SCATTERING FOR THE DIRICHLET-TO-NEUMANN MAP AND THE STEKLOV SPECTRUM

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Abstract. We develop scattering theory for Dirichlet-to-Neumann (DtN) maps on asymptotically conic, unbounded domains $\Omega \subset \mathbb{R}^d$, that are only Lipschitz inside a compact set and whose boundary extends to infinity. Specifically, we prove a limiting absorption principle and demonstrate existence of the scattering matrix.

As an application of the theory, we show that the eigenvalues of Steklov-type problems on curvilinear polygons, $\mathcal{P}$, are well approximated by the eigenvalues of a Dirac operator on $\partial\mathcal{P}$ whose transmission coefficients between edges meeting at a corner are given by the scattering matrix for DtN maps on asymptotically conic domains associated to the corners. Up to an arbitrary loss in the power of the eigenvalue, the error in the approximation is optimal among *all* possible asymptotics that use only local information about $\partial\mathcal{P}$.

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1. Introduction

Let $\mathcal{P} \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth, Lipschitz boundary. We study the following Steklov-type spectral problem on $\mathcal{P}$. Fix $M \geq 0$, and consider the non-trivial solutions $u_\sigma \in H^1(\mathcal{P})$ of

$$\begin{cases} (-\Delta + M^2)u_\sigma = 0 & \text{in } \mathcal{P} \\ \partial_\nu u_\sigma = \sigma u_\sigma & \text{on } \partial\mathcal{P}, \end{cases} \tag{1.1}$$

where ν is the outward pointing unit normal to $\partial\mathcal{P}$. For a Lipschitz domain, $\mathcal{P}$, there is a sequence $0 \leq \sigma_0(\mathcal{P}, M) \leq \sigma_1(\mathcal{P}, M) \leq \ldots$, $\sigma_j(\mathcal{P}, M) \to \infty$ as $j \to \infty$ such that (1.1) has a nontrivial solution if and only if $\sigma = \sigma_j(\mathcal{P}, M)$ for some j: see for example [AM12]. In fact, $\{\sigma_j(\mathcal{P}, M)\}_{j=0}^\infty$ are the eigenvalues of the associated Dirichlet-to-Neumann (DtN) map, $\Lambda_M^{\mathcal{P}} : H^1(\partial\mathcal{P}) \to L^2(\partial\mathcal{P})$ with the corresponding eigenfunctions $u_{\sigma_j(M)}|_{\partial\mathcal{P}}$.

While we develop tools which will be useful for the study of (1.1) in any dimension, our main application to this problem is in dimension 2.

Definition 1.1 (Curvilinear polygons)**.** We say $\mathcal{P} \subset \mathbb{R}^2$ is a *curvilinear polygon* if it is a bounded, piecewise smooth, Lipschitz domain. We then decompose the boundary as $\partial\mathcal{P} = \sqcup_{l=1}^L \Gamma^l$ and $\Gamma^l = \cup_{j=0}^{J_l - 1} \Gamma_j^l$. We then identify Γ_j^l with $[x_j^l, x_{j+1}^l]$ using the arc length parametrized map $\gamma_j^l : [x_j^l, x_{j+1}^l] \to \Gamma_j^l$ and denote by α_j^l the angle between $-(\gamma_j^l)'(x_j^l)$ and $(\gamma_{j-1}^l)'(x_j^l)$. Let

$$\alpha_+ = \max\{\alpha_j^l : 1 \leq l \leq L, 1 \leq j \leq J_l\} \tag{1.2}$$

be the maximal corner angle of $\partial\mathcal{P}$.

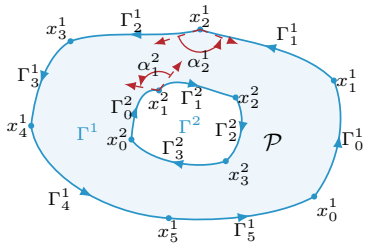


FIGURE 1. A curvilinear polygon $\mathcal{P}$ with $\partial\mathcal{P} = \Gamma^1 \sqcup \Gamma^2$ as parametrized arcs.

On the boundary of a curvilinear polygon, we define the Steklov Dirac operator.

Definition 1.2 (The Steklov Dirac operator)**.** Let $\mathcal{P}$ be a curvilinear polygon and $\{\Sigma_j^l(\lambda) : 1 \leq l \leq L, 1 \leq j \leq J_l\}$ be 2×2 unitary matrices such that for all $\lambda \in \mathbb{R}$,

$$\sigma_3 \overline{\Sigma_j^l(\lambda)} \sigma_3 = -\Sigma_j^l(-\lambda), \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

Define the following λ-dependent family of Dirac operators on $\partial\mathcal{P}$: $\mathscr{D}(\mathcal{P},\Sigma_j^l) : L^2(\partial\mathcal{P};\mathbb{C}^2) \to L^2(\partial\mathcal{P};\mathbb{C}^2)$ is the unbounded operator defined by

$$\mathscr{D}_\lambda(\mathcal{P},\Sigma_j^l)\begin{pmatrix}u_+\\u_-\end{pmatrix} = \begin{pmatrix}D_x & 0\\0 & -D_x\end{pmatrix}\begin{pmatrix}u_+\\u_-\end{pmatrix}, \tag{1.4}$$

$$\mathrm{Dom}(\mathscr{D}_\lambda(\mathcal{P},\Sigma_j^l)) = \Big\{u\in\bigoplus_{j,l} H^1((x_j^l,x_{j+1}^l);\mathbb{C}^2) : \Sigma_j^l(\lambda)\begin{pmatrix}u_+(x_j^l-0)\\u_-(x_j^l+0)\end{pmatrix} = \begin{pmatrix}u_-(x_j^l-0)\\u_+(x_j^l+0)\end{pmatrix}\Big\}. \tag{1.5}$$

Denote

$$\mathrm{char}(\mathscr{D}(\mathcal{P},\Sigma_j^l)) = \{\lambda\in\mathbb{R} : \ker\big(\mathscr{D}_\lambda(\mathcal{P},\Sigma_j^l)-\lambda\big)\neq\{0\}\}. \tag{1.6}$$

We say $\mathscr{D}(\mathcal{P},\Sigma_j^l)$ is a *Steklov Dirac operator with matching* Σ_j^l, if we can arrange

$$\mathrm{char}(\mathscr{D}(\mathcal{P},\Sigma_j^l)) = \{\cdots\le\lambda_{-2}(\mathscr{D}(\mathcal{P},\Sigma_j^l))\le\lambda_{-1}(\mathscr{D}(\mathcal{P},\Sigma_j^l))\le\lambda_0(\mathscr{D}(\mathcal{P},\Sigma_j^l))\le\lambda_1(\mathscr{D}(\mathcal{P},\Sigma_j^l))\le\cdots\} \tag{1.7}$$

as a countable bi-sequence where λ_n is repeated with multiplicity so that $\lambda_n=-\lambda_{-n-1}$ for $n\ge 0$.

If

$$\Sigma_j^l(\lambda) = \Sigma_{1,0}^{\mathcal{W}_{\alpha_j^l}},\ \Sigma_{1,0}^{\mathcal{W}_\alpha} = \begin{pmatrix}-i\cos\frac{\pi^2}{2\alpha} & \sin\frac{\pi^2}{2\alpha}\\ \sin\frac{\pi^2}{2\alpha} & -i\cos\frac{\pi^2}{2\alpha}\end{pmatrix}, \tag{1.8}$$

where $\Sigma_{1,0}^{\mathcal{W}_\alpha}$ is the scattering matrix for the infinite wedge of angle α with $M=0$ (see Theorem 1.8), then $\mathscr{D}_\lambda(\mathcal{P},\Sigma_j^l)$ is also λ-independent and we denote $\mathscr{D}^0=\mathscr{D}_\lambda(\mathcal{P},\Sigma_j^l)$ and call it the *basic Steklov Dirac operator on* $\mathcal{P}$. In this case, we have $\mathrm{spec}(\mathscr{D}^0)=\mathrm{char}(\mathscr{D}(\mathcal{P},\Sigma_j^l))$.

Theorem 1.3. *Let $\mathcal{P}\Subset\mathbb{R}^2$ be a curvilinear polygon, $M_0\ge 0$, and $k_0\in\mathbb{N}$. Then for all $\epsilon>0$, there is $C>0$ such that for $n\ge 0$, $M\in[0,M_0]$ we have*

$$|\sigma_n(\mathcal{P})-\lambda_n(\mathscr{D}^0)| < C\langle\sigma_n(\mathcal{P})\rangle^{-1+\epsilon}, \tag{1.9}$$

and there are $\{\Sigma_j^{l,k_0}(\lambda)\}$, with $\Sigma_j^{l,k_0}(\lambda)$ depending only on the first (k_0+1) derivatives of γ_j^l and γ_{j-1}^l at x_j^l such that for $n\ge 0$, $M\in[0,M_0]$ we have

$$\Big|\sigma_n(\mathcal{P})-\sqrt{\lambda_n^2(\mathscr{D}(\mathcal{P},\Sigma_j^{l,k_0}))+M^2}\Big| \le C\big(\langle\sigma_n(\mathcal{P})\rangle^{-\frac{2\pi}{\alpha_+}+\epsilon}+\langle\sigma_n(\mathcal{P})\rangle^{-1-k_0+\epsilon}+M^{2-\epsilon}\langle\sigma_n(\mathcal{P})\rangle^{-2+\epsilon}\big). \tag{1.10}$$

Furthermore we have:

(i) If the derivatives of order $2\le k\le k_0+1$ of γ_j^l and γ_{j-1}^l at x_j^l vanish, then $\Sigma_j^{l,k_0}=\Sigma_{1,0}^{\mathcal{W}_{\alpha_j^l}}$.

(ii) The estimate

$$\|\Sigma_j^{l,k_0}(\lambda)-\Sigma_j^{l,k_0'}(\lambda)\|\le C\langle\lambda\rangle^{-1-k_0+\epsilon},\qquad \big\|\partial_\lambda\big(\Sigma_j^{l,k_0}-\Sigma_j^{l,k_0'}\big)(\lambda)\big\|\le C\langle\lambda\rangle^{-(1+k_0)/2+\epsilon} \tag{1.11}$$

holds for any $0\le k_0<k_0'\le\lceil\frac{2\pi}{\alpha_+}\rceil-1$.

Theorem 1.3 implies that, when $M=0$, there is a Dirac operator on $\partial\mathcal{P}$ with matching conditions determined by the curvatures at corners so that the error in (1.10) is $O(\sigma^{-\frac{2\pi}{\alpha_+}+0})$. Moreover, if the curvatures vanish to high enough order at each corner, then the error in (1.9) can be improved to $O(\sigma^{-\frac{2\pi}{\alpha_+}+0})$.

In a recent paper, the first and last authors show that when $M = 0$ (1.10) is optimal (modulo the ϵ loss in the power) among *all* possible asymptotic expressions for σ_n that use only local information about the boundary of $\mathcal{P}$ [GW26, Theorem 1.2]. In fact, the authors construct a curvilinear polygon $\mathcal{P}_{\alpha_+}$ with a single corner of angle α_+ such that there are $r > 0$, $x_0, y_0 \in \mathcal{P}_{\alpha_+}$ satisfying

$$\limsup_{n\to\infty} \Big|\sigma_n\big(\mathcal{P}_{\alpha_+} \setminus B(x_0, r)\big) - \sigma_n\big(\mathcal{P}_{\alpha_+} \setminus B(y_0, r)\big)\Big|\sigma_n^{\frac{2\pi}{\alpha_+}} > 0, \tag{1.12}$$

where $B(x, r)$ denotes the ball of radius r centered at x. Figure 2 shows numerically computed differences between the DtN spectrum and Steklov Dirac spectrum in a variety of curvilinear polygons.

Previous work on the Steklov spectrum for curvilinear polygons.

When $M = 0$, and the angles of the curvilinear polygon are less than π, (1.1) was extensively studied in [LPPS22b, LPPS22a]. Under these additional restrictions the authors proved a substantially weaker version of Theorem 1.3 where the error was given by $\sigma_n^{-\frac{1}{2}(\min(\frac{\pi}{\alpha_+},\frac{3}{2})-1)+0}$. In order to do this, the authors of [LPPS22b, LPPS22a] used a classical result on the sloping beach problem from [Pet50] to construct a large family of quasimodes for (1.1) and, through a series of ingenious comparisons, were able to show that these quasimodes, in fact, form a basis for $L^2(\partial\mathcal{P})$ modulo a finite dimensional set. Thus, they concluded that all large enough Steklov eigenvalues on $\mathcal{P}$ lie close to one quasieigenvalue.

The method of proof used in [LPPS22b, LPPS22a] is highly tuned to exactly the Steklov problem and small perturbations of that problem destroy much of the proof. For instance, their methods do not provide any information when one takes $M > 0$. Moreover, there are a number of places in the proof which appear to be 'miracles'; for instance some poorly understood cancellations in certain quasimodes. Even for curvilinear polygons with $M = 0$, this sensitivity means that the methods of [LPPS22b, LPPS22a] could not treat angles $\alpha > \pi$.

To move toward a study of the Steklov problem on piecewise smooth Lipschitz domains in higher dimensions, more robust methods are needed. For instance, if one considers a domain with an $\mathbb{S}^1$ symmetry in $\mathbb{R}^3$, separation of variables with angular frequency, n, immediately leads to the Steklov problem with $M = |n|$. The methods developed in this paper are applicable much more broadly than those in [LPPS22b, LPPS22a]. Specifically for $M > 0$, they allow us to treat any planar, curvilinear polygon, and, although we do not pursue them here, even some other types of problems such as the interface problem of [BP13, (1.1)], [AK11, (5.3)] (see also [Bou92, (11)]) and Steklov problems on Riemannian manifolds.

Our methods are based on the scattering theory for the DtN map in non-compact domains, Ω, with boundary extending to infinity. We view (1.1) for large σ as a problem on smooth, open curves connected by the scattering problem for the DtN map on an infinite wedge at each corner of the domain (see Figure 3). While the DtN map is *far* from a local operator, it is nevertheless possible to separate the effects of the corners and the smooth sides at high energy.

Scattering theory for the Dirichlet-to-Neumann Map

To describe the geometry in which we work, we need two definitions.

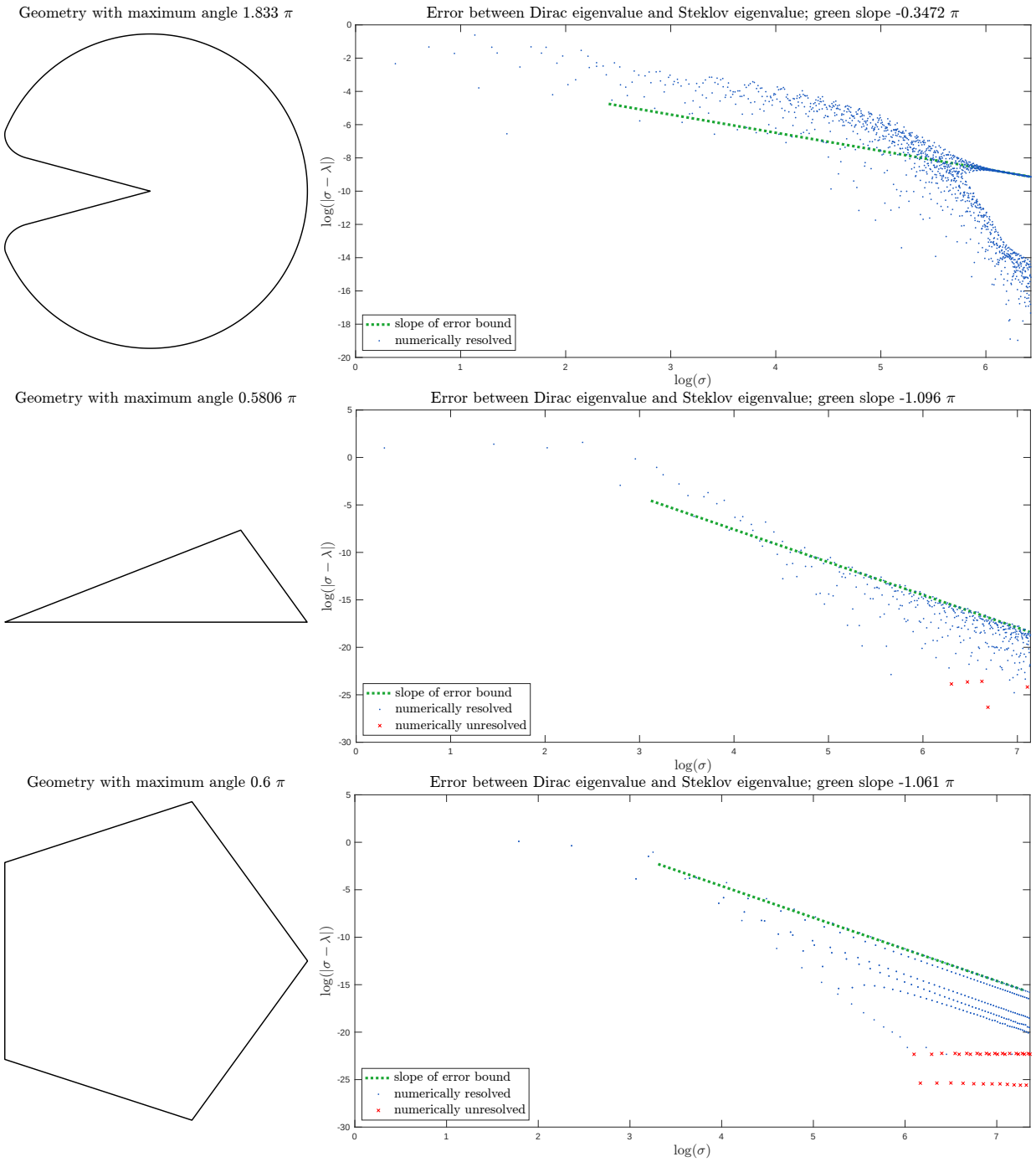


FIGURE 2. The left column depicts a curvilinear polygon and the right column shows the corresponding plot of $\log|\sigma_n(\mathcal{P}) - \lambda_n(\mathscr{D}^0)|$ v. $\log|\sigma_n(\mathcal{P})|$ (with $M = 0$) in the blue dots. Each curvilinear polygon has one corner at which all curvatures vanish. Any red x's are computed errors where the numerical resolution error was likely higher than the gap between the two spectra. In each figure, the green, dotted line is the upper bound on this error in Theorem 1.3.

