_\ell^{(1)} P_\ell^{-1} (1 - \chi_\ell^{(3)})\|_{(H_{\partial\Omega}^1(\Omega_\ell))^* \rightarrow \mathcal{W}^{p+1}} \|P(I - Q_{H_c}^0) v_h\|_{(H_{\partial\Omega}^1(\Omega_\ell))^*}.$$

Since $\text{supp}(1 - \chi_\ell^{(3)})$ is contained in the absorbing layer of Ω_ℓ (by (7.8)), by Lemma 6.1,

$$\|\chi_\ell P_\ell^{-1} (1 - \chi_\ell^{(3)})\|_{(H_{\partial\Omega}^1(\Omega_\ell))^* \rightarrow \mathcal{W}^{p+1}} \leq C \|\chi_\ell^{(2)} P_\ell^{-1} (1 - \chi_\ell^{(3)})\|_{(H_{\partial\Omega}^1(\Omega_\ell))^* \rightarrow \mathcal{W}^{1-p}} \leq C \langle k\delta_\ell \rangle^{-\infty},$$

and the bound (8.24) then follows. We have therefore proved (8.23) and (8.24) and the proof of Lemma 8.2 is complete.

8.5 End of the proof of Theorem 2.7

Recall from (8.6) that

$$\begin{aligned} (Q - I)v_h &= \sum_{\ell=1}^N \mathcal{R}_\ell^* \left[(\chi_\ell(Q^\ell - \mathcal{R}_\ell)(I - Q_{H_c}^0)v_h) - (\chi_\ell(Q^\ell - Q_h^\ell)(I - Q_{H_c}^0)v_h) \right. \\ &\quad \left. - (I - \mathcal{I}_\ell)(\chi_\ell(Q_h^\ell - \mathcal{R}_\ell)(I - Q_{H_c}^0)v_h) - (I - \mathcal{I}_\ell)\chi_\ell \mathcal{R}_\ell(I - Q_{H_c}^0)v_h \right]. \end{aligned}$$

By Lemma 8.2,

$$(Q - I)v_h = \sum_{\ell=1}^N \mathcal{R}_\ell^* \left[\chi_\ell \mathcal{A}_\ell + \chi_\ell \mathcal{B}_\ell + \chi_\ell \mathcal{C}_\ell + \mathcal{D}_\ell + \mathcal{E}_\ell \right] v_h$$

where

$$\mathcal{C}_\ell := -(Q^\ell - Q_h^\ell)(I - Q_{H_c}^0),$$

$$\mathcal{D}_\ell := -(I - \mathcal{I}_\ell)\chi_\ell(Q_h^\ell - \mathcal{R}_\ell)(I - Q_{H_c}^0) = -(I - \mathcal{I}_\ell)\chi_\ell(\mathcal{A}_\ell + \mathcal{B}_\ell + \mathcal{C}_\ell),$$

and

$$\mathcal{E}_\ell := -(I - \mathcal{I}_\ell)\chi_\ell \mathcal{R}_\ell(I - Q_{H_c}^0).$$

To prove (8.5), we apply the triangle inequality and Lemma 7.3 to obtain

$$\|(I - \Pi_{H_c})(Q - I)v_h\|_{\mathcal{Z}}^2 \leq C \left(\sum_{\mathcal{F} \in \{\mathcal{A}, \mathcal{B}\}} (H_c k)^{2p} \sum_{\ell=1}^N \|\mathcal{F}_\ell v_h\|_{\mathcal{W}^{p+1}}^2 + \sum_{\mathcal{F} \in \{\mathcal{C}, \mathcal{D}, \mathcal{E}\}} \sum_{\ell=1}^N \|\mathcal{F}_\ell v_h\|_{\mathcal{Z}}^2 \right), \quad (8.25)$$

and

$$\|(Q - I)v_h\|_{\mathcal{Z}}^2 \leq C \left(\sum_{\mathcal{F} \in \{\mathcal{A}, \mathcal{B}\}} \sum_{\ell=1}^N \|\mathcal{F}_\ell v_h\|_{\mathcal{W}^{p+1}}^2 + \sum_{\mathcal{F} \in \{\mathcal{C}, \mathcal{D}, \mathcal{E}\}} \sum_{\ell=1}^N \|\mathcal{F}_\ell v_h\|_{\mathcal{Z}}^2 \right), \quad (8.26)$$

where, importantly, C does not depend on N .