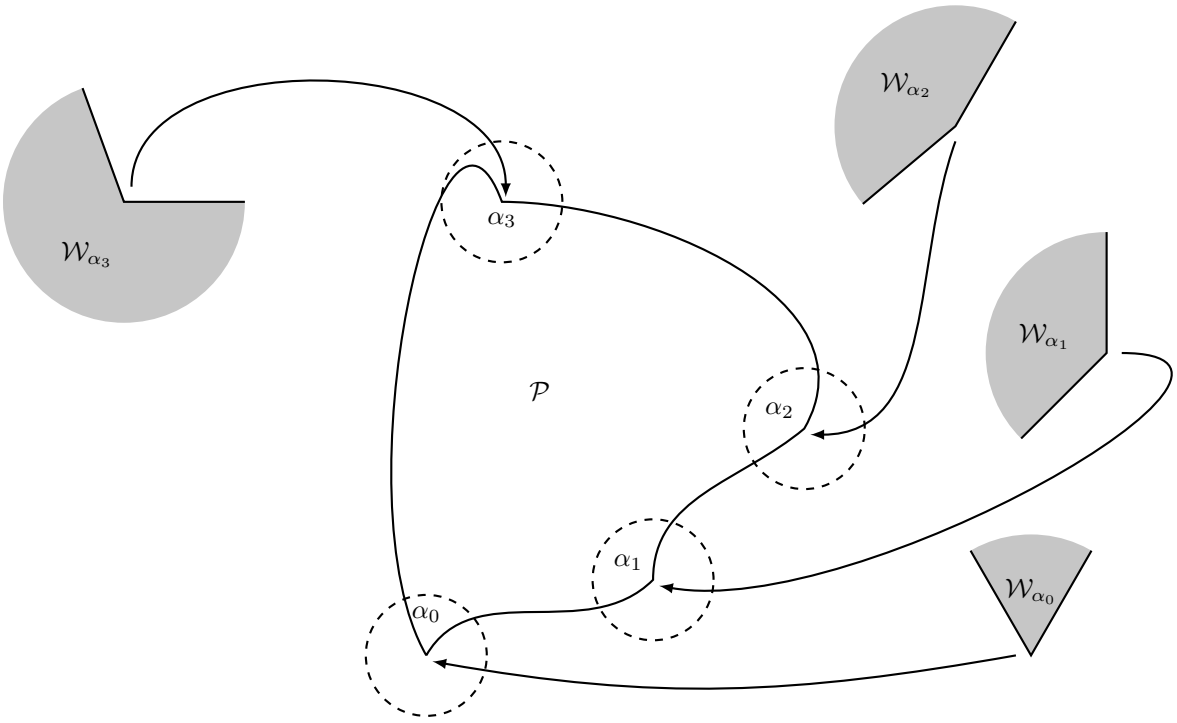


FIGURE 3. A curvilinear polygon $\mathcal{P}$ having angles $\alpha_0, \ldots, \alpha_3$ with the infinite wedge of angle α_j, $\mathcal{W}_{\alpha_j}$, inserted at the j^{th} corner.

Definition 1.4. Let (M, g) be a complete Riemannian manifold of dimension $d-1$, such that $M = \cup_{l=1}^{L} M_l \cup K$ where K is compact, M_l are diffeomorphic to $(0,\infty)_r \times Y_l$ for some smooth, compact $(d-2)$-dimensional manifolds, Y_l. Then, M can be compactified to $\overline{M}$ using the coordinate $\rho = 1/r$ in M_l. We allow K to be Lipschitz, that is, locally defined using charts that are only Lipschitz. We say that M is *a scattering manifold* if there is a choice ρ of defining function for $\partial\overline{M}_l$ and a collar neighborhood $[0,\epsilon)_\rho \times Y_l$ of $\partial\overline{M}_l$ such that

$$g = \frac{d\rho^2}{\rho^4} + \frac{h(\rho)}{\rho^2}, \tag{1.13}$$

where $h : [0,\epsilon) \to \mathrm{Sym}^2(T^*Y_l)$ is a smooth family of Riemannian metrics on Y_l.

Definition 1.5. We say that a domain Ω in $\mathbb{R}^d$ with boundary $\partial\Omega$ is an *asymptotically conic Lipschitz domain with cross section* Y if Ω is a Lipschitz domain, $Y \subset \mathbb{S}^{d-1}$ is open with smooth boundary ∂Y, and there are $R_0 > 0$ and $\gamma \in C^\infty((0,1] \times \overline{Y}; \mathbb{S}^{d-1})$, with $\gamma(\rho, \cdot)$ as an injective map, satisfying the following conditions:

(1) $\partial\Omega \setminus B(0,R_0) = \{\rho^{-1}\gamma(\rho,\omega) \,:\, \rho \in (0,1],\, \omega \in \partial Y\} \cap \mathbb{R}^d \setminus B(0,R_0)$,

(2) When $d = 2$, we ask $\gamma \in S^0$, that is for $\rho \in (0,1]$, $\omega \in \overline{Y}$,

$$|(\rho\partial_\rho)^\alpha \partial_\omega^\beta \gamma(\rho,\omega)| \leq C_{\alpha\beta},\ \lim_{\rho\to 0} \|D(\rho^{-1}\gamma)(\rho,\omega) - D(\rho^{-1}\iota)(\rho,\omega)\|_{L^\infty(Y)} = 0. \tag{1.14}$$

(3) When $d > 2$,

$$\gamma \in C^\infty([0,1]\times\overline{Y};\mathbb{S}^{d-1}),\ \gamma(0,\omega) = \omega \text{ for all } \omega \in \overline{Y}. \tag{1.15}$$

Remark 1.6. Notice that when $d = 2$, the requirement that $(\partial\Omega, g)$ be asymptotically conic is always satisfied since $\partial\Omega$ is 1-dimensional.

We now give two consequences of our second main theorem whose statement we postpone to Section 5. The first proves the existence of the limiting absorption resolvent and hence absolute continuity of the spectrum away from eigenvalues.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^d$ be an asymptotically conic domain and $\delta > 0$. Then there is $N \in \{0,1,\dots\}\cup\{\infty\}$ such that the set of eigenvalues, $\{\omega_n\}_{n=1}^N$ of Λ_M^Ω in $[M+\delta,\infty)$ is discrete. Moreover, for $\omega \notin \{\omega_n\}_{n=1}^N$, the limiting absorption resolvent, $(\Lambda_M^\Omega - \omega \pm i0)^{-1} : L^2_{\text{comp}} \to L^2_{\text{loc}}$ exists. Furthermore, if $d = 2$ and Ω is simply connected and smooth away from a single point, or $d > 2$ and Ω is scaling invariant, then $N < \infty$.*

Our next theorem demonstrates the existence of the scattering matrix.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^d$ be an asymptotically conic Lipschitz domain with cross-section Y and $\omega > 0$. Then for all $g^- \in L^2(\partial Y)$, there is a solution u to*

$$(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2})u = 0 \tag{1.16}$$

such that, in the coordinates of Definition 1.4,

$$u - \rho^{\frac{d-2}{2}}(e^{-i\omega/\rho}g^-(y) + e^{i\omega/\rho}g^+(y)) = O_{L^2(\partial\mathcal{W}_Y)}(\rho^{\frac{d}{2}}), \tag{1.17}$$

for some $g^+ \in L^2(\partial Y)$. Moreover, letting Π_ω denote the orthogonal projector onto the eigenspace of Λ_M^Ω at $\sqrt{\omega^2+M^2}$, the function $(I-\Pi_\omega)u$ is unique. In addition, the scattering matrix, *$\Sigma_{\omega,M}^\Omega : L^2(\partial Y) \to L^2(\partial Y)$ given by $\Sigma_{\omega,M}^\Omega : g^- \mapsto g^+$, is unitary.*

Remark 1.9. Theorems 1.7 and 1.8 are simplified versions of the main theorems from Part 1 on scattering theory, Proposition 5.18 and Proposition 6.3 respectively.

1.1. **Sketch of the strategy and outline of the paper.**

We now give a broad overview of the strategy of proof. We emphasize that, in addition, each of the sections of the main body starts with a sketch of what is done in the corresponding section.

Scattering for the Dirichlet-to-Neumann map on AC Lipschitz domains in any dimension

The first part of the article, Sections 2 to 6, is dedicated to the solution theory for the Dirichlet problem and the scattering theory of the DtN map on AC Lipschitz domains, Ω. When $M > 0$, it is clear that the equation

$$(-\Delta + M^2)u = f \text{ in } \Omega, \qquad u|_{\partial\Omega} = 0 \tag{1.18}$$

has a unique solution in $\cup_N \langle x\rangle^N H^1$ for any $f \in \cup_N \langle x\rangle^N H^{-1}$. However, when $M = 0$, this is no longer the case and, indeed, even solutions with $f \in H^{-1}$ are problematic.

Section 2 studies solutions to (1.18), for $M \in [0, M_0]$ on uniformly Lipschitz domains (i.e. those for which the Lipschitz norm of the boundary is controlled uniformly to infinity). To do this, we introduce spaces

$$\dot{X}^l_m(\Omega) = \Big\{ u \in H^1_{\mathrm{loc}}(\Omega) \,:\, \big\langle d(x,\partial\Omega)\big\rangle^{m-l}\langle x\rangle^l \nabla u \in L^2,\, \big\langle d(x,\partial\Omega)\big\rangle^{m-l-1}\langle x\rangle^l u \in L^2,\, u|_{\partial\Omega} = 0 \Big\},$$
$$Y^{-l}_{-m} = (\dot{X}^l_m)^*. \tag{1.19}$$

and show that for certain ranges of m and l depending on the geometry of Ω at infinity,

$$(-\Delta + M^2) : \{u \in \dot{X}^l_m \,:\, (-\Delta + M^2)u \in Y^l_m\} \to Y^l_m$$

is invertible for all $M \geq 0$ and has uniform estimates for $M \in [0, M_0]$ for any fixed $M_0 > 0$. We also show that the solution operator with $M = 0$, $\mathcal{R}^\Omega_0 : Y^l_m \to \dot{X}^l_m$, is a weak limit of those $\mathcal{R}^\Omega_M : Y^l_m \to \dot{X}^l_m$ with $M > 0$ and define the corresponding harmonic extension operators solving

$$(-\Delta + M^2)\mathcal{H}^\Omega_M f_\partial = 0, \qquad \mathcal{H}^\Omega_M f_\partial|_{\partial\Omega} = f_\partial. \tag{1.20}$$

We then prove boundedness of $\Lambda^\Omega_M : H^{\frac{1}{2},l}(\partial\Omega) \to H^{-\frac{1}{2},l}(\partial\Omega)$. The estimates on the solution operators $\mathcal{R}^\Omega_M$ (see Lemma 2.15) rely on Hardy inequalities controlling $\||x|^{-1}u\|_{L^2}$ and $\|d(x,\partial\Omega)^{-1}u\|_{L^2}$ by $\|\nabla u\|_{L^2}$ and the construction of a delicate Carleman weight. The range of allowed values of m and l is related to the optimal constants in these Hardy inequalities and we show that these ranges are optimal. For future reference, note that on AC Lipschitz domains, these ranges are

$$|m| < \mathfrak{r}^\infty_\Omega, \qquad |m - l| < \tfrac{1}{2}, \qquad \mathfrak{r}^\infty_\Omega = \lim_{r_0\to\infty} \inf \Big\{ \frac{\|\nabla u\|_{L^2(\Omega)}}{\|\frac{u}{|z|+r_0}\|_{L^2(\Omega)}} : u \in \dot{H}^1(\Omega) \setminus \{0\} \Big\}. \tag{1.21}$$

An important example to keep in mind is the half space $\mathbb{R}^d_+ = \{(x_0, x) \in \mathbb{R}^d : x_0 > 0\}$, where the DtN map $\Lambda^{\mathbb{R}^d_+}_M$ is a Fourier multiplier

$$\Lambda^{\mathbb{R}^d_+}_M = \sqrt{D^2 + M^2}. \tag{1.22}$$

It is then easy to check that

$$\Lambda^{\mathbb{R}^d_+}_0 : H^{s+1,l}(\mathbb{R}^{d-1}) \to H^{s,l}(\mathbb{R}^{d-1}),\ s \in \mathbb{R},\ |l| < \tfrac{d+1}{2} = \mathfrak{r}^\infty_{\mathbb{R}^d_+} + \tfrac{1}{2}, \tag{1.23}$$

and cannot be extended to a wider range of l. In particular, this shows that the thresholds for m and l in (1.21) cannot be improved.

With the basic solution theory for the Dirichlet problem in hand, Section 3 is devoted to the construction of a second microlocal pseudodifferential approximation to the DtN map on an AC Lipschitz domain.

Indeed, (1.22) becomes singular at $|\xi| = 0$, $|x| = \infty$ as $M \to 0$ and, as a result the DtN map cannot be pseudodifferential modulo a smoothing and rapidly decaying error. Because of this, we add an artificial mass term to $-\Delta$ and, for any $0 \leq \delta < 1$, construct a parametrix modulo a smoothing and decaying operator for an artificial DtN map, P, corresponding to

$$-\Delta + M^2 + \langle x\rangle^{-2\delta}. \tag{1.24}$$

In particular, using a factorization argument, we show that P is a pseudodifferential operator with symbol

$$\sigma(P) = \langle x\rangle^{-\delta}\big\langle \langle x\rangle^{\delta}\sqrt{|\xi|_g^2 + M^2}\big\rangle, \tag{1.25}$$

where g is the metric induced on $\partial\Omega$. (For the construction of the second microlocal calculus in which this lives, see Appendix A.) The solution theory from Section 2 then allows us to control the error between P and the true DtN map. One consequence of this approximation is that the harmonic extension and DtN maps on AC Lipschitz domains are, in fact, pseudodifferential away from the zero frequency and away from the Lipschitz singularities. Thus, they can be extended to data that may have any polynomial growth in x so long as it is away from the zero frequency.

Section 4 contains a more detailed analysis of solutions to (1.18) and (1.20) on AC Lipschitz domains. More precisely, we study the behavior of $\mathcal{R}_M^\Omega$ acting on functions that have non-trivial oscillations in the directions tangent to the boundary. To do this, we commute vector fields $\mathcal{V}$ that are tangent to $\partial\Omega$ and whose coefficients may grow like $\langle x\rangle$ through $\mathcal{R}_M^\Omega$. This analysis is ultimately used in Proposition 4.22 to control the error $\chi(\Lambda_M^\Omega - P)$. Specifically, we show that this error is rapidly decaying on high frequencies, maps high frequencies to high frequencies, low frequencies to low frequencies, and decays like $\langle x\rangle^{-\mathfrak{r}_\Omega^\infty - \frac{1}{2}}$ at low frequencies.

Remark 1.10. One can check $\langle x\rangle^{-\mathfrak{r}_\Omega^\infty - \frac{1}{2}}$ is the best possible decay rate for the low frequencies using explicit computations on the infinite planar wedge of any angle $0 < \alpha < 2\pi$.

At this point, we have a very accurate description of Λ_M^Ω microlocally near physical infinity and away from frequency 0. Except for the non-local part of Λ_M^Ω near 0 frequency, the Fredholm analysis of $\Lambda_M^\Omega - \omega$ is close to that of the Laplacian on an asymptotically flat manifold. In Section 5 we use our microlocal description together with radial point estimates to obtain Fredholm estimates and a limiting absorption resolvent in certain variable order Sobolev spaces (see Proposition 5.18). Then, in Section 6, we construct the scattering matrix associated to the DtN map (see Proposition 6.3).

Analysis of the Steklov problem on piecewise smooth, planar domains

Sections 7 to 10 apply the scattering theory developed in part 1 to the Steklov problem on piecewise smooth Lipschitz domains – resulting in Theorem 1.3.

The starting point is Section 7, which contains a more detailed study of the scattering matrix on simply connected AC Lipschitz domains in $\mathbb{R}^2$ whose boundary is smooth away from one single point (i.e. infinite wedges with possibly curved sides).

We start by showing that, at high energy, ω, the scattering matrix on a curved wedge $\mathcal{W}_\alpha^1$ with angle α is close to that of any other wedge $\mathcal{W}_\alpha^2$ of the same angle and that, if enough curvatures of $\mathcal{W}_\alpha^1$ and $\mathcal{W}_\alpha^2$ agree at the singularity, this difference is controlled by $\omega^{-\frac{2\pi}{\alpha}+0} + M^2\omega^{-2+0}$. To do this, we transplant the distorted plane waves from $\mathcal{W}_\alpha^1$ onto $\mathcal{W}_\alpha^2$. More precisely, let u_1 satisfy

$$(\Lambda_M^{\mathcal{W}_\alpha^1} - \sqrt{\omega^2 + M^2})u_1 = 0, \tag{1.26}$$

and let

$$u_1 = u_1^{\text{high}} + u_1^{\text{low}} \tag{1.27}$$

be the decomposition of u_1 into high and low frequencies. We then set

$$u_2(x) = \chi(x)\big(F^* u_1^{\text{low}}\big)(x) + \chi(\omega^{1-\delta}x)\big(F^* u_1^{\text{high}}\big)(x) + (1-\chi(\omega^{1-\delta}x))u_{\text{match}}^{\text{high}}(x), \tag{1.28}$$

where $F : \mathcal{W}_\alpha^2 \to \mathcal{W}_\alpha^1$, and $u_{\text{match}}^{\text{high}}$ are the distorted plane waves on the smooth edges of $\mathcal{W}_\alpha^2$ with scattering data matching that of $F^* u_1^{\text{high}}$. That is, we truncate the low frequencies of u_1 at distance from the corner 1 and match the high frequency parts at distance $\omega^{-1+\delta}$ for some $0 < \delta \ll 1$. We then show that $(\Lambda_M^{\mathcal{W}_\alpha^2} - \sqrt{\omega^2 + M^2})u_2$ is small and apply the limiting absorption resolvent to correct u_2 to an exact distorted plane wave and hence show that the scattering matrix on $\mathcal{W}_\alpha^2$ is close to that on $\mathcal{W}_\alpha^1$.

Next, we compare the scattering matrix on the wedge $\mathcal{W}_\alpha$ with some positive mass $M > 0$ to that with $M = 0$. In this analysis, we use a change of domain formula

$$\chi \Lambda_M^{\mathcal{W}_\alpha} = \Lambda_M^{\mathbb{R}^2_+}\chi + \partial_\nu \mathcal{R}_M^{\mathbb{R}^2_+}[-\Delta, \chi]\mathcal{H}_M^{\mathcal{W}_\alpha}, \tag{1.29}$$

proved in Section 4 together with very precise knowledge of Λ_M^Ω, when Ω is a smooth asymptotically conic domain, to study the perturbation $\Lambda_M^{\mathcal{W}_\alpha} - \Lambda_0^{\mathcal{W}_\alpha}$.

Section 8 determines the a priori structure of Steklov eigenfunctions on curvilinear polygons, showing in particular that they are, up to small low frequency errors (of size $\sigma^{-\frac{\pi}{\alpha}-1+0}$), (distorted) plane waves. This analysis proceeds in two steps. First, using the fact that, away from the corners of $\mathcal{P}$, the DtN map is a pseudodifferential operator with symbol $|\xi|_g$, we are able to show that Steklov eigenfunctions are (distorted) plane waves modulo an error of size σ^{-1} at low frequency. With this in hand, similar to how we constructed distorted plane waves on one infinite wedge from another, we are able to construct a distorted plane wave on a curved, infinite wedge that agrees with the domain near the corner from the Steklov eigenfunction near a corner. Our Fredholm estimates then allow us to obtain precise control on this distorted plane wave and hence on u. In particular, they show that parts of the eigenfunction that are 'incoming' and 'outgoing' from the corner are nearly related by the scattering matrix for the infinite wedge.

In Section 9, we construct a Grushin problem for the DtN map on a curvilinear polygon so that the effective Hamiltonian (approximately) characterizes the eigenvalues of the Dirac operator from Theorem 1.3. Since we know apriori that Steklov eigenfunctions behave like plane waves on smooth parts of the boundary, the key component in the analysis is using the relationship between the waves incoming to a corner and those outgoing from that corner. This is achieved by the same quasimode construction.

Section 10 then shows that this error does not significantly change the implied spectrum and uses a deformation argument to show that the labeling of the Dirac operator's eigenvalues exactly matches that of the Steklov eigenvalues.

Appendices

Section A is devoted to the construction of the second microlocal calculus $\Psi_{k,\delta}^{s,l}$ used in this paper. In Section B we prove a few results on the scattering geometry and properties of the Hardy and radial Hardy constants. Section C is devoted to the construction of DtN quasimodes in 2D AC smooth domains, and Section D is devoted to scattering matrices on 2D straight wedges.

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AI statement: This project began in 2023, before AI tools were widely available. In the later stages of the project ChatGPT 5 and 6 were used for proofreading, generating the numerical codes which produce Figure 2, and generating the TikZ code for most figures. In addition, in an earlier version of the proof the Carleman estimates in Lemma 2.15 were based only on Hardy inequalities of the form $\|u/\operatorname{dist}(x,\partial\Omega)\|_{L^2} \leq C\|\nabla u\|_{L^2}$. ChatGPT suggested replacing $1/\operatorname{dist}(x,\partial\Omega)$ by $1/|x|$. The authors then realized that this could be further improved to the current, optimal version. In addition, ChatGPT pointed out that one could replace Rouché's theorem, which was originally used in the proof of Lemma 10.5 when $k_0 = 0$, by homotopy invariance of the winding number [Hat02, p. 134 property (c)]. The manuscript was entirely written by the authors.

Part 1. The scattering theory for the DtN map on AC Lipschitz domains

2. The harmonic extension in uniformly Lipschitz domains

The goal of this section is to study solutions of

$$(-\Delta + M^2)u = f, \text{ in } \Omega \qquad u|_{\partial\Omega} = g, \tag{2.1}$$

uniformly as $M \to 0$, either for domains Ω that may grow or even be unbounded.

Outline of the strategy

We work on domains whose boundaries are uniformly Lipschitz in a sense made precise in Definition 2.1. When $M > 0$ or Ω is bounded, the solution theory for (2.1) is simple. Indeed (2.1) is uniquely solvable for f and g with any amount of polynomial growth/decay. However, as Ω grows and $M \to 0$, the implicit constants in this statement degenerate.

To develop a uniform solution theory, we introduce spaces in which derivatives of u decay faster than u itself (see Definition 2.8). Hardy inequalities of the form

$$\left\|\frac{u}{f}\right\|_{L^2(\Omega)} \leq \mathfrak{h}_f \|\nabla u\|_{L^2(\Omega)} \tag{2.2}$$

for different choices of (typically growing) f than allow us to prove apriori estimates in these spaces in Lemma 2.15 which in turn result in a uniform solution theory for (2.1) with $g = 0$ (see Proposition 2.16). These estimates are proved using a carefully designed Carleman-type weight that relates the sharp constants in the estimates (2.2) to the solution theory of (2.1). Specifically, the maximal growth/decay of f for which one can solve (2.1)uniquely are determined by these constants. We are then able to understand the case of $g \neq 0$ using arbitrary extension operators (see Proposition 2.19).

2.1. **Geometric Preliminaries and definition of solution operator.** We begin with a few geometric definitions

Definition 2.1 (Uniformly Lipschitz domains)**.** We say a Lipschitz domain $\Omega \subset \mathbb{R}^d$, is *uniformly Lipschitz* if its boundary consists of a countable family of codimension-1 hypersurfaces, $\partial\Omega = \cup_j \Gamma_j$. Here we ask each Γ_j to be, up to a rotation and translation, a Lipschitz hypograph with uniform control: that is, we assume there exist $c > 0$, $R_j > c$, $f_j : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

(1) injectivity radius: $\Gamma_j = \{(f_j(x), x) : x \in B(0, R_j)\}$, up to a rotation and translation;
(2) uniformly Lipschitz characters:$\|\nabla f_j\|_{L^\infty} \leq C$ independent of j;
(3) bounded local degree: $\sup_j \#\{k : \Gamma_j \cap \Gamma_k \neq \emptyset\} < \infty$;
(4) hypograph: $\{(x_0, x) \in \mathbb{R}^d : x \in B(0, R_j),\ f_j(x) < x_0 < c\} \subset \Omega$.

Bounded Lipschitz domains with a uniform Lipschitz constant are uniformly Lipschitz. Note all AC Lipschitz domains are uniformly Lipschitz. Throughout this section, $\Omega \subset \mathbb{R}^d$ denotes a uniformly Lipschitz domain.

We use $z \in \mathbb{R}^d$ as a global coordinate for Ω. We assume $\{z = 0\} \in \partial\Omega$. We consider set $\mathcal{V} \subset \partial\Omega$ to be a finite set of vertices that $\{z = 0\} \in \mathcal{V}$. Let the regularized vertex radius function $\mathbf{r}(z) \in C^\infty(\Omega; [\frac{1}{8}, \infty))$ that

$$\|\mathbf{r}(z) - (1 + \mathrm{dist}(z, \mathcal{V})^2)^{\frac{1}{2}}\|_{L^\infty(\Omega)} \leq \frac{1}{4},\ \mathbf{r} \geq \frac{1}{4} \text{ at } \mathcal{V},\ \|\nabla \mathbf{r}\|_{L^\infty} \leq 1,\ \ |\nabla^k \mathbf{r}(z)| \leq C_k \mathbf{r}^{1-k}(z) \quad (2.3)$$

for $k \in \mathbb{N}$. Let $\mathbf{d} : C^\infty(\Omega; [\frac{1}{8}, \infty))$ be a regularized distance function to $\partial\Omega$, that

$$\|\mathbf{d}(z) - (1 + \mathrm{dist}(z, \partial\Omega)^2)^{\frac{1}{2}}\|_{L^\infty(\Omega)} \leq \frac{1}{4},\ \mathbf{d} \geq \frac{1}{4} \text{ on } \partial\Omega,\ \|\nabla \mathbf{d}\|_{L^\infty} \leq 1,\ |\nabla^k \mathbf{d}(z)| \leq C_k \mathbf{d}^{1-k}(z) \quad (2.4)$$

for $k \in \mathbb{N}$. Without loss of generality $\mathrm{dist}(z, \partial\Omega) \leq \mathbf{d} \leq C\mathbf{r}$ for fixed $C \geq 1$. For $s, l \in \mathbb{R}$, we denote the Sobolev space by $H^s(\Omega)$, and denote the weighted Sobolev spaces $H^{s,l}(\Omega) = \mathbf{r}^{-l} H^s(\Omega)$. The trace-free spaces are denoted $\dot{H}^s(\Omega)$ and $\dot{H}^{s,l}(\Omega)$. The corresponding spaces over the boundary are $H^s(\partial\Omega)$ and $H^{s,l}(\partial\Omega) = \mathbf{r}^{-l} H^s(\partial\Omega)$. We define the Dirichlet Laplacian, $-\Delta : \dot{H}^1(\Omega) \to H^{-1}(\Omega)$, as the Riesz representation of the quadratic form $Q : \dot{H}^1(\Omega) \times \dot{H}^1(\Omega) \to \mathbb{C}$,

$$Q(u, v) := \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}. \quad (2.5)$$

We understand $\cdot|_{\partial\Omega}$ as the trace if the trace exists, or the non-tangential limit if the trace does not exist, in the latter case of which we abuse notation and call the non-tangential limit the trace. We define the harmonic extension and the Dirichlet resolvent:

Definition 2.2 (Harmonic extension)**.** Let Ω be a uniformly Lipschitz domain. Let $\mathcal{X}(\partial\Omega)$ and $\mathcal{Y}(\Omega)$ be Banach spaces of functions on $\partial\Omega$ and Ω. Let $M \geq 0$. We define $\mathcal{H}^\Omega_M : \mathcal{X}(\partial\Omega) \to \mathcal{Y}(\Omega)$ by sending f_∂ to $\mathcal{H}^\Omega_M f_\partial$ that

$$(-\Delta + M^2)\mathcal{H}^\Omega_M f_\partial = 0 \ \text{ in } \Omega,\ (\mathcal{H}^\Omega_M f_\partial)|_{\partial\Omega} = f_\partial, \quad (2.6)$$

if such $\mathcal{H}^\Omega_M f_\partial$ exists and is unique within $\mathcal{Y}(\Omega)$.

Definition 2.3 (Dirichlet resolvent)**.** Let Ω be a uniformly Lipschitz domain. Let $\mathcal{X}(\Omega)$ and $\mathcal{Y}(\Omega)$ be Banach spaces of functions on Ω. Let $M \geq 0$. We define $\mathcal{R}^\Omega_M : \mathcal{X}(\Omega) \to \mathcal{Y}(\Omega)$ by sending f to $\mathcal{R}^\Omega_M f$ that

$$(-\Delta + M^2)\mathcal{R}^\Omega_M f = f \ \text{ in } \Omega,\ (\mathcal{R}^\Omega_M f)|_{\partial\Omega} = 0, \quad (2.7)$$

if such $\mathcal{R}^\Omega_M f$ exists and is unique within $\mathcal{Y}(\Omega)$.

We start with a formula that corrects any extension into a harmonic extension:

Lemma 2.4 (Harmonic extension via correction)**.** *Let $\Omega \subset \mathbb{R}^d$ be a uniformly Lipschitz domain. Let $\mathcal{X}(\partial\Omega)$ and $\mathcal{Y}(\Omega)$ be Banach spaces of functions on $\partial\Omega$ and Ω. Let $M \geq 0$. Given any extension operator $E : \mathcal{X}(\partial\Omega) \to \mathcal{Y}(\Omega)$ that $E \cdot |_{\partial\Omega} = \mathrm{I}$, we have*

$$\mathcal{H}^\Omega_M g = Eg - \mathcal{R}^\Omega_M(-\Delta + M^2)Eg, \tag{2.8}$$

if all three terms exist and are unique in $\mathcal{Y}(\Omega)$. Here we understand the $\cdot|_{\partial\Omega}$ as the non-tangential limit of $\cdot$ approaching $\partial\Omega$, which we require it to agree with the trace when the trace exists.

Proof. It suffices to note $(-\Delta + M^2)\mathcal{R}^\Omega_M = \mathrm{I}$ by definition. □

Our main goals in this section are to establish uniform boundedness of $\mathcal{R}^\Omega_M : Y^l_m(\Omega) \to \dot{X}^l_m(\Omega)$ in Proposition 2.16, and $\mathcal{H}^\Omega_M : H^{\frac{1}{2},l}(\partial\Omega) \to X^l_m(\Omega)$ in Proposition 2.19, as $M \to 0$, where X^l_m, Y^l_m (defined in Definition 2.8) are suitably weighted Sobolev-type spaces with weight parameters m, l in intervals controlled by the Hardy constants of Ω.

2.2. **The elliptic theory on bounded uniformly Lipschitz domains.** In this subsection, we include the properties of harmonic extension and Dirichlet/Neumann resolvents on bounded uniformly Lipschitz domains.

Lemma 2.5 (Dirichlet resolvent on bounded domains)**.** *Let $\Omega \Subset \mathbb{R}^d$ be a bounded uniformly Lipschitz domain. Let $M_0 > 0$. Let $0 < s < 1$. There is $C > 0$ such that for all $f \in H^{s-\frac{3}{2}}(\Omega)$ and $f_\partial \in H^s(\partial\Omega)$, there is a unique $u \in H^{s+\frac{1}{2}}(\Omega)$ satisfying*

$$(-\Delta + M^2)u = f,\ u|_{\partial\Omega} = f_\partial, \tag{2.9}$$

and

$$\|u\|_{H^{s+\frac{1}{2}}(\Omega)} \leq C\|f\|_{H^{s-\frac{3}{2}}(\Omega)} + C\|f_\partial\|_{H^s(\partial\Omega)}, \tag{2.10}$$

uniformly for all such f, f_∂ and for all $M \in [0, M_0]$. When $f = 0$, the above is true for any $0 \leq s \leq 1$. This implies $\mathcal{H}^\Omega_M : H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega)$ for $0 \leq s \leq 1$, and $\mathcal{R}^\Omega_M : H^{s-\frac{3}{2}}(\Omega) \to H^{s+\frac{1}{2}}(\Omega)$ for $0 < s < 1$.

Proof. 1. Let $M = 0$. The $0 < s < 1$ case follows from [JK95, Theorem 0.5(b)] and the extension operators from $H^s(\partial\Omega)$ to $H^{s+\frac{1}{2}}(\Omega)$ is well-defined. For $f = 0$, the endpoint case $s = 0$ follows from [JK81a] and [JK95, Corollary 5.5], where we understand $u|_{\partial\Omega}$ as the non-tangential limit of u towards $\partial\Omega$. The case $s = 1$ follows from [JK81b].

2. Let $M > 0$. We first note that Lax–Milgram gives that $\mathcal{R}^\Omega_M : H^{-1}(\Omega) \to \dot{H}^1(\Omega)$ uniformly in $M \in [0, M_0]$. The correction formula (2.8) gives a uniform bound in $\mathcal{H}^\Omega_M = \mathcal{H}^\Omega_0 - M^2\mathcal{R}^\Omega_M\mathcal{H}^\Omega_0 : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$. Now for f, f_∂ as stated, note $\mathcal{H}^\Omega_0 f_\partial, \mathcal{R}^\Omega_0 f \in H^{s+\frac{1}{2}}(\Omega) \subset H^{\frac{1}{2}}(\Omega)$. The basic bounds above then give

$$u = \mathcal{R}^\Omega_M f + \mathcal{H}^\Omega_M f_\partial = \mathcal{R}^\Omega_0 f - M^2\mathcal{R}^\Omega_M\mathcal{R}^\Omega_0 f + \mathcal{H}^\Omega_0 f_\partial - M^2\mathcal{R}^\Omega_M\mathcal{H}^\Omega_0 f_\partial \in H^{\frac{1}{2}}(\Omega) \tag{2.11}$$

to exist. Since [JK95, Theorem B] implies $\mathcal{R}_0^\Omega : L^2(\Omega) \to \dot{H}^{\frac{3}{2}}(\Omega)$ bounded, we have $\mathcal{R}_M^\Omega = \mathcal{R}_0^\Omega - M^2\mathcal{R}_0^\Omega\mathcal{R}_M^\Omega$ also bounded on $L^2(\Omega) \to \dot{H}^{\frac{3}{2}}(\Omega)$. Then

$$\mathcal{H}_M^\Omega f_\partial = \mathcal{H}_0^\Omega f_\partial - M^2\mathcal{R}_0^\Omega\mathcal{H}_M^\Omega f_\partial \in H^{s+\frac{1}{2}}(\Omega) \tag{2.12}$$

for any $0 \le s \le 1$. Similarly we have

$$\mathcal{R}_M^\Omega f = \mathcal{R}_0^\Omega f - M^2\mathcal{R}_0^\Omega\mathcal{R}_M^\Omega f \in H^{s+\frac{1}{2}}(\Omega) \tag{2.13}$$

for any $0 < s < 1$. They conclude the proof. □

Lemma 2.6 (Neumann resolvent on bounded domains)**.** *Let $\Omega \Subset \mathbb{R}^d$ be a bounded uniformly Lipschitz domain. Let $M_0 > 0$. Let $0 < s < 1$. Let $u \in H^{s+\frac{1}{2}}(\Omega)$, $f \in \dot{H}^{s-\frac{3}{2}}(\Omega)$, and $f_\partial \in H^{s-1}(\partial\Omega)$, satisfying*

$$(-\Delta + M^2)u = f,\ \partial_\nu u|_{\partial\Omega} = f_\partial. \tag{2.14}$$

Then

$$\|u\|_{H^{s+\frac{1}{2}}(\Omega)} \le C\|f\|_{H^{s-\frac{3}{2}}(\Omega)} + C\|f_\partial\|_{H^{s-1}(\partial\Omega)} + C\|u\|_{H^{-\frac{1}{2}+0}(\Omega)}, \tag{2.15}$$

uniformly for all such f, f_∂ and for all $M \in [0, M_0]$. When $f = 0$, the above is true for any $0 \le s \le 1$.

Remark 2.7. One here can replace the last term in (2.15) by $C|\langle u, 1\rangle|$ and show

$$\|u\|_{H^s(\partial\Omega)} \le C\|f\|_{H^{s-\frac{3}{2}}(\Omega)} + C\|f_\partial\|_{H^{s-1}(\partial\Omega)} + C\|u\|_{H^{-1}(\partial\Omega)}, \tag{2.16}$$

in the above-mentioned scenarios, by using [BGM25, Theorem 3.6, Corollary 5.7].

Proof. 1. We first address $M = 0$. The cases $f = 0$, and $f_\partial = 0$ are known to hold. The homogeneous problem $s = 0$ was treated in [Ver05, Proposition 4.2], and $s = 1$ was treated in [JK81b] under the condition $\langle f_\partial, 1\rangle_{\partial\Omega} = 0$. The inhomogeneous problem was treated in [FMM98, Theorem 9.2] for $\langle f, 1\rangle = 0$. We will use those results to allow general f, f_∂ with $\langle f, 1\rangle + \langle f_\partial, 1\rangle_{\partial\Omega} = 0$ when $0 < s < 1$. Denoting the volume of Ω by $|\Omega|$, we build the average corrector $h(z) \in C_c^\infty(\overline{\Omega})$ by

$$h(z) = -\frac{1}{2d}|z|^2,\ -\Delta h = 1,\ h_\partial = \partial_\nu h \in L^2(\partial\Omega),\ \langle h_\partial, 1\rangle_{\partial\Omega} = \langle \Delta h, 1\rangle = -|\Omega|. \tag{2.17}$$

The cases discussed above imply there exist $u_1, u_2 \in H^{s+\frac{1}{2}}(\Omega)$ that

$$-\Delta u_1 = f - \frac{1}{|\Omega|}\langle f, 1\rangle,\ \partial_\nu u_1 = 0, \tag{2.18}$$

$$-\Delta u_2 = 0,\ \partial_\nu u_2 = f_\partial - \frac{1}{|\Omega|}\langle f, 1\rangle h_\partial. \tag{2.19}$$

Now setting $\tilde{u} = u_1 + u_2 + \frac{1}{|\Omega|}\langle f, 1\rangle h - \frac{1}{|\Omega|}\langle u_1 + u_2 + \frac{1}{|\Omega|}\langle f, 1\rangle h, 1\rangle$ gives $u = \tilde{u} + \frac{1}{|\Omega|}\langle u, 1\rangle$ and thus

$$\|u\|_{H^{s+\frac{1}{2}}(\Omega)} \le C\|f\|_{H^{s-\frac{3}{2}}(\Omega)} + C\|f_\partial\|_{H^{s-1}(\partial\Omega)} + C\|u\|_{H^{-\frac{1}{2}+0}(\Omega)}. \tag{2.20}$$

2a. For $M > 0$, note $-\Delta u = f - M^2 u$ and applying the case $M = 0$ gives the proof for $0 < s < 1$. It remains to address the endpoint cases.