We first bound the terms involving $\mathcal{D}_\ell$ in terms of $\mathcal{A}_\ell$ and $\mathcal{B}_\ell$ and $\mathcal{C}_\ell$: by Lemma 7.1 with $w = \mathcal{A}_\ell v_h + \mathcal{B}_\ell v_h$,

$$\begin{aligned} \sum_{\ell=1}^N \|\mathcal{D}_\ell v_h\|_{\mathcal{Z}}^2 &\leq C(hk)^{2p} \sum_{\ell=1}^N \|\chi_\ell^{(1)}(\mathcal{A}_\ell + \mathcal{B}_\ell)v_h\|_{\mathcal{W}^{p+1}}^2 + C \sum_{\ell=1}^N \mu_\ell^2 \|\chi_\ell^{(1)} \mathcal{C}_\ell\|_{\mathcal{Z}}^2 \\ &\leq C \left((hk)^{2p} \sum_{\ell=1}^N \|\mathcal{A}_\ell v_h\|_{\mathcal{W}^{p+1}}^2 + (hk)^{2p} \sum_{\ell=1}^N \|\mathcal{B}_\ell v_h\|_{\mathcal{W}^{p+1}}^2 + \sum_{\ell=1}^N \mu_\ell^2 \|\mathcal{C}_\ell\|_{\mathcal{Z}}^2 \right). \end{aligned} \quad (8.27)$$

For $\chi_\ell^{(1)} \mathcal{A}_\ell$, by (8.8), the Cauchy–Schwarz inequality, Lemmas 7.2 and 7.4, and (5.2), if $\rho(H_c k)^{2p} \leq c$ then, since $\Lambda(\{\Omega_\ell\}, k) \leq \Upsilon$,

$$\begin{aligned}
& \sum_{\ell=1}^N \|\chi_\ell^{(1)} \mathcal{A}_\ell v_h\|_{\mathcal{W}^{p+1}}^2 \\
& \leq \sum_{\ell=1}^N \left(C \rho_\ell^2 (k \delta_\ell)^{-2} \|\chi_\ell^{(3)} (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}}^2 + C \rho_\ell^2 (H_c k)^{4p} \left(\sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_j)} \right)^2 \right) \\
& \leq C \sum_{\ell=1}^N \left(\rho_\ell^2 (k \delta_\ell)^{-2} \|\chi_\ell^{(3)} (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}}^2 \right. \\
& \quad \left. + C \rho_\ell^2 (H_c k)^{4p} \left(\sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \right) \left(\sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_j)}^2 \right) \right) \\
& \leq \sum_{\ell=1}^N \left(C \rho_\ell^2 (k \delta_\ell)^{-2} \|\chi_\ell^{(3)} (I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}}^2 + C \rho_\ell^2 (H_c k)^{4p} \sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_j)}^2 \right) \\
& \leq C \left(\max_\ell (\rho_\ell^2 (k \delta_\ell)^{-2}) \|(I - Q_{H_c}^0) v_h\|_{\mathcal{W}^{1-p}}^2 + \max_\ell \rho_\ell^2 (H_c k)^{4p} \|(I - Q_{H_c}^0) v_h\|_X^2 \right) \\
& \leq C \left(\max_\ell (\rho_\ell^2 (k \delta_\ell)^{-2}) (H_c k)^{2p} \rho^2 \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}^2 + \max_\ell \rho_\ell^2 (H_c k)^{2p} \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}^2 \right) \tag{8.28}
\end{aligned}$$

For $\chi_\ell^{(1)} \mathcal{B}_\ell$, by (8.9), (7.9), and (5.1), if $\rho(H_c k)^{2p} \leq c$ then, again using that $\Lambda(\{\Omega_\ell\}, k) \leq \Upsilon$,

$$\begin{aligned}
\sum_{\ell=1}^N \|\chi_\ell^{(1)} \mathcal{B}_\ell v_h\|_{\mathcal{W}^{p+1}}^2 & \leq C \max_\ell \langle k \delta_\ell \rangle^{-M} (H_c k)^{2p} \sum_{\ell=1}^N \sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_j)}^2 \\
& \leq C \max_\ell \langle k \delta_\ell \rangle^{-M} (H_c k)^{2p} \sum_{j=1}^N \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_j)}^2 \\
& \leq C \max_\ell \langle k \delta_\ell \rangle^{-M} (H_c k)^{2p} \|(I - Q_{H_c}^0) v_h\|_X^2 \leq C \max_\ell \langle k \delta_\ell \rangle^{-M} \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}^2. \tag{8.29}
\end{aligned}$$