2b. Let $f = 0$ and $s = 1$. We have $-\Delta u = -M^2 u$ for $u \in H^{\frac{3}{2}}$. First let $u_3 = -M^2 \mathcal{R}_0^\Omega u \in \dot{H}^{\frac{3}{2}}(\Omega)$. Then Step 1 with $M = 0, s = \frac{1}{2}$ together with [BGM25, Theorem 5.4] implies

$$\|u_3\|_{H^{\frac{3}{2}}} \le CM^2\|u\| \le CM^2\|f_\partial\|_{H^{-\frac{1}{2}}(\partial\Omega)} + CM^2\|u\|_{H^{-\frac{1}{2}+0}}, \tag{2.21}$$

$$\|\partial_\nu u_3\|_{L^2(\partial\Omega)} \le C\|u_3\|_{H^{\frac{3}{2}}} + CM^2\|u\| \le CM^2\|f_\partial\|_{H^{-\frac{1}{2}}(\partial\Omega)} + CM^2\|u\|_{H^{-\frac{1}{2}+0}}. \tag{2.22}$$

Set $u_4 = u - u_3$, we have $-\Delta u_4 = 0$, $\partial_\nu u_4 = f_\partial - \partial_\nu u_3$. Apply Step 1 with $M = 0, s = 1$ to see

$$\|u_4\|_{H^{\frac{3}{2}}} \le C\|f_\partial\|_{L^2(\partial\Omega)} + C\|u\|_{H^{-\frac{1}{2}+0}}. \tag{2.23}$$

2c. Let $f = 0$ and $s = 0$, where $u \in H^{\frac{1}{2}}$. Similar to (2.19), there exists a unique $u_5 \in H^{\frac{1}{2}}(\Omega)$ with $\langle u_5, 1\rangle = 0$ that

$$-\Delta u_5 = 0,\ \partial_\nu u_5 = f_\partial - \frac{1}{|\partial\Omega|}\langle f_\partial, 1\rangle_{\partial\Omega}, \tag{2.24}$$

by [Ver05, Proposition 4.2], and we have

$$\|u_5\|_{H^{\frac{1}{2}}} \le C\|f_\partial - \frac{1}{|\partial\Omega|}\langle f_\partial, 1\rangle_{\partial\Omega}\|_{H^{-1}(\partial\Omega)} \le C\|f_\partial\|_{H^{-1}(\partial\Omega)}. \tag{2.25}$$

On another hand, $u_6 = u - u_5 \in H^{\frac{1}{2}}$ satisfies

$$(-\Delta + M^2)u_6 = -M^2 u_5 \in H^{\frac{1}{2}},\ \partial_\nu u_6 = \frac{1}{|\partial\Omega|}\langle f_\partial, 1\rangle_{\partial\Omega} \in H^1(\partial\Omega). \tag{2.26}$$

We can use Step 2a with $M > 0, s = 0 + 0$ to estimate

$$\|u_6\|_{H^{\frac{1}{2}+0}} \le CM^2\|u_5\|_{H^{-\frac{3}{2}+0}} + \frac{C}{|\partial\Omega|}\|\langle f_\partial, 1\rangle_{\partial\Omega}\|_{H^{-1+0}} + C\|u_6\|_{H^{-\frac{1}{2}+0}} \le C\|f_\partial\|_{H^{-1}(\partial\Omega)} + C\|u\|_{H^{-\frac{1}{2}+0}} \tag{2.27}$$

as desired. □

2.3. **Spaces with Hardy-type weights.** In order to specify the boundedness of $\mathcal{H}_M^\Omega$ and $\mathcal{R}_M^\Omega$ on unbounded uniformly Lipschitz domains, we introduce the following space that contains a Hardy-type weight that relates ∇ to the regularized distance $\mathbf{d}$ to the boundary:

Definition 2.8 (Spaces with Hardy-type weights)**.** Let $\Omega \subset \mathbb{R}^d$ be a uniformly Lipschitz domain. Let $m, l \in \mathbb{R}$. We define

$$X_m^l(\Omega; \mathcal{V}) = \{u \in \mathbf{r}^{-l}\mathbf{d}^{l-m+1}H^1(\Omega) : \mathbf{r}^l\mathbf{d}^{m-l}\nabla u \in L^2(\Omega),\ \mathbf{r}^l\mathbf{d}^{m-1-l}u \in L^2(\Omega)\}, \tag{2.28}$$

$$\dot{X}_m^l(\Omega; \mathcal{V}) = \{u \in \mathbf{r}^{-l}\mathbf{d}^{l-m+1}\dot{H}^1(\Omega) : \mathbf{r}^l\mathbf{d}^{m-l}\nabla u \in L^2(\Omega),\ \mathbf{r}^l\mathbf{d}^{m-1-l}u \in L^2(\Omega)\}, \tag{2.29}$$

where $\dot{H}^1(\Omega)$ denote $H^1(\Omega)$-elements with vanishing traces over $\partial\Omega$. We drop $\mathcal{V}$ and write $X_m^l(\Omega)$ whenever there is no confusion about $\mathcal{V}$. The spaces are equipped with norm

$$\|u\|_{X_m^l}^2 = \|\mathbf{r}^l\mathbf{d}^{m-l-1}u\|_{L^2}^2 + \|\mathbf{r}^l\mathbf{d}^{m-l}\nabla u\|_{L^2}^2. \tag{2.30}$$

We define their dual spaces with respect to $L^2(\Omega)$ by

$$Y_m^l(\Omega) = (\dot{X}_{-m}^{-l}(\Omega))^*,\ \dot{Y}_m^l(\Omega) = (X_{-m}^{-l}(\Omega))^*. \tag{2.31}$$

We also introduce the following Hardy-type constants for Ω:

Definition 2.9 (Radial Hardy constants)**.** Let $\Omega \subset \mathbb{R}^d$ be a uniformly Lipschitz domain. For each $r_0 \geq 0$, we define the *r_0-regularized radial Hardy constant* by

$$\mathfrak{r}_\Omega^{r_0} = \inf\Big\{ \frac{\|\nabla u\|_{L^2(\Omega)}}{\|\frac{u}{\operatorname{dist}(z,\mathcal{V})+r_0}\|_{L^2(\Omega)}} : u \in \dot{H}^1(\Omega) \setminus \{0\} \Big\}. \tag{2.32}$$

We call $\mathfrak{r}_\Omega^0$ *the (strong) radial Hardy constant.* We then define *the regularized radial Hardy constant* by

$$\mathfrak{r}_\Omega^\infty = \lim_{r_0 \to \infty} \mathfrak{r}_\Omega^{r_0}. \tag{2.33}$$

Note that $\mathfrak{r}_\Omega^{r_0} > 0$ is non-decreasing in r_0 and hence the limit either exists or is ∞. Given $\Omega \subset \tilde{\Omega}$, we have $\mathfrak{r}_\Omega^\infty \geq \mathfrak{r}_{\tilde{\Omega}}^\infty$; we further have $\mathfrak{r}_\Omega^{r_0} \geq \mathfrak{r}_{\tilde{\Omega}}^{r_0}$ for each $r_0 \geq 0$ if they have same choice of $\mathcal{V}$. We require any uniformly Lipschitz domain Ω considered in this paper to satisfy $\mathfrak{r}_\Omega^\infty > 0$.

Definition 2.10 (Hardy constants)**.** Let $\Omega \subset \mathbb{R}^d$ be a uniformly Lipschitz domain, and U an open subset of Ω. For each $r_0 \geq 0$, we define the *r_0-regularized Hardy constant of U with respect to $\partial\Omega$* by

$$\mathfrak{h}_{U,\partial\Omega}^{r_0} = \inf\Big\{ \frac{\|\nabla u\|_{L^2(\Omega)}}{\|\frac{u}{\operatorname{dist}(z,\partial\Omega)+r_0}\|_{L^2(\Omega)}} : u \in \dot{H}^1(U) \setminus \{0\}, u|_{\partial\Omega} = 0 \Big\}. \tag{2.34}$$

We define the *regularized Hardy constant of U with respect to $\partial\Omega$* by

$$\mathfrak{h}_{U,\partial\Omega}^\infty = \lim_{r_0 \to \infty} \mathfrak{h}_{U,\partial\Omega}^{r_0}. \tag{2.35}$$

Note that $\mathfrak{h}_{U,\partial\Omega}^{r_0} > 0$ is non-decreasing in r_0 and hence the limit either exists or is ∞. Also for $U \subset \tilde{U}$, we have $\mathfrak{h}_{U,\partial\Omega}^{r_0} \geq \mathfrak{h}_{\tilde{U},\partial\Omega}^{r_0}$. We define the *$r_0$-regularized Hardy constant of Ω* by

$$\mathfrak{h}_\Omega^{r_0} := \mathfrak{h}_{\Omega,\partial\Omega}^{r_0} = \inf\{\mathfrak{h}_{U,\partial\Omega}^{r_0} : U \subset \Omega\} > 0. \tag{2.36}$$

We call $\mathfrak{h}_\Omega^0$ the *(strong) Hardy constant* and $\mathfrak{h}_\Omega^\infty$ the *regularized Hardy constant.* For each $r_0 \geq 0$, we have $\mathfrak{r}_\Omega^{r_0} \geq \mathfrak{h}_\Omega^{r_0}$. For each $\gamma > 0$ we define the conic collar

$$U_\gamma^{r_0} = \{z \in \Omega, \operatorname{dist}(z,\partial\Omega) + r_0 < \gamma(\operatorname{dist}(z,\mathcal{V}) + r_0)\}. \tag{2.37}$$

The constant $\mathfrak{h}_{U_\gamma^{r_0},\partial\Omega}^{r_0}$ will appear as the threshold for $|m-l|$ in this paper.

Remark 2.11. Throughout the paper, we fix some $r_0 \geq 0$ if not specified otherwise.

Remark 2.12. Here $\mathfrak{r}_\Omega^\infty + \frac{1}{2} - 0$ is the maximal decay one can track at the zero frequency for one specific domain Ω individually, and $\mathfrak{r}_\Omega^{r_0} + \mathfrak{h}_{U_\gamma^{r_0},\partial\Omega}^{r_0} - 0$ is the maximal decay one can trace at the zero frequency uniformly for all domains with controlled $\mathfrak{r}_\Omega^{r_0}$, γ. For $\mathcal{W}_\alpha$, an infinite planar wedge of corner angle $\alpha \in (0, 2\pi)$, we have $\mathfrak{r}_{\mathcal{W}_\alpha}^0 = \frac{\pi}{\alpha}$ (see Lemma B.7).

Remark 2.13. The Hardy inequality implies

$$\dot{X}_m^l = \{u \in \mathbf{r}^{-l}\mathbf{d}^{l-m+1}\dot{H}^1(\Omega) : \mathbf{d}^{m-l}\mathbf{r}^l\nabla u \in L^2(\Omega)\},\ Y_m^l = \operatorname{div}(\mathbf{d}^{l-m}\mathbf{r}^{-l}L^2(\Omega; T\Omega)). \tag{2.38}$$

Lemma 2.14 (Properties for X^l_m-spaces)**.** *Let Ω be a uniformly Lipschitz domain. Let $m, l \in \mathbb{R}$. The following statements are true:*

$$X^l_m(\Omega) = \mathbf{r}^{-l}\mathbf{d}^{l-m}X^0_0(\Omega),\ Y^l_m(\Omega) = \mathbf{r}^{-l}\mathbf{d}^{l-m}Y^0_0(\Omega), \tag{2.39}$$

$$\mathbf{r}^{-l}\mathbf{d}^{l-m}H^1(\Omega) \subset X^l_m(\Omega) \subset \mathbf{r}^{-l}\mathbf{d}^{l-m+1}H^1(\Omega), \tag{2.40}$$

$$\mathbf{r}^{-l}\mathbf{d}^{l-m-1}H^{-1}(\Omega) \subset Y^l_m(\Omega) \subset \mathbf{r}^{-l}\mathbf{d}^{l-m}H^{-1}(\Omega), \tag{2.41}$$

$$X^l_m(\Omega) \subset Y^l_{m-2}(\Omega). \tag{2.42}$$

The statements are also true for $\dot{X}^l_m$ and $\dot{Y}^l_m$ when H^1, H^{-1} are replaced with $\dot{H}^1, \dot{H}^{-1}$.

Proof. Note $\mathbf{r}^l\mathbf{d}^{m-l}\nabla\mathbf{r}^{-l}\mathbf{d}^{-m+l} = \nabla + O(\mathbf{d}^{-1})$ to see (2.39). The rest follows similarly. □

2.4. **The Poisson problem.** We study the solution $u \in \dot{H}^{1,l}(\Omega)$ to the Poisson problem

$$(-\Delta + M^2)u = f, \tag{2.43}$$

for $f \in H^{-1,l}(\Omega)$. For each $M > 0$, the solution u to (2.43) exists by the Lax–Milgram Theorem. However, as $M \to 0$, estimates for $u \in \dot{H}^{1,l}$ degenerate. In this subsection, we study uniform estimates for (2.43) in X^l_m instead of $\dot{H}^{1,l}$ as $M \to 0$.

Lemma 2.15 (Stability for the Poisson problem)**.** *Let $r_0 \geq 0$, $\gamma \in (0, \frac{1}{2})$, $\delta_0 \in (0,1)$. Then for all $\Omega \subset \mathbb{R}^d$ a uniformly Lipschitz domain with $|m| \leq \delta_0\mathfrak{r}^{r_0}_\Omega$, $|m-l| \leq \delta_0\mathfrak{h}^{r_0}_{U^{r_0}_\gamma,\partial\Omega}$, $m_1, m_2 \in \mathbb{R}$, we have*

$$\|\langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^{m_2}u\|_{X^l_m} + M\|\langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^{m_2}\mathbf{r}^l\mathbf{d}^{m-l}u\|_{L^2} \leq C\|\langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^{m_2}(-\Delta+M^2)u\|_{Y^l_m}. \tag{2.44}$$

uniformly for $u \in \langle M\mathbf{r}\rangle^{-m_1}\langle M\mathbf{d}\rangle^{-m_2}\mathbf{r}^{-l}\mathbf{d}^{l-m}\dot{H}^1(\Omega)$ and all $M \geq 0$.

Proof. 1. Fix $r_0' > 0$ that $r_0' \geq r_0$. Only in this proof, we use the following notation for convenience:

$$r = \operatorname{dist}(z, \mathcal{V}) + r_0',\ d = \operatorname{dist}(z, \partial\Omega) + r_0'. \tag{2.45}$$

Note r, d are only $C^{0,1}(\Omega)$, and $r \sim \mathbf{r}$, $d \sim \mathbf{d}$. Let $\delta_1 = 1 - \delta_0 \in (0,1)$ for convenience. By Definition 2.10, Definition 2.9, and $\mathfrak{r}^{r_0'}_\Omega \geq \mathfrak{r}^{r_0}_\Omega$, $\mathfrak{h}^{r_0'}_{U^{r_0'}_\gamma,\partial\Omega} \geq \mathfrak{h}^{r_0}_{U^{r_0}_\gamma,\partial\Omega}$, we have

$$(1-\frac{1}{4}\delta_1)^2(\mathfrak{r}^{r_0}_\Omega)^2\|\frac{u}{r}\|^2_{L^2(\Omega)} \leq (1-\frac{1}{8}\delta_1)\|\nabla u\|^2_{L^2(\Omega)},\ u \in \dot{H}^1(\Omega) \tag{2.46}$$

$$(1-\frac{1}{4}\delta_1)^2(\mathfrak{h}^{r_0}_{U^{r_0}_\gamma,\partial\Omega})^2\|\frac{u}{d}\|^2_{L^2(\Omega)} \leq (1-\frac{1}{8}\delta_1)\|\nabla u\|^2_{L^2(\Omega)},\ u \in \dot{H}^1(U^{r_0}_\gamma),\ u|_{\partial\Omega} = 0. \tag{2.47}$$

All constants in the lemma only depend on δ_0, r_0, γ, the choices of m, l, m_1, m_2, and lower bounds on $\mathfrak{h}^{r_0}_{U^{r_0}_\gamma,\partial\Omega}$, $\mathfrak{r}^{r_0}_\Omega$.

2. We construct the weight functions $\varphi_M \in C^{0,1}(\Omega)$. Set $K > 1$, $K_1 > \gamma^{-1}$ large to be determined later. Let

$$\varphi_M = m\log r + (m-l)\log(\frac{d/r}{1+Kd/r}) + m_1\log(Mr + K(|m_1|+1)) + m_2\log(Md + K(|m_2|+1)). \tag{2.48}$$

Noting $1+Kd/r$ is bounded from above and below by positive constants we have

$$e^{\varphi_M} \underset{r_0,\delta_0}{\sim} \langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^{m_2}\mathbf{r}^l\mathbf{d}^{m-l},\ |\nabla e^{\varphi_M}| \underset{r_0,\delta_0}{\leq} \langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^{m_2}\mathbf{r}^l\mathbf{d}^{m-l-1}. \tag{2.49}$$

Let $\chi_0,\chi_1\in C^{0,1}(\Omega)$ that $\chi_0^2+\chi_1^2\equiv 1$ and $\chi_1\equiv 0$ on $\{\log(\frac{r}{d})\geq K_1+\log(K_1)\}=\{\frac{d}{r}\leq K_1^{-1}e^{-K_1}\}$ and $\chi_1\equiv 1$ on $\{\log(\frac{r}{d})\leq \log(K_1)\}=\{\frac{d}{r}\geq K_1^{-1}\}$, and

$$|\nabla\chi_0|\leq \frac{C}{K_1 d},\ |\nabla\chi_1|\leq \frac{C}{K_1 d} \tag{2.50}$$

for some $C>0$ uniformly. This makes $\operatorname{supp}\chi_0\subset\{d\leq \gamma r\}$. We claim choosing large K, K_1 gives

$$|\nabla\varphi_M|^2\leq (1-\frac{1}{4}\delta_1)^2\Big(\frac{\chi_0^2(\mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega})^2}{d^2}+\frac{\chi_1^2(\mathfrak{r}_\Omega^{r_0})^2}{r^2}\Big)+\frac{1}{4}M^2 \tag{2.51}$$

in this step. To see it, we compute

$$\nabla\varphi_M=\frac{m\nabla r}{r}+\frac{m-l}{1+Kd/r}\nabla\log(\frac{d}{r})+\frac{m_1M\nabla r}{Mr+K(|m_1|+1)}+\frac{m_2M\nabla d}{Md+K(|m_2|+1)}, \tag{2.52}$$

the last two terms of which can be made small by a choosing a large K since

$$\Big|\frac{m_1M}{Mr+K(|m_1|+1)}\nabla r\Big|+\Big|\frac{m_2M}{Md+K(|m_2|+1)}\nabla d\Big|\leq\frac{2M}{K}. \tag{2.53}$$

We now analyze the first two terms. On $\operatorname{supp}\chi_0$ we have $\frac{d}{r}\leq K_1^{-1}$ and thus

$$\begin{aligned}\chi_0^2|\nabla\varphi_M|^2&\leq\chi_0^2\Big((1+\epsilon)\Big|\frac{m\nabla r}{r}+\frac{m-l}{1+Kd/r}\nabla\log(\frac{d}{r})\Big|^2+\frac{C_\epsilon M^2}{K^2}\Big)\\&\leq\chi_0^2\Big((1+2\epsilon)(1+K_1^{-1})^2\frac{(m-l)^2}{d^2}+\frac{C_\epsilon m^2}{K_1^2d^2}+\frac{C_\epsilon M^2}{K^2}\Big),\end{aligned} \tag{2.54}$$

where we used $|\nabla d|\leq 1$, $|\nabla r|\leq 1$ and on $\operatorname{supp}\chi_0$ we have

$$\Big|\frac{\nabla r}{r}\Big|\leq\frac{1}{K_1d},\ \Big|\frac{1}{1+Kd/r}\nabla\log(\frac{d}{r})\Big|\leq\Big|\frac{\nabla d}{d}-\frac{\nabla r}{r}\Big|\leq\frac{1}{d}+\frac{1}{r}\leq\frac{1+K_1^{-1}}{d}. \tag{2.55}$$

On $\operatorname{supp}\chi_1$ we have $\frac{d}{r}\geq K_1^{-1}e^{-K_1}$ and

$$\begin{aligned}\chi_1^2|\nabla\varphi_M|^2&\leq\chi_1^2\Big((1+\epsilon)\Big|\frac{m\nabla r}{r}+\frac{m-l}{1+Kd/r}\nabla\log(\frac{d}{r})\Big|^2+\frac{C_\epsilon M^2}{K^2}\Big)\\&\leq\chi_1^2\Big((1+2\epsilon)\frac{m^2}{r^2}+\frac{C_\epsilon(1+K_1e^{K_1})^2(m-l)^2}{(1+KK_1^{-1}e^{-K_1})^2r^2}+\frac{C_\epsilon M^2}{K^2}\Big),\end{aligned} \tag{2.56}$$

where we used

$$\begin{aligned}\Big|\frac{1}{1+Kd/r}\nabla\log(\frac{d}{r})\Big|&\leq\frac{1}{1+KK_1^{-1}e^{-K_1}}\Big|\frac{\nabla d}{d}-\frac{\nabla r}{r}\Big|\leq\frac{1}{1+KK_1^{-1}e^{-K_1}}(\frac{1}{d}+\frac{1}{r})\\&\leq\frac{1+K_1e^{K_1}}{(1+KK_1^{-1}e^{-K_1})r}.\end{aligned} \tag{2.57}$$

Now sequentially choosing small ϵ, large K_1, large K gives (2.51).

3. Consider the conjugated operator

$$P_{\varphi_M} := e^{\varphi_M}(-\Delta + M^2)e^{-\varphi_M}. \tag{2.58}$$

We claim P_{φ_M} is elliptic in the sense that there is $c > 0$ small that

$$\Re\langle P_{\varphi_M} v, v\rangle \geq c(\|v\|_{X_0^0}^2 + M^2\|v\|_{L^2}^2) \tag{2.59}$$

uniformly for all $v \in \dot{H}^1(\Omega)$ and $M \geq 0$. Indeed,

$$\begin{aligned} 2\Re\langle P_{\varphi_M} v, v\rangle &= 2\|\nabla v\|^2 + 2\langle\langle \nabla\varphi_M, \nabla v\rangle, v\rangle + 2\langle v, \langle \nabla\varphi_M, \nabla v\rangle\rangle + 2\langle (M^2 - \nabla^*\nabla\varphi_M - |\nabla\varphi_M|^2)v, v\rangle \\ &= 2\|\nabla v\|^2 + 2\langle (M^2 - |\nabla\varphi_M|^2)v, v\rangle = 2\|\nabla v\|^2 + 2M^2\|v\|^2 - 2\||\nabla\varphi_M|v\|^2. \end{aligned} \tag{2.60}$$

Making $K_1 > \gamma^{-1}$ larger if necessary, we note Step 1 and 2 give

$$\begin{aligned} &\||\nabla\varphi_M|v\|^2 \leq (1 - \frac{1}{4}\delta_1)^2\Big((\mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega})^2\Big\|\frac{\chi_0 v}{d}\Big\|^2 + (\mathfrak{r}_\Omega^{r_0})^2\Big\|\frac{\chi_1 v}{r}\Big\|^2\Big) + \frac{1}{4}M^2\|v\|^2 \\ &\leq (1 - \frac{1}{8}\delta_1)\Big(\Big\|\nabla(\chi_0 v)\Big\|^2 + \Big\|\nabla(\chi_1 v)\Big\|^2\Big) + \frac{1}{4}M^2\|v\|^2 \leq (1 - \frac{1}{8}\delta_1)\|\nabla v\|^2 + \frac{C}{K_1}\Big\|\frac{v}{d}\Big\|^2 + \frac{1}{4}M^2\|v\|^2 \\ &\leq (1 - \frac{1}{16}\delta_1)\|\nabla v\|^2 + \frac{1}{4}M^2\|v\|^2, \end{aligned} \tag{2.61}$$

where we used (2.50), (2.51), and $\chi_0 v, (\nabla\chi_0)v, (\nabla\chi_1)v$ are supported inside $U_\gamma^{r_0} \cup (\overline{U_\gamma^{r_0}} \cap \partial\Omega)$. We now have for some $C > 0$ large independent of M that

$$2\Re\langle P_{\varphi_M} v, v\rangle \geq \frac{\delta_1}{8}\|\nabla v\|^2 + \frac{3M^2}{2}\|v\|^2 \geq C^{-1}\Big(\|\nabla v\|_{L^2}^2 + \Big\|\frac{v}{\mathbf{d}}\Big\|_{L^2}^2 + M^2\|v\|_{L^2}^2\Big). \tag{2.62}$$

This gives (2.59) as desired by noting $\|v\|_{X_0^0}^2 = \|\nabla v\|_{L^2}^2 + \|\mathbf{d}^{-1}v\|_{L^2}^2$.

4. Now we take $v = e^{\varphi_M}u \in \dot{H}^1(\Omega)$. Step 3 gives

$$\|v\|_{X_0^0}^2 + M^2\|v\|_{L^2}^2 \leq C|\langle e^{\varphi_M}(-\Delta + M^2)u, v\rangle| \leq C\epsilon^{-1}\|e^{\varphi_M}(-\Delta + M^2)u\|_{Y_0^0}^2 + \epsilon\|v\|_{X_0^0}^2. \tag{2.63}$$

The absorption of the last term on the right concludes the proof with (2.49). □

We now show $\mathcal{R}_M^\Omega$, the Dirichlet resolvent, is uniformly bounded in $M \geq 0$ as an operator from Y_m^l to $\dot{X}_m^l$, and show $\mathcal{R}_M^\Omega \to \mathcal{R}_0^\Omega$ as a weak limit.

Proposition 2.16 (Zero mass limit of Dirichlet resolvent)**.** *Let $r_0 \geq 0$, $\gamma \in (0, \frac{1}{2})$. Let $m, l \in \mathbb{R}$. Let $\Omega \subset \mathbb{R}^d$ be any uniformly Lipschitz domain with $|m| < \mathfrak{r}_\Omega^{r_0}$, $|m - l| < \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}$. Let $m_1, m_2 \in \mathbb{R}$. Then*

$$\mathcal{R}_M^\Omega : \langle M\mathbf{r}\rangle^{-m_1}\langle M\mathbf{d}\rangle^{-m_2}Y_m^l(\Omega) \to \langle M\mathbf{r}\rangle^{-m_1}\langle M\mathbf{d}\rangle^{-m_2}\dot{X}_m^l(\Omega) \tag{2.64}$$

exists and is uniformly bounded for all $M \geq 0$, where $\mathcal{R}_M^\Omega f$ is the unique element in that class such that $(-\Delta + M^2)\mathcal{R}_M^\Omega f = f$. Indeed, there is $C > 0$ such that for all $M \geq 0$, all $f \in \langle M\mathbf{r}\rangle^{-m_1}\langle M\mathbf{d}\rangle^{-m_2}Y_m^l(\Omega)$, we have

$$\|\langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^{m_2}\mathcal{R}_M^\Omega f\|_{X_m^l} + M\|\langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^{m_2}\mathbf{r}^l\mathbf{d}^{m-l}\mathcal{R}_M^\Omega f\|_{L^2} \leq C\|\langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^{m_2}f\|_{Y_m^l}. \tag{2.65}$$

Moreover $\mathcal{R}_0^\Omega : Y_m^l(\Omega) \to \dot{X}_m^l(\Omega)$ *agrees with the zero mass limit of* $\mathcal{R}_M^\Omega : Y_m^l(\Omega) \to \dot{X}_m^l(\Omega)$ *as* $M \to 0$,

$$Y_m^l(\Omega) \ni f \mapsto \mathcal{R}_0^\Omega f = \lim_{M\to 0} \mathcal{R}_M^\Omega f \in \dot{X}_m^l(\Omega) \tag{2.66}$$

as weak convergence in $\dot{X}_m^l$*, and in the norm sense in* $\dot{X}_{m-0}^{l-0}$*. The only dependence of bounds on* Ω *are through the lower bounds on* $\mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}, \mathfrak{r}_\Omega^{r_0}$.

Remark 2.17. For $u \in \dot{X}_m^l(\Omega)$ within the specified limits for $l, m, \mathfrak{r}_\Omega^{r_0}, \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}$, we have $\mathcal{R}_M^\Omega(-\Delta + M^2)u = u$ by definition, for all $M \geq 0$.

Proof. 1. Set φ_M as in (2.48). For each $M > 0$, $\mathbf{r} \sim \langle M\mathbf{r}\rangle$, $\mathbf{d} \sim \langle M\mathbf{d}\rangle$. Let

$$f \in \langle M\mathbf{r}\rangle^{-m_1}\langle M\mathbf{d}\rangle^{-m_2} Y_m^l \subset \langle M\mathbf{r}\rangle^{-m_1}\langle M\mathbf{d}\rangle^{-m_2}\mathbf{r}^{-l}\mathbf{d}^{-m+l}H^{-1}. \tag{2.67}$$

Then, we have $e^{\varphi_M} f \in H^{-1}(\Omega)$ per (2.49). By the Lax–Milgram theorem and (2.59), there is a unique solution $v \in \dot{H}^1(\Omega)$ to $P_{\varphi_M} v = e^{\varphi_M} f$. Now, setting $u = e^{-\varphi_M} v$, we have

$$(-\Delta + M^2)u = f,\ u \in \langle M\mathbf{r}\rangle^{-m_1}\langle M\mathbf{d}\rangle^{-m_2}\mathbf{d}^{l-m}\mathbf{r}^{-l}\dot{H}^1. \tag{2.68}$$

Thus, $u = \mathcal{R}_M^\Omega f$, and applying Lemma 2.15, we obtain (2.65) and (2.64).

2. Let $\delta > 0$, $f \in Y_m^l$ and let $u_M = \mathcal{R}_M^\Omega f$ for $M \geq 0$. By (2.64) we have $u_M \in \dot{X}_m^l$ and

$$\sup_M \|u_M\|_{X_m^l} \leq C\|f\|_{Y_m^l} \tag{2.69}$$

uniformly. There is a sequence of $M \to 0$ and there is $u_0 \in \dot{X}_m^l$ such that

$$u_M \rightharpoonup u_0 \text{ weakly in } X_m^l,\ \mathbf{d}^{m-l-1}\mathbf{r}^{l-\delta}u_M \to \mathbf{d}^{m-l-1}\mathbf{r}^{l-\delta}u_0 \text{ in } L^2. \tag{2.70}$$

It follows that $-\Delta u_0 = f$ and $u_0 = \mathcal{R}_0^\Omega f$ via uniqueness. Here we used the fact that the embedding $\dot{X}_m^l \hookrightarrow \mathbf{d}^{l-m+1}\mathbf{r}^{-l+\delta}L^2$ is compact. Indeed, note that $\mathbf{d} \leq C\mathbf{r}$, we have that $\mathbf{d}^{l-m}\mathbf{r}^{-l} \leq C\mathbf{d}^{l-m-\frac{1}{2}\delta}\mathbf{r}^{-l+\frac{1}{2}\delta}$. We then have

$$\dot{X}_m^l = \mathbf{d}^{l-m}\mathbf{r}^{-l}\dot{X}_0^0 \subset \mathbf{d}^{l-m+1}\mathbf{r}^{-l}\dot{H}^1 \hookrightarrow \mathbf{d}^{l-m+1}\mathbf{r}^{-l+\delta}L^2 \tag{2.71}$$

compactly by the Rellich theorem. It remains to show the norm convergence of $u_M \to u_0$ in $\dot{X}_{m-\delta}^{l-\delta}$ in each $\delta > 0$, by showing $\mathbf{d}^{m-l}\mathbf{r}^{l-\delta}\nabla u_M \to \mathbf{d}^{m-l}\mathbf{r}^{l-\delta}\nabla u_0$ in L^2. This is equivalent to showing $\mathbf{d}^{m-l}\mathbf{r}^{l-\delta}\nabla w_M \to 0$ for $w_M = u_M - u_0$.

2a. For each $R > 1$, let $\chi_R(z) \in C^\infty(\mathbb{R}^d)$ that $\chi_R \equiv 1$ in $\{\mathbf{r} \leq R-1\}$ and $\chi_R \equiv 0$ in $\{\mathbf{r} \geq R+1\}$ such that $|\nabla\chi_R| \leq C$ independently of R. In this step, we claim for each $\epsilon > 0$, there is $R > 1$ that $\|\mathbf{d}^{m-l}\mathbf{r}^{l-\delta}(1-\chi_R)\nabla w_M\|_{L^2} < \epsilon$ for all M along the sequence. Indeed,

$$\|\mathbf{d}^{m-l}\mathbf{r}^{l-\delta}(1-\chi_R)\nabla w_M\| \leq CR^{-\delta}\|\mathbf{d}^{m-l}\mathbf{r}^{l}\nabla w_M\| \leq CR^{-\delta}\|w_M\|_{X_m^l} \leq CR^{-\delta}\|f\|_{Y_m^l} \tag{2.72}$$

yielding the claim.

2b. We now claim for such ϵ and its corresponding R given in Step 2a,

$$\|\mathbf{d}^{m-l}\mathbf{r}^{l-\delta}\chi_R\nabla w_M\| \to 0,\ M \to 0. \tag{2.73}$$

Showing this would imply the norm convergence in $\dot{X}_{m-\delta}^{l-\delta}$ as desired. For so, compute

$$\|\mathbf{r}^{l-\delta}\mathbf{d}^{m-l}\chi_R\nabla w_M\|_{L^2}^2 = |\langle\nabla w_M, \nabla\mathbf{r}^{2l-2\delta}\mathbf{d}^{2m-2l}\chi_R^2 w_M\rangle + \langle\nabla w_M, [\mathbf{r}^{2l-2\delta}\mathbf{d}^{2m-2l}\chi_R^2, \nabla]w_M\rangle|. \tag{2.74}$$

Noting that

$$[\mathbf{r}^{2l-2\delta}\mathbf{d}^{2m-2l}\chi_R^2, \nabla] = O(\mathbf{r}^{2l-2\delta}\mathbf{d}^{2m-2l-1}), \tag{2.75}$$

we can estimate the second term of (2.74) by

$$\begin{aligned}|\langle \nabla w_M, [\mathbf{r}^{2l-2\delta}\mathbf{d}^{2m-2l}\chi_R^2, \nabla] w_M\rangle| &\leq C\|\mathbf{r}^{l-\delta}\mathbf{d}^{m-l}\nabla w_M\|_{L^2}\|\mathbf{r}^{l-\delta}\mathbf{d}^{m-l-1}w_M\|_{L^2} \\ &\leq C\|f\|_{Y_m^l}\|\mathbf{r}^{l-\delta}\mathbf{d}^{m-l-1}w_M\|_{L^2} \to 0 \end{aligned} \tag{2.76}$$

as $M \to 0$. For the first term, we note $-\Delta w_M = -M^2 u_M$ gives

$$\begin{aligned}|\langle \nabla w_M, \nabla \mathbf{r}^{2l-2\delta}\mathbf{d}^{2m-2l}\chi_R^2 w_M\rangle| &\leq CR^2M^2\|\mathbf{r}^{l-\delta}\mathbf{d}^{m-l-1}u_M\|\|\mathbf{r}^{l-\delta}\mathbf{d}^{m-l-1}w_M\| \\ &\leq CR^2M^2\|f\|_{Y_m^l}\|\mathbf{r}^{l-\delta}\mathbf{d}^{m-l-1}w_M\| \to 0. \end{aligned} \tag{2.77}$$

Now we have $u_M \to u_0$ in $\dot{X}^{l-\delta}_{m-\delta}$ for every $\delta > 0$. □

2.5. **The harmonic extension.** We are now in a position to study the harmonic extension operator for $-\Delta + M^2$.

Lemma 2.18 (Extension operator)**.** *Let Ω be a uniformly Lipschitz domain. Then there is an extension operator E_0 such that for all $0 \leq s \leq 1$, $l \in \mathbb{R}$, and $N \in \mathbb{R}$, $E_0 : H^{s,l}(\partial\Omega) \to \mathbf{d}^{-N}H^{s+\frac{1}{2},l}(\Omega)$ is continuous, and for all $f_\partial \in H^{s,l}(\partial\Omega)$, $(E_0 f_\partial)|_{\partial\Omega} = f_\partial$ and $\operatorname{supp} E_0 f_\partial \subset \{\mathbf{d} \leq 2\}$.*

Proof. Consider a partition of unity $\chi_k \in C_c^\infty(\partial\Omega)$ of bounded local degree, with $\tilde{\chi}_k \in C_c^\infty(\partial\Omega)$ that $\operatorname{supp}\chi_k \cap \operatorname{supp}(1-\tilde{\chi}_k) = \emptyset$, $\operatorname{supp}\tilde{\chi}_k \subset \{\mathbf{d} \leq 2\}$. We do not distinguish functions on $\partial\Omega$ with their extensions in Ω, and without loss of generality assume $\operatorname{supp}\tilde{\chi}_k \subset \tilde{\Omega}_k$ for some bounded uniformly Lipschitz domain $\tilde{\Omega}_k$ that $\partial\Omega = \partial\tilde{\Omega}_k$ over $\operatorname{supp}\tilde{\chi}_k$, where the Lipschitz constants and diameters of such $\tilde{\Omega}_k$ does not depend on k. Note the above makes $\mathcal{H}_M^{\tilde{\Omega}_k} : H^s(\partial\tilde{\Omega}_k) \to H^{s+\frac{1}{2}}(\tilde{\Omega}_k)$ is bounded independently of k. Now set up $E_M = \sum_k \tilde{\chi}_k \mathcal{H}_M^{\tilde{\Omega}_k}\chi_k$ to conclude with E_0. □

We now show $\mathcal{H}_M^\Omega$ exists for each $M \geq 0$, and $\mathcal{H}_0^\Omega$ agrees with $\lim_{M\to 0}\mathcal{H}_M^\Omega$.