For $\mathcal{C}_\ell$, by (7.6), (7.9), and (5.1), if $(H_c k)^{2p} \rho \leq c$ then

$$\begin{aligned}
\sum_{\ell=1}^N \|(I - \mathcal{I}_\ell) \chi_\ell (I - Q_{H_c}^0) v_h\|_{H_k^1(\Omega_\ell)}^2 & \leq C \sum_{\ell=1}^N \left((h_\ell k)^p + (H_c k)^p \mu_\ell \right)^2 \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_\ell)}^2 \\
& \leq C \max_\ell \left((h_\ell k)^p + (H_c k)^p \mu_\ell \right)^2 \|(I - Q_{H_c}^0) v_h\|_X^2 \\
& \leq C \max_\ell \left((h_\ell k)^p + (H_c k)^p \mu_\ell \right)^2 (H_c k)^{-2p} \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}^2. \tag{8.30}
\end{aligned}$$

Finally, we estimate $\mathcal{C}_\ell$. Since $d(\text{supp} \chi_\ell^{(1)}, \partial \Omega_\ell \setminus \partial \Omega) \geq c \delta_\ell$, we may apply the “local quasioptimality modulo low frequencies” result of [1, Theorem 4.2] (see also [1, Theorem 1.2 and Example 3.1]), to obtain

$$\begin{aligned}
& \|\chi_\ell (Q^\ell - Q_h^\ell) (I - Q_{H_c}^0) v_h\|_{\mathcal{Z}} \\
& \leq C \inf_{w_h, \ell \in \mathcal{V}_{\ell, h}} \|\chi_\ell^{(1)} Q^\ell (I - Q_{H_c}^0) v_h - w_h, \ell\|_{\mathcal{Z}} + C \|\chi_\ell^{(1)} (Q^\ell - Q_h^\ell) (I - Q_{H_c}^0) v_h\|_{\mathcal{Y}^{-p}} \\
& \leq C \inf_{w_h, \ell \in \mathcal{V}_{\ell, h}} \|\chi_\ell^{(1)} (Q^\ell - I) (I - Q_{H_c}^0) v_h - w_h, \ell\|_{\mathcal{Z}} + C \inf_{w_h, \ell \in \mathcal{V}_{\ell, h}} \|\chi_\ell^{(1)} (I - Q_{H_c}^0) v_h - w_h, \ell\|_{\mathcal{Z}} \\
& \quad + C \|\chi_\ell^{(1)} (Q^\ell - Q_h^\ell) (I - Q_{H_c}^0) v_h\|_{\mathcal{Y}^{-p}}.
\end{aligned}$$

Now, by Lemmas 8.1 and 8.2, (7.6), (7.7), and Lemma 6.3, for any $0 < s < \frac{1}{2}$,