Proposition 2.19 (Zero mass limit of harmonic extension)**.** *Let $r_0 \geq 0$, $\gamma \in (0, \frac{1}{2})$. Let $m, l \in \mathbb{R}$. Let $\Omega \subset \mathbb{R}^d$ be a uniformly Lipschitz domain with $|m| < \mathfrak{r}_\Omega^{r_0}$, $|m-l| < \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}$. Let $m_1, N \in \mathbb{R}$. Let $M_0 > 0$. Then*

$$\mathcal{H}_M^\Omega = E_0 - \mathcal{R}_M^\Omega(-\Delta + M^2)E_0 : H^{\frac{1}{2},l}(\partial\Omega) \to X_m^l(\Omega) \tag{2.78}$$

is bounded uniformly in $M \in [0, M_0]$. Indeed, there is $C > 0$ such that

$$\|\langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^N \mathcal{H}_M^\Omega f_\partial\|_{X_m^l} + M\|\langle M\mathbf{r}\rangle^{m_1}\langle M\mathbf{d}\rangle^N \mathbf{r}^l\mathbf{d}^{m-l}\mathcal{H}_M^\Omega f_\partial\|_{L^2} \leq C\|\langle M\mathbf{r}\rangle^{m_1} f_\partial\|_{H^{\frac{1}{2},l}(\partial\Omega)}, \tag{2.79}$$

uniformly in $f_\partial \in \langle M\mathbf{r}\rangle^{-m_1}H^{\frac{1}{2},l}(\partial\Omega)$ and all $M \in [0, M_0]$. Moreover, $\mathcal{H}_0^\Omega = \lim_{M\to 0}\mathcal{H}_M^\Omega$ in the operator norm $H^{\frac{1}{2},l}(\partial\Omega) \to X_m^l(\Omega)$, with

$$\|(\mathcal{H}_M^\Omega - \mathcal{H}_0^\Omega)f_\partial\|_{X_m^l} \leq CM^{\min(\mathfrak{r}_\Omega^{r_0} - m - 0, \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega} - (m-l') - 0, 2)}\|f_\partial\|_{H^{\frac{1}{2},l'}(\partial\Omega)} \tag{2.80}$$

uniformly for all $M \in [0, M_0]$, where $l' \geq l$ satisfies $|l'| < \mathfrak{r}_\Omega^{r_0} + \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}$.

Remark 2.20. Given fixed $|l| < \mathfrak{r}_\Omega^{r_0} + \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}$, one can choose m, l' stated above that

$$\min(\mathfrak{r}_\Omega^{r_0} - m - 0, \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega} - (m - l') - 0, 2) = \min(2\mathfrak{r}_\Omega^{r_0} - 0, \mathfrak{r}_\Omega^{r_0} + \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega} - l - 0, 2). \tag{2.81}$$

Proof. 1. Note that Lemma 2.18 gives

$$(-\Delta + M^2)E_0 f_\partial \in \langle M\mathbf{r}\rangle^{-m_1}\mathbf{d}^{-N-1}H^{-1,l} \subset \langle M\mathbf{r}\rangle^{-m_1}Y^l_{l+N}, \tag{2.82}$$

uniformly in $M \in [0, M_0]$. Applying Proposition 2.16 to (2.78) gives (2.79) and (2.78) bounded.

2. It remains to show the convergence of $\mathcal{H}^\Omega_M \to \mathcal{H}^\Omega_0$ in the operator norm as $M \to 0$. Consider

$$-\Delta(\mathcal{H}^\Omega_M - \mathcal{H}^\Omega_0)f_\partial = -M^2\mathcal{H}^\Omega_M f_\partial,\ (\mathcal{H}^\Omega_M - \mathcal{H}^\Omega_0)f_\partial|_{\partial\Omega} = 0. \tag{2.83}$$

By the uniqueness of $\mathcal{R}^\Omega_0$, we have

$$(\mathcal{H}^\Omega_M - \mathcal{H}^\Omega_0)f_\partial = -\mathcal{R}^\Omega_0 M^2\mathcal{H}^\Omega_M f_\partial \tag{2.84}$$

Let $\epsilon = \min(\mathfrak{r}_\Omega^{r_0} - m - 0, \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega} - (m - l') - 0, 2)$. Now, for $|m| < \mathfrak{r}_\Omega^{r_0}$, $|m - l| < \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}$, we have some $m' > m+\epsilon$ such that $|m'| < \mathfrak{r}_\Omega^{r_0}$, $|m'-l'| < \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}$ as well. Note that $X^{l'}_{m'+2-\epsilon} \subset X^l_{m+2} \subset Y^l_m$ implies

$$\|M^{2-\epsilon}\mathcal{H}^\Omega_M f_\partial\|_{Y^l_m} \le C\|M^{2-\epsilon}\mathcal{H}^\Omega_M f_\partial\|_{X^{l'}_{m'+2-\epsilon}}. \tag{2.85}$$

We can use (2.79) to further estimate

$$\|M^{2-\epsilon}\mathcal{H}^\Omega_M f_\partial\|_{X^{l'}_{m'+2-\epsilon}} \le C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\mathcal{H}^\Omega_M f_\partial\|_{X^{l'}_{m'}} \le C\|f_\partial\|_{H^{\frac12,l'}_\partial}. \tag{2.86}$$

Then using Proposition 2.16, we can estimate (2.84) by

$$\|(\mathcal{H}^\Omega_M - \mathcal{H}^\Omega_0)f_\partial\|_{X^l_m} \le \frac{CM^2}{M^{2-\epsilon}}\|M^{2-\epsilon}\mathcal{H}^\Omega_M f_\partial\|_{Y^l_m} \le CM^\epsilon\|f_\partial\|_{H^{\frac12,l'}_\partial}. \tag{2.87}$$

This is (2.80). We have $\|\mathcal{H}^\Omega_M - \mathcal{H}^\Omega_0\|_{H^{\frac12,l}_\partial \to X^l_m}$ goes to 0 as $M \to 0$ as desired, by taking $l' = l$. □

2.6. **The DtN map and elliptic regularity.** As an immediate corollary of Proposition 2.19, we give a definition of the DtN map.

Definition 2.21 (DtN map)**.** Let $\Omega \subset \mathbb{R}^d$ be a uniformly Lipschitz domain, $|l| < \mathfrak{r}_\Omega^{r_0} + \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}$. For each $M \ge 0$, we define the Dirichlet-to-Neumann map, $\Lambda^\Omega_M : H^{\frac12,l}(\partial\Omega) \to H^{-\frac12,l}(\partial\Omega)$ as a bounded map given by the pairing

$$\langle\Lambda^\Omega_M u, \phi\rangle_{\partial\Omega} = \langle\nabla\mathcal{H}^\Omega_M u, \nabla\mathcal{H}^\Omega_M\phi\rangle_\Omega + M^2\langle\mathcal{H}^\Omega_M u, \mathcal{H}^\Omega_M\phi\rangle_\Omega,\ u \in H^{\frac12,l},\ \phi \in H^{\frac12,-l}, \tag{2.88}$$

or equivalently $\Lambda^\Omega_M = \partial_\nu\mathcal{H}^\Omega_M$. Here Λ^Ω_M is self-adjoint on $L^2(\partial\Omega)$. Moreover $\Lambda^\Omega_0 = \lim_{M\to 0}\Lambda^\Omega_M$ in norm.

We prove some useful elliptic estimates to be used later.

Lemma 2.22. *Let $\Omega \subset \mathbb{R}^d$ be a uniformly Lipschitz domain. Let $M_0 > 0$. Let $\psi, \tilde{\psi} \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \cap \operatorname{supp}(1-\tilde{\psi}) = \emptyset$ and $\psi(\mathbf{d}) \equiv 1$ near $\{\mathbf{d} \le 2\}$. Then, for $0 < s < 1$, and $l \in \mathbb{R}$,*

$$\|\mathbf{r}^l \psi(\mathbf{d}) u\|_{H^{s+\frac{1}{2}}(\Omega)} \le C\|\mathbf{r}^l \tilde{\psi}(\mathbf{d})(-\Delta + M^2)u\|_{H^{s-\frac{3}{2}}(\Omega)} + C\|\mathbf{r}^l u\|_{H^s(\partial\Omega)} + C\|\mathbf{r}^l \tilde{\psi}(\mathbf{d}) u\|_{H^{-\frac{1}{2}+0}(\Omega)}, \tag{2.89}$$

uniformly for all $u \in \mathbf{r}^\infty H^{\frac{1}{2}+0}(\Omega)$. When $s = 1$, we have

$$\|\mathbf{r}^l \psi(\mathbf{d}) u\|_{H^{\frac{3}{2}}(\Omega)} \le C\|\mathbf{r}^l \tilde{\psi}(\mathbf{d})(-\Delta + M^2)u\|_{L^2(\Omega)} + C\|\mathbf{r}^l u\|_{H^1(\partial\Omega)} + C\|\mathbf{r}^l \tilde{\psi}(\mathbf{d}) u\|_{H^{-\frac{1}{2}+0}(\Omega)}, \tag{2.90}$$

for all $u \in \mathbf{r}^\infty H^{\frac{1}{2}+0}(\Omega)$. When $s = 0$, we have

$$\|\mathbf{r}^l \psi(\mathbf{d}) u\|_{H^{\frac{1}{2}}(\Omega)} \le C\|\mathbf{r}^l \tilde{\psi}(\mathbf{d})(-\Delta + M^2)u\|_{H^{-\frac{3}{2}+0}(\Omega)} + C\|\mathbf{r}^l u\|_{L^2(\partial\Omega)} + C\|\mathbf{r}^l \tilde{\psi}(\mathbf{d}) u\|_{H^{-\frac{1}{2}+0}(\Omega)}, \tag{2.91}$$

for all $u \in \mathbf{r}^\infty H^{\frac{1}{2}}(\Omega)$ with non-tangential limits almost everywhere on $\partial\Omega$. The above estimates are uniform in $M \in [0, M_0]$. Moreover, the above are true for all $0 \le s \le 1$ by replacing $\|\mathbf{r}^l u\|_{H^s(\partial\Omega)}$ on the right by $\|\mathbf{r}^l \partial_\nu u\|_{H^{s-1}(\partial\Omega)}$.

Proof. 1. Since Ω is uniformly Lipschitz, we may find χ_k uniformly bounded in $C_c^\infty(\overline{\Omega})$ a locally finite partition of unity on $\operatorname{supp}\psi(\mathbf{d})$ with uniformly bounded multiplicity and diameters of $\operatorname{supp}\chi_k$. There are also $\tilde{\chi}_k \in C_c^\infty(\overline{\Omega})$ that has uniformly bounded multiplicities and diameters of $\operatorname{supp}\tilde{\chi}_k$, that $\operatorname{supp}\chi_k \cap \operatorname{supp}(1-\tilde{\chi}_k) = \emptyset$. Fix k and let $\tilde{\Omega}$ be a bounded uniformly Lipschitz domain such that $\partial\tilde{\Omega}$ agrees with $\partial\Omega$ on $\operatorname{supp}\tilde{\chi}_k$. Let $f_\partial = \mathbf{r}^l \psi(\mathbf{d})\chi_k u|_{\partial\Omega} \in \mathbf{r}^l H^s(\partial\tilde{\Omega})$ in the sense of non-tangential limits: see further in [JK95, Corollary 5.5]. Then we aim to estimate

$$\|\mathbf{r}^l \psi(\mathbf{d})\chi_k u\|_{H^{s+\frac{1}{2}}} \le C\|\mathcal{H}_M^{\tilde{\Omega}} f_\partial\|_{H^{s+\frac{1}{2}}} + \|\mathcal{R}_M^{\tilde{\Omega}} \mathbf{r}^l \psi(\mathbf{d})\chi_k(-\Delta + M^2)u + \mathcal{R}_M^{\tilde{\Omega}}[-\Delta, \mathbf{r}^l \psi(\mathbf{d})\chi_k]u\|_{H^{s+\frac{1}{2}}}, \tag{2.92}$$

the first term of which can be bounded by Lemma 2.5,

$$\|\mathcal{H}_M^{\tilde{\Omega}} f_\partial\|_{H^{s+\frac{1}{2}}} \le C\|f_\partial\|_{H^s_\partial} = C\|\mathbf{r}^l \chi_k u\|_{H^s_\partial}. \tag{2.93}$$

For $0 < s < 1$, we utilize $\mathcal{R}_M^{\tilde{\Omega}} : H^{s-\frac{3}{2}} \to H^{s+\frac{1}{2}}$ bounded from Lemma 2.5 to see

$$\|\mathbf{r}^l \psi(\mathbf{d})\chi_k u\|_{H^{s+\frac{1}{2}}} \le C\|\mathbf{r}^l \psi(\mathbf{d})\chi_k(-\Delta + M^2)u\|_{H^{s-\frac{3}{2}}} + C\|\mathbf{r}^l \chi_k u\|_{H^s_\partial} + C\|\mathbf{r}^l \tilde{\psi}(\mathbf{d})\tilde{\chi}_k u\|_{H^{s-\frac{1}{2}}}. \tag{2.94}$$

One can choose χ_k that the diameters and Lipschitz constants of $\tilde{\Omega}$ are uniformly bounded for each k: this makes all constants C uniformly in k. Now since χ_k have uniformly bounded multiplicity, we have

$$\|\mathbf{r}^l \psi(\mathbf{d}) u\|_{H^{s+\frac{1}{2}}} \le C\|\mathbf{r}^l \psi(\mathbf{d})(-\Delta + M^2)u\|_{H^{s-\frac{3}{2}}} + C\|\mathbf{r}^l u\|_{H^s_\partial} + C\|\mathbf{r}^l \tilde{\psi}(\mathbf{d}) u\|_{H^{s-\frac{1}{2}}}. \tag{2.95}$$

Applying it first to $s = 0 + 0$ and then to any $s < 1$ justifies (2.89).

2. When $s = 0$, by Lemma 2.5 the second term of (2.92) can be bounded by

$$\|\mathcal{R}_M^{\tilde{\Omega}} \mathbf{r}^l \psi(\mathbf{d})\chi_k(-\Delta + M^2)u\|_{H^{\frac{1}{2}}} \le C\|\mathbf{r}^l \psi(\mathbf{d})\chi_k(-\Delta + M^2)u\|_{H^{-\frac{3}{2}+0}}, \tag{2.96}$$

$$\|\mathcal{R}_M^{\tilde{\Omega}}[-\Delta, \mathbf{r}^l \psi(\mathbf{d})\chi_k]u\|_{H^{\frac{1}{2}}} \le C\|[-\Delta, \mathbf{r}^l \psi(\mathbf{d})\chi_k]u\|_{H^{-\frac{3}{2}+0}} \le C\|\mathbf{r}^l \tilde{\psi}(\mathbf{d})\tilde{\chi}_k u\|_{H^{-\frac{1}{2}+0}}. \tag{2.97}$$

Similar arguments gluing up estimates in k yield the case. When $s=1$, we have $\mathcal{R}_M^{\tilde{\Omega}}: L^2 \to H^{\frac{3}{2}}$ from Lemma 2.5 instead

$$\|\mathcal{R}_M^{\tilde{\Omega}} \mathbf{r}^l \psi(\mathbf{d})\chi_k(-\Delta+M^2)u\|_{H^{\frac{3}{2}}} \leq C\|\mathbf{r}^l\psi(\mathbf{d})\chi_k(-\Delta+M^2)u\|_{L^2}, \tag{2.98}$$

$$\|\mathcal{R}_M^{\tilde{\Omega}}[-\Delta, \mathbf{r}^l\psi(\mathbf{d})\chi_k]u\|_{H^{\frac{3}{2}}} \leq C\|[-\Delta, \mathbf{r}^l\psi(\mathbf{d})\chi_k]u\|_{L^2} \leq C\|\mathbf{r}^l\tilde{\psi}(\mathbf{d})\tilde{\chi}_k u\|_{H^1}. \tag{2.99}$$

One can apply (2.89) with $s=\frac{1}{2}$ to the second line to conclude the proof for the Dirichlet case.

3. We address the Neumann cases where $0<s<1$. We further ask that $\partial_\nu \chi_k \equiv 0$. For each k, the uniformly bounded diameter of $\operatorname{supp}\tilde{\chi}_k$ implies that $C^{-1}K(k) \leq \mathbf{r} \leq CK(k)$ on $\operatorname{supp}\tilde{\chi}_k$ for some $C>0$ independent of k. Step 1 in Lemma 2.6 then implies

$$\begin{aligned}\|\psi(\mathbf{d})\chi_k u\|_{H^{s+\frac{1}{2}}} &\leq C\|(-\Delta+M^2)\psi(\mathbf{d})\chi_k u\|_{H^{s-\frac{3}{2}}} + C\|\chi_k\partial_\nu u\|_{H_\partial^{s-1}} + C\|\psi(\mathbf{d})\chi_k u\|_{H^{-\frac{1}{2}+0}} \\ &\leq C\|\psi(\mathbf{d})\chi_k(-\Delta+M^2)u\|_{H^{s-\frac{3}{2}}} + C\|\chi_k\partial_\nu u\|_{H_\partial^{s-1}} + C\|\tilde{\psi}(\mathbf{d})\tilde{\chi}_k u\|_{H^{s-\frac{1}{2}}},\end{aligned} \tag{2.100}$$

by looking at $[-\Delta, \psi(\mathbf{d})\chi_k]$. Now noting that $\mathbf{r}$ is comparable to a constant on $\operatorname{supp}\tilde{\chi}_k$, we have

$$\begin{aligned}&\|\mathbf{r}^l\psi(\mathbf{d})\chi_k u\|_{H^{s+\frac{1}{2}}} \leq CK^l\|\psi(\mathbf{d})\chi_k u\|_{H^{s+\frac{1}{2}}} \leq CK^l\|\psi(\mathbf{d})\chi_k(-\Delta+M^2)u\|_{H^{s-\frac{3}{2}}} + CK^l\|\chi_k\partial_\nu u\|_{H_\partial^{s-1}} \\ &+CK^l\|\tilde{\psi}(\mathbf{d})\tilde{\chi}_k u\|_{H^{s-\frac{1}{2}}} \leq C\|\mathbf{r}^l\psi(\mathbf{d})\chi_k(-\Delta+M^2)u\|_{H^{s-\frac{3}{2}}} + C\|\mathbf{r}^l\chi_k\partial_\nu u\|_{H_\partial^{s-1}} + C\|\mathbf{r}^l\tilde{\psi}(\mathbf{d})\tilde{\chi}_k u\|_{H^{s-\frac{1}{2}}},\end{aligned} \tag{2.101}$$

for $C>0$ independent of k. This is the counterpart to (2.94) and the rest follows similarly. The cases $s=0$ and $s=1$ are also similar. □

As a consequence of Lemma 2.22, we can control the finer regularity gain of $\mathcal{R}_M^\Omega$.