$$\|\chi_\ell (Q^\ell - Q_h^\ell) (I - Q_{H_c}^0) v_h\|_{\mathcal{Z}}$$

$$\begin{aligned}
&\leq C(h_\ell k)^p \left(\|\chi_\ell^{(1)} \mathcal{A}_\ell v_h\|_{\mathcal{W}^{p+1}} + \|\chi_\ell^{(1)} \mathcal{B}_\ell v_h\|_{\mathcal{W}^{p+1}} \right) + \left((h_\ell k)^p + (H_c k)^p \mu_\ell \right) \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_\ell)} \\
&\quad + C((h_\ell k)^p \rho_\ell + (h_\ell k)^s) \inf_{w_{h,\ell} \in \mathcal{V}_{\ell,h}} \|Q_\ell^\ell (I - Q_{H_c}^0) v_h - w_{h,\ell}\|_{H_k^1(\Omega_\ell)} \\
&\leq C(h_\ell k)^p \left(\|\chi_\ell^{(1)} \mathcal{A}_\ell v_h\|_{\mathcal{W}^{p+1}} + \|\chi_\ell^{(1)} \mathcal{B}_\ell v_h\|_{\mathcal{W}^{p+1}} \right) + \left((h_\ell k)^p + (H_c k)^p \mu_\ell \right) \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_\ell)} \\
&\quad + C((h_\ell k)^p \rho_\ell + (h_\ell k)^s) \left((h_\ell k)^p \|\mathcal{L}_\ell v_h\|_{\mathcal{W}^{p+1}} + \|\mathcal{H}_\ell v_h\|_{H_k^1} \right) \\
&\leq C(h_\ell k)^p \left(\|\chi_\ell^{(1)} \mathcal{A}_\ell v_h\|_{\mathcal{W}^{p+1}} + \|\chi_\ell^{(1)} \mathcal{B}_\ell v_h\|_{\mathcal{W}^{p+1}} \right) + \left((h_\ell k)^p + (H_c k)^p \mu_\ell \right) \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_\ell)} \\
&\quad + C((h_\ell k)^p \rho_\ell + (h_\ell k)^s) \left(\rho_\ell (h_\ell k)^p + (k\delta_\ell)^{-1} \rho_\ell + 1 \right) (H_c k)^p \sum_{j=1}^N \langle kd(\Omega_j, \Omega_\ell) \rangle^{-M} \|(I - Q_{H_c}^0) v_h\|_{X(\Omega_j)}.
\end{aligned}$$

Hence, arguing as in (8.28) and (8.29),

$$\begin{aligned}
&\sum_{\ell=1}^N \|\chi_\ell (Q_\ell^\ell - Q_h^\ell) (I - Q_{H_c}^0) v_h\|_{\mathcal{Z}}^2 \\
&\leq C \max_\ell (h_\ell / H_c)^{2p} \left(\max_\ell (\rho_\ell^2 (k\delta_\ell)^{-2}) (\rho (H_c k)^{2p})^2 + \max_\ell \rho_\ell^2 (H_c k)^{4p} \right) \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}^2 \\
&\quad + C \max_\ell \langle k\delta_\ell \rangle^{-M} \max_\ell (h_\ell k)^{2p} \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}^2 + C \max_\ell \left((h_\ell / H_c)^{2p} + \mu_\ell^2 \right) \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}^2 \\
&\quad + C \max_\ell \left((h_\ell k)^{2p} \rho_\ell^2 + (h_\ell k)^{2s} \right) \left(\rho_\ell^2 (h_\ell k)^{2p} + \rho_\ell^2 (k\delta_\ell)^{-2} + 1 \right) \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}^2.
\end{aligned} \tag{8.31}$$

Putting together (8.25), (8.27), (8.28) (8.29), (8.30), and (8.31), we obtain

$$\begin{aligned}
&\|(I - \Pi_{H_c})(Q - I) v_h\|_{\mathcal{Z}} \\
&\leq C \left[\max_\ell (\rho_\ell (k\delta_\ell)^{-1}) \rho (H_c k)^{2p} + \max_\ell \rho_\ell (H_c k)^{2p} \right. \\
&\quad \left. + \max_\ell \langle k\delta_\ell \rangle^{-M} (H_c k)^p + (h/H_c)^p + \max_\ell \mu_\ell \right. \\
&\quad \left. + \max_\ell ((h_\ell k)^p \rho_\ell + (h_\ell k)^s) \left(\rho_\ell (h_\ell k)^p + \rho_\ell (k\delta_\ell)^{-1} + 1 \right) \right] \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}.
\end{aligned} \tag{8.32}$$

Also, putting together (8.26), (8.27), (8.28) (8.29), (8.30), and (8.31), we obtain

$$\begin{aligned}
\|(Q - I) v_h\|_{\mathcal{Z}} &\leq C \left(\max_\ell (\rho_\ell (k\delta_\ell)^{-1}) \rho (H_c k)^p + \max_\ell \rho_\ell (H_c k)^p \right. \\
&\quad \left. + \max_\ell \langle k\delta_\ell \rangle^{-M} + (h/H_c)^p + \max_\ell \mu_\ell \right. \\
&\quad \left. + \max_\ell ((h_\ell k)^p \rho_\ell + (h_\ell k)^s) \left(\rho_\ell (h_\ell k)^p + \rho_\ell (k\delta_\ell)^{-1} + 1 \right) \right) \|(I - \Pi_{H_c}) v_h\|_{\mathcal{Z}}.
\end{aligned} \tag{8.33}$$

The estimates (8.32) and (8.33) then imply (8.5), and the proof is complete.

A Coercive estimates in the absorbing layer

Lemma A.1 *Let $c_1 > 0$. Then there is $c_0 > 0$ such that for all $k > 0$, all $U \subset \mathbb{R}^d$ bounded with Lipschitz boundary and $\Gamma \subset U$ a Lipschitz hypersurface, all $A \in L^\infty(U; \mathbb{M}(d \times d))$ symmetric matrices, $b \in L^\infty(U \setminus \Gamma; \mathbb{C}^d)$ and $n \in L^\infty(U; \mathbb{C})$, and $\theta \in \mathbb{R}$ with*

$$q(x, \xi) := \Re \left(e^{i\theta} (A^{ij}(x) \xi_i \xi_j + b^i(x) \xi_i - n(x)) \right) > c_1 (1 + |\xi|^2) \quad \text{for all } x \in U \setminus \Gamma, \xi \in \mathbb{R}^d, \tag{A.1}$$

the following is true. For all $u \in H^1(U)$

$$Q(u) \geq c_0 \|u\|_{H_k^1(U)}^2$$

where

$$Q(u) := \Re e^{i\theta} \left(\langle Ak^{-1}\nabla u, k^{-1}\nabla u \rangle_{L^2(U)} - \frac{i}{2} \langle b \cdot k^{-1}\nabla u, u \rangle_{L^2(U)} + \frac{i}{2} \langle bu, k^{-1}\nabla u \rangle_{L^2(U)} - \langle nu, u \rangle_{L^2(U)} \right).$$

Proof. Set

$$G(x) := \Re(e^{i\theta} A(x)), \quad \beta(x) := \Re(e^{i\theta} b(x)), \quad \text{and} \quad \gamma(x) := -\Re(e^{i\theta} n(x))$$

so that

$$q(x, \xi) = \xi^T G(x) \xi + \beta(x) \cdot \xi + \gamma(x). \quad (\text{A.2})$$

Since A is symmetric, G is a real symmetric matrix. Moreover, by (A.1), there is $c_0 > 0$ such that $c_1 > 2c_0$ and $\xi^T G(x) \xi \geq 2c_0 |\xi|^2$. Set $G_1(x) := G(x) - c_0 I$ so that

$$G_1(x) \geq c_0 I > 0. \quad (\text{A.3})$$

Define

$$v(x) := \frac{1}{2} G_1(x)^{-1} \beta(x) \quad \text{and} \quad \delta(x) := \gamma(x) - c_0 - \frac{1}{4} \beta(x)^T G_1(x)^{-1} \beta(x).$$

Completing the square gives

$$q(x, \xi) = c_0 (\xi^T \xi + 1) + (\xi + v(x))^T G_1(x) (\xi + v(x)) + \delta(x) \quad \text{for all } x \in U, \xi \in \mathbb{R}^d. \quad (\text{A.4})$$

On the other hand, (A.1) and the fact that $c_1 > 2c_0$ implies that

$$q(x, \xi) - c_0 (|\xi|^2 + 1) \geq c_0 (|\xi|^2 + 1) \quad \text{for all } x \in U, \xi \in \mathbb{R}^d.$$

Evaluating this inequality at $\xi = -v(x)$ (which is allowed since v is real) and using (A.4) gives

$$\delta(x) \geq c_0 (|v(x)|^2 + 1) \geq c_0 \quad \text{for all } x \in U. \quad (\text{A.5})$$

Now let

$$C(x) := G_1(x)^{1/2}, \quad \text{and} \quad d(x) := C(x)v(x).$$

With this notation (A.4) becomes

$$q(x, \xi) = c_0 (\xi^T \xi + 1) + |C(x)\xi + d(x)|^2 + \delta(x) \quad \text{for all } \xi \in \mathbb{R}^d. \quad (\text{A.6})$$