Lemma 2.23. *Let $\Omega \subset \mathbb{R}^d$ be a uniformly Lipschitz domain with $|m| < \mathfrak{r}_\Omega^{r_0}$, $|m-l| < \mathfrak{h}^{r_0}_{U_\gamma^{r_0},\partial\Omega}$. Let $0<s<1$. Let $M_0>0$. There is $C>0$ such that*

$$\|\mathbf{r}^l\mathbf{d}^{m-1-l}\mathcal{R}_M^\Omega f\|_{H^{s+\frac{1}{2}}} \leq C\|\mathbf{r}^l\mathbf{d}^{m+1-l}f\|_{H^{s-\frac{3}{2}}}, \tag{2.102}$$

uniformly for all $f \in \mathbf{r}^{-l}\mathbf{d}^{-m-1+l}H^{s-\frac{3}{2}}(\Omega)$ and $M \in [0, M_0]$.

Proof. 1. Consider first the case $s \geq \frac{1}{2}$. We begin with $f \in \mathcal{S}(\Omega)$ consisting of the restriction to Ω of elements in $\mathcal{S}(\mathbb{R}^d)$. Since $\mathcal{R}_M^\Omega : Y_m^l \to \dot{X}_m^l$ bounded, Lemma 2.22 implies

$$\begin{aligned}\|\mathbf{r}^l\psi(\mathbf{d})\mathcal{R}_M^\Omega f\|_{H^{s+\frac{1}{2}}} &\leq C\|\mathbf{r}^l\tilde{\psi}(\mathbf{d})f\|_{H^{s-\frac{3}{2}}} + C\|\mathbf{r}^l\tilde{\psi}(\mathbf{d})\mathcal{R}_M^\Omega f\|_{H^{-\frac{1}{2}+0}} \\ &\leq C\|\mathbf{r}^l\tilde{\psi}(\mathbf{d})f\|_{H^{s-\frac{3}{2}}} + C\|f\|_{Y_m^l} \leq C\|\mathbf{r}^l\mathbf{d}^{m+1-l}f\|_{H^{s-\frac{3}{2}}},\end{aligned} \tag{2.103}$$

where we used $\mathcal{R}_M^\Omega f \in \mathbf{r}^\infty \dot{H}^{\frac{1}{2}+0}$, $\mathbf{r}^{-l}\mathbf{d}^{-m-1+l}H^{-1} \subset Y_m^l$ from Lemma 2.14, $s \geq \frac{1}{2}$. We also have

$$\|\mathbf{r}^l(1-\psi(\mathbf{d}))\mathbf{d}^{m-1-l}\mathcal{R}_M^\Omega f\|_{H^1} \leq \|(1-\psi(\mathbf{d}))\mathcal{R}_M^\Omega f\|_{X_m^l} \leq C\|f\|_{Y_m^l} \leq C\|\mathbf{r}^l\mathbf{d}^{m+1-l}f\|_{H^{-1}}. \tag{2.104}$$

Noting

$$(1-\psi(\mathbf{d}))\nabla\mathcal{R}_M^\Omega f = \mathcal{R}_M^\Omega(1-\psi(\mathbf{d}))\nabla f + \mathcal{R}_M^\Omega[\Delta, \psi(\mathbf{d})]\nabla\mathcal{R}_M^\Omega f \tag{2.105}$$

and $[\Delta, \psi(\mathbf{d})]\nabla : X^l_m \to Y^l_m$ bounded, we have

$$\begin{aligned}\|(1-\psi(\mathbf{d}))\nabla\mathcal{R}^\Omega_M f\|_{X^l_m} &\le \|(1-\psi(\mathbf{d}))\nabla f\|_{Y^l_m} + \|f\|_{Y^l_m} \\ &\le \|\mathbf{r}^l(1-\psi(\mathbf{d}))\mathbf{d}^{m+1-l}\nabla f\|_{H^{-1}} + \|\mathbf{r}^l\mathbf{d}^{m+1-l}f\|_{H^{-1}} \le C\|\mathbf{r}^l\mathbf{d}^{m+1-l}f\|_{H^0}. \quad (2.106)\end{aligned}$$

We end up with

$$\|\mathbf{r}^l(1-\psi(\mathbf{d}))\mathbf{d}^{m-1-l}\mathcal{R}^\Omega_M f\|_{H^2} \le C\|\mathbf{r}^l\mathbf{d}^{m+1-l}f\|_{H^0}. \quad (2.107)$$

Note $H^\cdot(\Omega)$ is an interpolation scale: see for example [CWHM15, Lemma 4.2]. Interpolation now gives

$$\|(1-\psi(\mathbf{d}))\mathbf{r}^l\mathbf{d}^{m-1-l}\mathcal{R}^\Omega_M f\|_{H^{s+\frac{1}{2}}} \le C\|\mathbf{r}^l\mathbf{d}^{m+1-l}f\|_{H^{s-\frac{3}{2}}}, \quad (2.108)$$

and with (2.103),

$$\|\mathbf{r}^l\mathbf{d}^{m-1-l}\mathcal{R}^\Omega_M f\|_{H^{s+\frac{1}{2}}} \le C\|\mathbf{r}^l\mathbf{d}^{m+1-l}f\|_{H^{s-\frac{3}{2}}}, \quad (2.109)$$

uniformly for each $f \in \mathcal{S}'(\Omega)$. Now by the density of $\mathcal{S}(\Omega)$ in $\mathcal{S}'(\Omega)$, we have (2.102) as desired for $\frac{1}{2} \le s < 1$.

2. The case $0 < s < \frac{1}{2}$ remains. For any $\phi \in \mathcal{S}$, since $\mathcal{R}^\Omega_M$ is self-adjoint on L^2 we have

$$\langle \mathbf{r}^l\mathbf{d}^{m-1-l}\mathcal{R}^\Omega_M f, \phi\rangle = \langle f, \mathcal{R}^\Omega_M\mathbf{r}^l\mathbf{d}^{m-1-l}\phi\rangle = \langle \mathbf{r}^l\mathbf{d}^{m+1-l}f, \mathbf{r}^{-l}\mathbf{d}^{l-m-1}\mathcal{R}^\Omega_M\mathbf{r}^l\mathbf{d}^{m-1-l}\phi\rangle, \quad (2.110)$$

where we can use Step 1 to estimate

$$\|\mathbf{r}^{-l}\mathbf{d}^{l-m-1}\mathcal{R}^\Omega_M\mathbf{r}^l\mathbf{d}^{m-1-l}\phi\|_{H^{\frac{3}{2}-s}} \le C\|\phi\|_{H^{-s-\frac{1}{2}}} \quad (2.111)$$

by noting that $\frac{3}{2} - s > 1$. This implies

$$\|\mathbf{r}^l\mathbf{d}^{m-1-l}\mathcal{R}^\Omega_M f\|_{H^{s+\frac{1}{2}}} \le C\|\mathbf{r}^l\mathbf{d}^{m+1-l}f\|_{H^{s-\frac{3}{2}}} \quad (2.112)$$

as desired. □

We can now improve the mapping properties of $\mathcal{H}^\Omega_M$.

Lemma 2.24 (Elliptic regularity near the Lipschitz boundary)**.** *Let $\Omega \subset \mathbb{R}^d$ be a uniformly Lipschitz domain. Let $0 \le s \le 1$. Let $M_0 > 0$. Let $|l| < \mathfrak{r}^{r_0}_\Omega + \mathfrak{h}^{r_0}_{U^{r_0}_\gamma,\partial\Omega}$. Let $\chi \in C^\infty_c(\overline{\Omega})$. Let $\tilde{\chi} \in C^\infty_c(\overline{\Omega})$ that $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$. Then there is $C > 0$ such that*

$$\|\chi\mathcal{H}^\Omega_M f\|_{H^{s+\frac{1}{2}}(\Omega)} \le C\|\tilde{\chi}f\|_{H^s(\partial\Omega)} + C\|f\|_{H^{0,l}(\partial\Omega)}, \quad (2.113)$$

uniformly for all $f \in H^{s,l}(\partial\Omega)$ and $M \in [0, M_0]$. If $\partial_\nu\chi \equiv 0$, we further have for all $f \in H^{s,l}(\partial\Omega)$,

$$\|\chi\mathcal{H}^\Omega_M f\|_{H^{s+\frac{1}{2}}(\Omega)} \le C\|\tilde{\chi}\Lambda^\Omega_M f\|_{H^{s-1}(\partial\Omega)} + C\|f\|_{H^{0,l}(\partial\Omega)}. \quad (2.114)$$

Proof. 1. We first work with (2.113). Let $\psi \in C^\infty_c(\mathbb{R})$ that $\operatorname{supp}\chi \cap \operatorname{supp}(1-\psi(\mathbf{d})) = \emptyset$. For $s \in [0, 1)$, Lemma 2.22 implies

$$\begin{aligned}\|\psi(\mathbf{d})\chi\mathcal{H}^\Omega_M f\|_{H^{s+\frac{1}{2}}} &\le C\|\chi f\|_{H^s_\partial} + C\|[-\Delta,\chi]\mathcal{H}^\Omega_M f\|_{H^{s-\frac{3}{2}+0}} + C\|\tilde{\psi}(\mathbf{d})\chi\mathcal{H}^\Omega_M f\|_{H^{-\frac{1}{2}+0}} \\ &\le C\|\chi f\|_{H^s_\partial} + C\|\tilde{\psi}(\mathbf{d})\tilde{\chi}\mathcal{H}^\Omega_M f\|_{H^{s-\frac{1}{2}+0}} + C\|\tilde{\psi}(\mathbf{d})\chi\mathcal{H}^\Omega_M f\|_{H^{-\frac{1}{2}+0}}, \quad (2.115)\end{aligned}$$

repeated use of which gives

$$\|\psi(\mathbf{d})\chi\mathcal{H}^\Omega_M f\|_{H^{s+\frac{1}{2}}} \le C\|\tilde{\chi}f\|_{H^s_\partial} + C\|\tilde{\psi}(\mathbf{d})\tilde{\chi}\mathcal{H}^\Omega_M f\|_{H^{-\frac{1}{2}+0}}. \quad (2.116)$$

We aim to bound the last term here. We remind the reader of the extension $E_0 : H^{0,l}_{\partial} \to H^{\frac{1}{2},l}$ with $\operatorname{supp} E_0 f \subset \{\mathbf{d} \leq 2\}$ constructed in Lemma 2.18, by $E_0 = \sum_k \tilde{\chi}_k \mathcal{H}_0^{\tilde{\Omega}_k} \chi_k$. Note we have

$$\mathcal{H}^{\Omega}_M f = E_0 f - \mathcal{R}^{\Omega}_M(-\Delta + M^2)E_0 f = E_0 f - M^2 \mathcal{R}^{\Omega}_M E_0 f - \sum_k \mathcal{R}^{\Omega}_M[-\Delta, \tilde{\chi}_k]\mathcal{H}_0^{\tilde{\Omega}_k}\chi_k f \tag{2.117}$$

from the correction formula (2.8). Then $\mathcal{R}^{\Omega}_M : Y^l_m \to \dot{X}^l_m$ bounded and compact $\operatorname{supp}\tilde{\chi}$ implies

$$\|\tilde{\psi}(\mathbf{d})\tilde{\chi}\sum_k \mathcal{R}^{\Omega}_M[-\Delta, \tilde{\chi}_k]\mathcal{H}_0^{\tilde{\Omega}_k}\chi_k f\|_{H^{-\frac{1}{2}+0}} \leq C\|\mathbf{r}^l \mathbf{d}^{-N}\sum_k [-\Delta, \tilde{\chi}_k]\mathcal{H}_0^{\tilde{\Omega}_k}\chi_k f\|_{H^{-1}} \leq C\|f\|_{H^{0,l}_{\partial}}. \tag{2.118}$$

Bounding other two terms similarly, we have

$$\|\tilde{\psi}(\mathbf{d})\tilde{\chi}\mathcal{H}^{\Omega}_M f\|_{H^{-\frac{1}{2}+0}} \leq C\|f\|_{H^{0,l}_{\partial}}, \tag{2.119}$$

bringing which into (2.116) concludes the case $s \in [0, 1)$. For the case $s = 1$, (2.115) now reads

$$\|\psi(\mathbf{d})\chi\mathcal{H}^{\Omega}_M f\|_{H^{\frac{3}{2}}} \leq C\|\chi f\|_{H^1_{\partial}} + C\|\tilde{\psi}(\mathbf{d})\tilde{\chi}\mathcal{H}^{\Omega}_M f\|_{H^1} + C\|\tilde{\psi}(\mathbf{d})\chi\mathcal{H}^{\Omega}_M f\|_{H^{-\frac{1}{2}+0}} \leq C\|\tilde{\chi}f\|_{H^1_{\partial}} + C\|f\|_{H^{0,l}_{\partial}}, \tag{2.120}$$

by applying the case $s = \frac{1}{2}$ to get rid of the H^1-term and renaming the cutoffs if necessary. This shows (2.113).

2. We show (2.114). Lemma 2.22 gives the alternative to (2.115), in the case of $0 \leq s < 1$,

$$\|\chi\mathcal{H}^{\Omega}_M f\|_{H^{s+\frac{1}{2}}} \leq C\|\chi\Lambda^{\Omega}_M f\|_{H^{s-1}_{\partial}} + C\|[-\Delta, \chi]\mathcal{H}^{\Omega}_M f\|_{H^{s-\frac{3}{2}+0}} + C\|\tilde{\psi}(\mathbf{d})\chi\mathcal{H}^{\Omega}_M f\|_{H^{-\frac{1}{2}+0}}, \tag{2.121}$$

and the one with $C\|[-\Delta, \chi]\mathcal{H}^{\Omega}_M f\|_{L^2}$ in place of $C\|[-\Delta, \chi]\mathcal{H}^{\Omega}_M f\|_{H^{s-\frac{3}{2}+0}}$ for $s = 1$. Repeat Step 1 and 2 to see

$$\|\chi\mathcal{H}^{\Omega}_M f\|_{H^{s+\frac{1}{2}}} \leq C\|\tilde{\chi}\Lambda^{\Omega}_M f\|_{H^{s-1}_{\partial}} + C\|\tilde{\psi}(\mathbf{d})\tilde{\chi}\mathcal{H}^{\Omega}_M f\|_{H^{-\frac{1}{2}+0}}, \tag{2.122}$$

which gives (2.114). □

Remark 2.25. All constants in this section can depend on Ω in the following way: the upper bound on uniform Lipschitz characters of $\partial\Omega$, the upper bound on the choice of $r_0 \geq 0$, the lower bound on the choice of $\gamma > 0$, and the lower bound on Hardy constants $\mathfrak{r}^{r_0}_{\Omega}, \mathfrak{h}^{r_0}_{U^{r_0}_{\gamma},\partial\Omega}$. We show in Lemma B.2 that for all AC Lipschitz domains and all $\epsilon > 0$ there are γ, r_0 such that $\mathfrak{h}^{r_0}_{U^{r_0}_{\gamma},\partial\Omega} > \frac{1}{2} - \epsilon$. Moreover, if the only Lipschitz singularity of $\partial\Omega$ is a conic singularity with smooth cross-sections that occurs at 0, then for all $\epsilon > 0$ there is $\gamma > 0$ such that for *all* $r_0 \geq 0$ we have $\mathfrak{h}^{r_0}_{U^{r_0}_{\gamma},\partial\Omega} > \frac{1}{2} - \epsilon$. In particular, this means that in the rest of the paper, we may replace $\mathfrak{h}^{r_0}_{U^{r_0}_{\gamma},\partial\Omega}$ by $\frac{1}{2}$ and disregard the dependency on γ, without any loss of generality or control on constants, without further comment.

3. Structure of the DtN map on AC Lipschitz domains

As a preparation for our study of the DtN map on AC Lipschitz domains, we first find a microlocal approximation of it on AC Lipschitz domains.

Outline of the strategy

It is easy to see that on $\mathbb{R}^d_+$, the DtN map at $M=0$ is given by the Fourier multiplier $\Lambda_0^{\mathbb{R}^d_+} = |D|$ and therefore is a bounded operator from $H^{s,l}(\mathbb{R}^{d-1})$ to $H^{s-1,l}(\mathbb{R}^{d-1})$ only for $|l| < \frac{d-1}{2} + 1$. Moreover, it is easy to check that $|D|$ is far from local. For example, when $d=2$ and $\psi_\pm \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\psi_\pm \subset \{\pm x > 0\}$ and $\operatorname{supp}(1-\psi_\pm) \subset \{\pm x < 1\}$, we have

$$\psi_- |D| \psi_+ : H^{-N,l} \to H^{N,l+1} \tag{3.1}$$

for any N and $-\frac{3}{2} < l < l+1 < \frac{3}{2}$, but not better. Both of these remarks demonstrate that $|D|$ is far from pseudodifferential. This failure to be pseudodifferential is caused by the singularity of $|\xi|$ at $\xi = 0$ and, in fact, one can show that away from $\xi = 0$, $\Lambda_0^{\mathbb{R}_+}$ is pseudodifferential.

Motivated by the above discussion, the goal of this section is to find a (second microlocal) pseudodifferential approximation to the DtN map on an AC Lipschitz domain that is accurate as close to $\xi = 0$ as possible.

In Fermi normal coordinates $\bar{x} = (x_0, x)$, the Laplacian takes the form

$$-\Delta = D_{x_0}^2 + b(\bar{x}) D_{x_0} + R(\bar{x}, D_x). \tag{3.2}$$

One well-known method for finding an approximation to the DtN map with mass M is to factor

$$D_{x_0}^2 + b(\bar{x}) D_{x_0} + R(\bar{x}, D_x) + M^2 = (D_{x_0} + iP + b(\bar{x}))(D_{x_0} - iP) + \text{small}, \tag{3.3}$$

where P is pseudodifferential with symbol $\sqrt{R(\bar{x},\xi) + M^2}$ (see e.g. [LU89]). This is straightforward when $M > 0$. However, when $M = 0$, one immediately arrives at the problem that $R(\bar{x}, D_x)$ is *not* elliptic in the scattering calculus. In particular, its symbol vanishes at $\{\xi = 0\} \cap \{|x| = \infty\}$. As a result, $\sqrt{R(\bar{x},\xi)}$ is *not* a scattering symbol and the factorization fails.