Set $z := -ik^{-1}\nabla u$ and write

$$e^{i\theta} A = G + iH,$$

where G and H are real symmetric matrices. Since $z^* H z$ is real,

$$\Re(e^{i\theta} A^{ij} k^{-1} \partial_i u \overline{k^{-1} \partial_j u}) = G^{ij} z_i \overline{z_j}. \quad (\text{A.7})$$

Next, we consider the symmetrized first-order term. Write

$$e^{i\theta} b = \beta + i\alpha, \quad \alpha := \Im(e^{i\theta} b),$$

where α and β are real-valued. Observe then that

$$\begin{aligned} & \Re \left(-\frac{i}{2} e^{i\theta} \langle b \cdot k^{-1} \nabla u, u \rangle_{L^2(U)} + \frac{i}{2} e^{i\theta} \langle bu, k^{-1} \nabla u \rangle_{L^2(U; \mathbb{C}^d)} \right) \\ &= \frac{1}{2} \Re \left(\langle (\beta + i\alpha) \cdot z, u \rangle_{L^2(U)} + \langle (\beta + i\alpha) u, z \rangle_{L^2(U)} \right) \\ &= \frac{1}{2} \Re \left(\overline{\langle (\beta - i\alpha) u, z \rangle_{L^2(U)}} + \langle (\beta + i\alpha) u, z \rangle_{L^2(U)} \right) \\ &= \Re \langle \beta u, z \rangle = \Re \langle \beta \cdot z, u \rangle. \end{aligned} \quad (\text{A.8})$$

Because $C^T C = G_1$ and $C^T d = G_1 v = \frac{1}{2} \beta$,

$$\|Cz + du\|_{L^2(U)}^2 = \int_U G_1^{ij} z_i \overline{z_j} dx + \Re \int_U \beta \cdot z \overline{u} dx + \int_U |d|^2 |u|^2 dx. \quad (\text{A.9})$$

Combining (A.6)–(A.9) gives

$$Q(u) = c_0 \left(\|k^{-1} \nabla u\|_{L^2(U)}^2 + \|u\|_{L^2(U)}^2 \right) + \| -iCk^{-1} \nabla u + du \|_{L^2(U)}^2 + \int_U \delta(x) |u(x)|^2 dx. \quad (\text{A.10})$$

Finally, using (A.5) in (A.10), we obtain that

$$Q(u) \geq c_0 \|k^{-1} \nabla u\|_{L^2(U)}^2 + c_0 \|u\|_{L^2(U)}^2.$$

and the result follows. $\blacksquare$

Acknowledgements

All three authors were supported by ERC Synergy Grant “PSINumScat” (101167139). JG was supported by EPSRC grants EP/V001760/1 and EP/V051636/1 and Leverhulme Research Project Grant RPG-2023-325.

This work was granted access to the HPC resources of GENCI–TGCC, CEA, under the allocation 2026-AD010618004.

ChatGPT version 5.6 was used to proofread the paper and to help with the algebraic manipulations in Appendix A.

References

- [1] M. Averseng, J. Galkowski, and E. A. Spence. Helmholtz FEM solutions are locally quasi-optimal modulo low frequencies. *Advances in Computational Mathematics*, 50(6):112, 2024.
- [2] M. Averseng, J. Galkowski, and E. A. Spence. Non-uniform finite-element meshes defined by ray dynamics for Helmholtz problems. *arXiv preprint arXiv:2506.15630*, 2025.
- [3] J.-P. Berenger. A perfectly matched layer for the absorption of electromagnetic waves. *J. Comp. Phys.*, 114:185–200, October 1994.
- [4] M. Bernkopf, S. Sauter, C. Torres, and A. Veit. Solvability of discrete Helmholtz equations. *IMA Journal of Numerical Analysis*, 43(3):1802–1830, 2023.
- [5] M. Bonazzoli, V. Dolean, I. G. Graham, E. A. Spence, and P.-H. Tournier. A two-level domain-decomposition preconditioner for the time-harmonic Maxwell’s equations. In *International Conference on Domain Decomposition Methods*, pages 149–157. Springer, 2017.
- [6] M. Bonazzoli, V. Dolean, I. G. Graham, E. A. Spence, and P.-H. Tournier. Two-level preconditioners for the Helmholtz equation. In *International Conference on Domain Decomposition Methods*, pages 139–147. Springer, 2017.
- [7] M. Bonazzoli, V. Dolean, I. G. Graham, E. A. Spence, and P.-H. Tournier. Domain decomposition preconditioning for the high-frequency time-harmonic Maxwell equations with absorption. *Math. Comp.*, 88:2559–2604, 2019.
- [8] N. Bootland, V. Dolean, P. Jolivet, and P.-H. Tournier. A comparison of coarse spaces for Helmholtz problems in the high frequency regime. *Computers & Mathematics with Applications*, 98:239–253, 2021.
- [9] N. Burq. Semi-classical estimates for the resolvent in nontrapping geometries. *International Mathematics Research Notices*, 2002(5):221–241, 2002.
- [10] T. Chaumont-Frelet and E. A. Spence. The geometric error is less than the pollution error when solving the high-frequency Helmholtz equation with high-order FEM on curved domains. *IMA Journal of Numerical Analysis*, 46(3):1643–1682, 2026.
- [11] F. Collino and P. Monk. The perfectly matched layer in curvilinear coordinates. *SIAM Journal on Scientific Computing*, 19(6):2061–2090, 1998.
- [12] V. Dolean, M. Fry, M. Langer, E. Parolin, and P.-H. Tournier. Achieving wavenumber robustness in domain decomposition for heterogeneous helmholtz equation: An overview of spectral coarse spaces. *Computers & Mathematics with Applications*, 218:361–385, 2026.
- [13] V. Dolean, P. Jolivet, and F. Nataf. *An introduction to domain decomposition methods: algorithms, theory, and parallel implementation*. SIAM, 2015.
- [14] Y. Du and H. Wu. Preasymptotic error analysis of higher order FEM and CIP-FEM for Helmholtz equation with high wavenumber. *SIAM J. Numer. Anal.*, 53:782–804, 2015.

- [15] A. Ern and J.-L. Guermond. *Finite elements I: Approximation and interpolation*, volume 72 of *Texts in Applied Mathematics*. Springer Nature, 2021.
- [16] S. Fu, S. Gong, G. Li, and Y. Wang. On Edge Multiscale Space based Hybrid Schwarz Preconditioner for Helmholtz Problems with Large Wavenumbers. *arXiv preprint arXiv:2408.08198*, 2024.
- [17] J. Galkowski, S. Gong, I. G. Graham, D. Lafontaine, and E. A. Spence. Convergence of overlapping domain decomposition methods with PML transmission conditions applied to nontrapping Helmholtz problems. *arXiv preprint arXiv:2404.02156*, 2024.
- [18] J. Galkowski, D. Lafontaine, and E. A. Spence. Perfectly-matched-layer truncation is exponentially accurate at high frequency. *SIAM Journal on Mathematical Analysis*, 55(4):3344–3394, 2023.
- [19] J. Galkowski and E. A. Spence. Sharp preasymptotic error bounds for the Helmholtz h -FEM. *SIAM J. Numer. Anal.*, 63(1):1–23, 2025.
- [20] J. Galkowski and E. A. Spence. Convergence theory for two-level hybrid Schwarz preconditioners for high-frequency Helmholtz problems. *SIAM J. Numer. Anal.*, 64(1):29–54, 2026.
- [21] J. Galkowski and E. A. Spence. Numerical analysis of the high-frequency Helmholtz equation using semiclassical analysis. *Acta Numerica*, 35:1–171, 2026.
- [22] J. Galkowski, E. A. Spence, and J. Wunsch. Optimal constants in nontrapping resolvent estimates. *Pure and Applied Analysis*, 2(1):157–202, 2020.
- [23] S. Gong, I. G. Graham, and E. A. Spence. Domain decomposition preconditioners for high-order discretisations of the heterogeneous Helmholtz equation. *IMA J. Numer. Anal.*, 41(3):2139–2185, 2021.
- [24] I. G. Graham and E. A. Spence. Theory of two-level Schwarz preconditioners with piecewise-polynomial coarse spaces for the high-frequency Helmholtz equation. *arXiv preprint arXiv:2501.15976*, 2025.
- [25] I. G. Graham, E. A. Spence, and E. Vainikko. Domain decomposition preconditioning for high-frequency Helmholtz problems with absorption. *Math. Comp.*, 86(307):2089–2127, 2017.
- [26] I. G. Graham, E. A. Spence, and E. Vainikko. Recent results on domain decomposition preconditioning for the high-frequency Helmholtz equation using absorption. In D. Lahaye, J. Tang, and K. Vuik, editors, *Modern Solvers for Helmholtz Problems*, pages 3–26. Springer International Publishing, Cham, 2017.
- [27] L. Hörmander. *The analysis of linear partial differential operators III: pseudo-differential operators*. Springer, 1985.
- [28] Q. Hu and Z. Li. A Hybrid Two-Level Weighted Schwarz Method for Helmholtz Equations. *SIAM Journal on Numerical Analysis*, 63(2):716–743, 2025.
- [29] F. Ihlenburg and I. Babuška. Finite element solution of the Helmholtz equation with high wave number Part I: The h -version of the FEM. *Computers & Mathematics with Applications*, 30(9):9–37, 1995.
- [30] F. Ihlenburg and I. Babuska. Finite element solution of the Helmholtz equation with high wave number part II: the hp version of the FEM. *SIAM J. Numer. Anal.*, 34(1):315–358, 1997.
- [31] Dmitry Jakobson, Yuri Safarov, and Alexander Strohmaier. The semiclassical theory of discontinuous systems and ray-splitting billiards. *Amer. J. Math.*, 137(4):859–906, 2015. With an appendix by Yves Colin de Verdière.

- [32] D. Jerison and C. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130:161–219, 1995.
- [33] D. Lafontaine, E. A. Spence, and J. Wunsch. For most frequencies, strong trapping has a weak effect in frequency-domain scattering. *Communications on Pure and Applied Mathematics*, 74(10):2025–2063, 2021.
- [34] P. Lu, X. Xu, B. Zheng, and J. Zou. Two-level hybrid Schwarz Preconditioners for the Helmholtz Equation with high wave number. *SIAM Journal on Numerical Analysis*, 63(6):2187–2220, 2025.
- [35] C. Ma, C. Alber, R. Scheichl, and Y. Zhang. Two-level Restricted Additive Schwarz preconditioner based on Multiscale Spectral Generalized FEM for Heterogeneous Helmholtz Problems. *Journal of Scientific Computing*, 105(3):99, 2025.
- [36] W. McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. CUP, 2000.
- [37] U. V. Riss and H.-D. Meyer. Calculation of resonance energies and widths using the complex absorbing potential method. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 26(23):4503, 1993.
- [38] Yu. Safarov and D. Vassiliev. *The asymptotic distribution of eigenvalues of partial differential operators*, volume 155 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1997. Translated from the Russian manuscript by the authors.
- [39] G. Savaré. Regularity results for elliptic equations in Lipschitz domains. *Journal of Functional Analysis*, 152(1):176–201, 1998.
- [40] A. H. Schatz. An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. *Math. Comp*, 28(128):959–962, 1974.
- [41] A. H. Schatz and J. Wang. Some new error estimates for Ritz–Galerkin methods with minimal regularity assumptions. *Mathematics of Computation*, 65(213):19–27, 1996.
- [42] E. Sieburgh, A. Heinlein, V. Dwarka, and C. Vuik. Coarse spaces based on higher-order interpolation for Schwarz preconditioners for Helmholtz problems. In Petter Bjørstad, X.-C. Cai, V. Dolean, D. E. Keyes, R. Kornhuber, and J. Xu, editors, *Domain Decomposition Methods in Science and Engineering XXVIII*, volume 155 of *Lecture Notes in Computational Science and Engineering*, pages 319–326, Cham, 2026. Springer.
- [43] A. Toselli and O. Widlund. *Domain Decomposition Methods: Algorithms and Theory*. Springer, 2005.
- [44] O. B. Widlund. The development of coarse spaces for domain decomposition algorithms. In *Domain Decomposition methods in science and engineering XVIII*, pages 241–248. Springer, 2009.
- [45] M. Zworski. *Semiclassical analysis*. American Mathematical Society Providence, RI, 2012.