To remedy this, we allow an error in the factorization that has a fixed, polynomial decay in x at $\xi = 0$ but rapidly becomes residual in both x and ξ away from $\xi = 0$ (see Lemma 3.2). One way to think of this is that we add an artificial damping to the Laplacian. Thus, instead of factoring the Laplacian, we factor

$$D_{x_0}^2 + b(\bar{x}) D_{x_0} + R(\bar{x}, D_x) + \operatorname{Op}(\langle x \rangle^{-2\delta} \langle \langle x \rangle^{\delta} |\xi| \rangle^{-N}), \tag{3.4}$$

for an arbitrarily large N.

This factorization involves a pseudodifferential calculus that is second microlocalized at $\{\xi = 0\} \cap \{|x| = \infty\}$ (see (3.5) for the symbols in this class and Appendix A for the construction of the calculus).

Once we have factored $-\Delta$ appropriately, in Lemma 3.4, we use energy estimates to construct the corresponding approximate Poisson extension operator, E_P, satisfying $(D_{x_0} - iP)E_P = 0$.

In section 3.4, we finally show that E_P is a good approximation of $\mathcal{H}_M^\Omega$ (and hence also P is a good approximation of Λ_M^Ω) using microlocally weighted estimates on $\mathcal{R}_M^\Omega$ in X_m^l spaces. Building on the theory developed in Section 2, these estimates are proved in Lemma 3.9.

3.1. **Pseudodifferential calculus and symbol class.** Given an AC smooth domain $\Omega \subset \mathbb{R}^d$, its boundary $\partial\Omega$ is a scattering manifold of dimension $(d-1)$, embedded in $\mathbb{R}^d$. Denote the half-space collar of $\partial\Omega$ by $\Omega_+ = \{(x_0, x) : x_0 \in [0,\infty), x \in \partial\Omega\}$, where x is the local coordinates on $\partial\Omega$. We will use the second-microlocal pseudodifferential calculus $\Psi_{k,\delta}^{s,l}(\Omega_+)$ with $M_0 > 0$ fixed,

developed in the Appendix A. To make it easier to read, we briefly describe the calculus in this subsection.

Consider the phase space $(\bar{x},\xi)\in\Omega_+\times\mathbb{R}^{d-1}$, with position variable $\bar{x}=(x_0,x)\in\Omega_+$ and momentum variable $\xi\in\mathbb{R}^{d-1}$ dual to x. One heuristically sees x_0 as the normal variable to $\partial\Omega_x$ and the phase space is $\mathbb{R}_{x_0}\times T^*\partial\Omega_x$ equivalently.

Let $s,l,k\in\mathbb{R}$ and $\delta\in[0,1)$. Fix $M_0>0$. We consider the symbol class $S^{s,l}_{k,\delta}(\Omega_+;\mathbb{R}^{d-1}_\xi)$ that contains all $a(x_0,x,\xi;h)\in C^\infty(\Omega_+\times\mathbb{R}^{d-1}_\xi\times(0,h_0))$ that for some $C_{\alpha\beta}>0$,

$$\left|(\langle x\rangle\partial_{\bar{x}})^\alpha(\langle x\rangle^{-\delta}\langle\langle x\rangle^\delta\sqrt{|\xi|^2+h^2M^2}\rangle\partial_\xi)^\beta a(\bar{x},\xi;h)\right|\le C_{\alpha\beta}\langle\xi\rangle^s\langle x\rangle^l\langle\langle x\rangle^\delta\sqrt{|\xi|^2+h^2M^2}\rangle^k,\quad(3.5)$$

uniform in all $(\bar{x},\xi;h)\in\Omega_+\times\mathbb{R}^{d-1}_\xi\times(0,h_0)$ and $M\in[0,M_0]$. This is essentially a tangential calculus in x with global control in x_0, and a second microlocalization in $\langle x\rangle^\delta\xi$. Here we interpret s as the regularity in x, l as the decay in x, and k as the weighted regularity in x. The residual symbol class is denoted $S^{-\infty,-\infty}_{-\infty,\delta}(\Omega_+;\mathbb{R}^{d-1}_\xi)$.

We have an associated quantization map $\mathrm{Op}_h:S^{s,l}_{k,\delta}(\Omega_+;\mathbb{R}^{d-1}_\xi)\to\Psi^{s,l}_{k,\delta}(\Omega_+)$ locally given by

$$\mathrm{Op}_h(a)(x_0,x,y_0,y)=\frac{\delta(y_0-x_0)}{(2\pi h)^{d-1}}\int_{\mathbb{R}^{d-1}_\xi}e^{\frac{i}{h}\langle x-y,\xi\rangle}a(x_0,x,\xi)\,d\xi,\quad(3.6)$$

and a principal symbol map $\sigma_h:\Psi^{s,l}_{k,\delta}\to S^{s,l}_{k,\delta}/hS^{s,l-(1-\delta)}_{k-1,\delta}$. The residual operators are $h^\infty\Psi^{-\infty,-\infty}_{-\infty,\delta}(\Omega_+):\langle x\rangle^NL^2_{x_0}H^{-N}_x\to h^N\langle x\rangle^{-N}L^2_{x_0}H^N_x$ for any $N\ge0$. For every $A\in\Psi^{s,l}_{k,\delta}(\Omega_+)$, there exists $a\in S^{s,l}_{k,\delta}(\Omega_+;\mathbb{R}^{d-1}_\xi)$ that $A=\mathrm{Op}_h(a)+O(h^\infty\Psi^{-\infty,-\infty}_{-\infty,\delta})$, unique modulo $S^{-\infty,-\infty}$.

We say $\chi(x_0,x)\in S^l(\Omega_+)$ if $\langle x\rangle^{-l}\chi(x_0,x)$ extends to a function in $C^\infty([0,\infty)_{x_0}\times\overline{\partial\Omega_x})$, where $\overline{\partial\Omega_x}$ is the radial compactification of $\partial\Omega_x$ using the boundary defining function $\langle x\rangle^{-1}$. This is equivalent to say there are $C_\alpha>0$ that

$$|\partial^\alpha_{\bar{x}}\chi(\bar{x})|\le C_\alpha\langle x\rangle^{l-|\alpha|}\quad(3.7)$$

uniformly in $\bar{x}\in\Omega_+$ and $\chi(x_0,x)$ is polyhomogeneous in x.

We introduce a convenient notation

$$O_\Psi=O(\langle\bar{x}\rangle^{-N}\Psi^{-\infty,-\infty}_{-\infty,\delta}(\Omega_+)):\langle\bar{x}\rangle^NL^2_{x_0}H^{-N}_x\to\langle\bar{x}\rangle^{-N}L^2_{x_0}H^N_x.\quad(3.8)$$

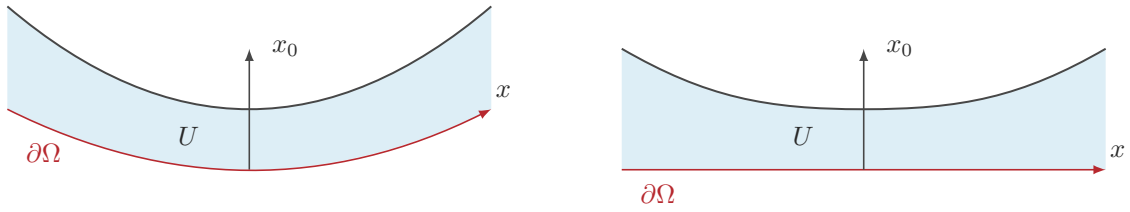
For $\chi(\bar{x})\in S^0(\Omega_+)$ such that $\operatorname{supp}\chi\subset\{x_0\le C\langle x\rangle\}$ for some $C>0$, we have an improved remainder estimate:

$$A\in\Psi^{-\infty,-\infty}_{-\infty,\delta}\Rightarrow A\chi=O_\Psi\chi,\quad(3.9)$$

by that $\langle\bar{x}\rangle\sim\langle x\rangle$ on the support of χ.

The operators in $\Psi^{s,l}_{k,\delta}(\Omega_+)$ are by nature tangential to $\partial\Omega_x=\{x_0=0\}$, in the sense that they are bounded on $\mathcal{S}'(\partial\Omega_x)$. We denote their restriction on $\partial\Omega_x$ by $\Psi^{s,l}_{k,\delta}(\partial\Omega)$.

Remark 3.1. A heuristic way to understand why we use such a calculus, is to see $\Lambda^{\mathbb{R}^2_+}_M=\sqrt{D^2+M^2}$ converging to $\Lambda^{\mathbb{R}^2_+}_0=|D|$ as Fourier multipliers. The limit $|D|$ has a $H^{\frac{3}{2}-0}$-singularity



(A) Ω (B) Ω_+

FIGURE 4. Illustration for Ω and Ω_+.

at $\xi = 0$ and thus maps $H^{s,l}$ to $H^{s-1,l}$ for $|l| \leq \frac{3}{2} - 0$, $s \in \mathbb{R}$. This decay threshold only happens at $\xi = 0$, and our calculus captures that key feature.

We denote by $\Gamma^{s,l}_{k,\delta} \in \Psi^{s,l}_{k,\delta}(\Omega_+)$ the scaling operators built in Lemma A.12 in Appendix A and corresponding second-microlocal weighted Sobolev spaces $H^{s,l}_{k,\delta}(\Omega_+) = \Gamma^{-s,-l}_{-k,\delta} L^2(\Omega_+)$, and $H^{s,l}_{k,\delta}(\partial\Omega) = \Gamma^{-s,-l}_{-k,\delta} L^2(\partial\Omega)$. We brief the readers that $\Gamma^{s,l}_{k,\delta}$ has classical principal symbol $\langle h\xi\rangle^s \langle x\rangle^l \langle h\langle x\rangle^\delta (|\xi|^2 + M^2)^{\frac{1}{2}}\rangle^k$. Note that for $k \geq 0$, we have

$$H^{s+k,l+\delta k}_{0,\delta} \subset H^{s,l}_{k,\delta}. \tag{3.10}$$

3.2. **The boundary factorization.** In this subsection, we factor the Laplacian near $\partial\Omega$ and derive its tangential part, $P \in \Psi^{0,-\delta}_{1,\delta}$, as a microlocal representation of Λ^Ω_M. We also identify an extension operator E_P such that $(D_{x_0} - iP)E_P = 0$.

We first identify a conic collar of $\partial\Omega$ in Ω by that in Ω_+. Fix some $c > 0$ small, we choose a finite set of local Fermi normal coordinates $\bar{x} = (x_0, x)$ in an asymptotically conic set U identified as $\{(x_0, x) \in \Omega_+ : x_0 < c\langle x\rangle\}$ near $\partial\Omega$, such that $\partial\Omega$ is identified as $\{x_0 = 0\}$, and the metric on Ω in those coordinates, g_+, satisfies $(g_+)_{00}(x_0, x) \equiv 1$, $(g_+)_{0j}(x_0, x) \equiv 0$ for each $1 \leq j \leq d-1$. Such U is an open subset of $\overline{\Omega}$. See Figure 4.

Here we interpret x_0 as the normal coordinates to the $\partial\Omega$ and x as the local coordinates that parametrize $\partial\Omega$. Our strategy it to perform analysis in $U \subset \Omega_+$ using the pseudodifferential tools, and then transplant the analysis to $U \subset \Omega$. We say $\chi(\bar{x}) \in S^l(U)$ if $\chi(\bar{x}) \in S^l(\Omega_+)$ and $\operatorname{supp}\chi \subset U$.

Furthermore g_+ satisfies the symbolic estimate in $\bar{x}$:

$$|\partial^\alpha_{\bar{x}} (g_+)_{ij}(\bar{x})| \leq C_\alpha \langle x\rangle^{-|\alpha|}, \ 0 \leq i, j \leq d-1, \tag{3.11}$$

for all $\bar{x} \in U$. We remind the reader that $\langle \bar{x}\rangle \sim \langle x\rangle$ on U, and we can extend the metric into $(g_+)_{ij} \in S^0(\Omega_+)$ that retains $(g_+)_{00} \equiv 1$ and $(g_+)_{0j} \equiv 0$, and $\det((g_+)_{ij}) \geq C^{-1}$ uniformly for some $C > 0$. In particular, one can extend in the way that there is $b(\bar{x}) \in S^{-1}(\Omega_+)$ that

$$-\Delta_{g_+} = D^2_{x_0} + b(\bar{x})D_{x_0} + R_2(\bar{x}, D_x) + R_1(\bar{x}, D_x), \tag{3.12}$$

where $R_k(\bar{x},\xi)$ are homogeneous polynomials of degree k in ξ and

$$|\partial_{\bar{x}}^{\alpha} R_k(\bar{x},\xi)| \leq C_\alpha \langle x\rangle^{-2+k-|\alpha|}|\xi|^k,\ R_2(\bar{x},\xi) = \sum_{i,j=1}^{d-1} (g_+)^{ij}(\bar{x})\xi_i\xi_j = |\xi|^2_{g_+(\bar{x})}, \tag{3.13}$$

where $(g_+^{ij})_{i,j=0}^{d-1} = ((g_+)_{ij})^{-1}$. Here we abbreviated $|(0,\xi_1,\cdots,\xi_{d-1})|^2_{g_+(x_0,x)} = |\xi|^2_{g_+(x_0,x)}$. At $x_0 = 0$, we have $|\xi|^2_{g_+(0,x)} = |\xi|^2_g$. Note that $R_1(\bar{x},D_x) \in h^{-1}\Psi^{0,-1-\delta}_{1,\delta}$, $R_2(\bar{x},D_x) \in h^{-2}\Psi^{0,-2\delta}_{2,\delta}$.

Our first goal is to factor $-\Delta_{g_+} + M^2$ uniformly in $M \geq 0$ inside U, into tangential parts and conormal parts with respect to $\partial\Omega$. However, at $M = 0$, $-\Delta_{g_+}$ is not uniformly elliptic in the scattering calculus or its second-microlocal variant. Therefore one expects to accept an error in the factorization in order to work in a well-behaved calculus of pseudodifferential operators.

Lemma 3.2 (Factorization of $-\Delta_{g_+} + M^2$ on Ω_+)**.** *Let $M_0 > 0$. Let $\delta \in [0,1)$. Let $h_0 > 0$ be small enough. There then exists a family of invertible operators $P \in h^{-1}\Psi^{0,-\delta}_{1,\delta}(\Omega_+)$ smoothly dependent on the parameter $h \in (0,h_0]$ and $M \in [0,M_0]$, such that the remainder*

$$\mathcal{E}_0^\circ = (-\Delta_{g_+} + M^2) - (D_{x_0} + iP + b(x_0,x))(D_{x_0} - iP) \in h^{-2}\Psi^{0,-2\delta}_{-\infty,\delta}(\Omega_+) \subset h^{-N-2}\Psi^{-N,-2\delta}_{-\infty,\delta}(\Omega_+), \tag{3.14}$$

for each $N \geq 0$. Moreover we have

$$\sigma_h(P) = h^{-1}\langle x\rangle^{-\delta}\big\langle \langle x\rangle^{\delta}\sqrt{|\xi|^2_{g_+(\bar{x})} + h^2M^2}\big\rangle,\ \sigma(P) = h^{-1}\langle x\rangle^{-\delta}\big\langle h\langle x\rangle^{\delta}\sqrt{|\xi|^2_{g_+(\bar{x})} + M^2}\big\rangle. \tag{3.15}$$

The construction of P only depends locally on g_+.

Proof. 1. Set

$$P_0 = h^{-1}\operatorname{Op}_h\left(\langle x\rangle^{-\delta}\langle\langle x\rangle^{\delta}\sqrt{|\xi|^2_{g_+} + h^2M^2}\rangle\right) \in h^{-1}\Psi^{0,-\delta}_{1,\delta},\ a_0 = \sigma_h(P_0) \in h^{-1}S^{0,-\delta}_{1,\delta}, \tag{3.16}$$

where P_0 is elliptic. We have from (3.12)

$$(D_{x_0} + iP_0 + b)(D_{x_0} - iP_0) = D_{x_0}^2 + bD_{x_0} - ibP_0 + i[P_0,D_{x_0}] + P_0^2 = -\Delta_{g_+} + M^2 + \mathcal{E}_1^\circ,$$

where

$$\mathcal{E}_1^\circ = (-R_2(x_0,x,D_x) - M^2 + P_0^2) - ibP_0 + i[P_0,D_{x_0}] - R_1(x_0,x,D_x) \in h^{-2}\Psi^{0,-2\delta}_{1,\delta}, \tag{3.17}$$

since

$$[P_0,D_{x_0}] \in h^{-1}\Psi^{0,-1-\delta}_{1,\delta},\ bP_0 \in h^{-1}\Psi^{0,-1-\delta}_{1,\delta},\ -R_2(x_0,x,D_x) - M^2 + P_0^2 \in h^{-2}\Psi^{0,-2\delta}_{1,\delta}. \tag{3.18}$$

2. Iteratively, assume that we have a_j for $j = 0,\dots N-1$, $a_j \in h^{-1}S^{0,-\delta}_{1-j,\delta}$, such that with $P_{N-1} = \sum_{j=0}^{N-1}\operatorname{Op}_h(a_j)$, we have

$$(D_{x_0} + iP_{N-1} + b)(D_{x_0} - iP_{N-1}) = -\Delta_{g_+} + M^2 + \mathcal{E}_N^\circ, \tag{3.19}$$

with $\mathcal{E}_N^\circ \in h^{-2}\Psi_{2-N,\delta}^{0,-2\delta}$. Then,

$$\begin{aligned}(D_{x_0}+iP_{N-1}+i\operatorname{Op}_h(a_N)+b)(D_{x_0}-iP_{N-1}-i\operatorname{Op}_h(a_N)) &= -\Delta_{g_+}+M^2+\mathcal{E}_N^\circ\\ -i(D_{x_0}+iP_{N-1}+b)\operatorname{Op}_h(a_N)+i\operatorname{Op}_h(a_N)(D_{x_0}-iP_{N-1}-i\operatorname{Op}_h(a_N)) &= -\Delta_{g_+}+M^2+\mathcal{E}_N^\circ\\ +2P_{N-1}\operatorname{Op}_h(a_N)+[\operatorname{Op}_h(a_N),P_{N-1}]-i[D_{x_0},\operatorname{Op}_h(a_N)]-ib\operatorname{Op}_h(a_N)+\operatorname{Op}_h(a_N)^2. &\quad (3.20)\end{aligned}$$

Setting

$$a_N = -\frac{\sigma_h(\mathcal{E}_N^\circ)}{2a_0} \in h^{-1}S_{1-N,\delta}^{0,-\delta}, \tag{3.21}$$

we have

$$\mathcal{E}_N^\circ + 2P_{N-1}\operatorname{Op}_h(a_N), [\operatorname{Op}_h(a_N),P_{N-1}], [D_{x_0},\operatorname{Op}_h(a_N)] \in h^{-2}\Psi_{1-N,\delta}^{0,-1-\delta},\ \operatorname{Op}_h(a_N)^2 \in h^{-2}\Psi_{2-2N,\delta}^{0,-2\delta}. \tag{3.22}$$

Then, with $P_N = P_{N-1}+\operatorname{Op}_h(a_N)$, we have

$$(D_{x_0}+iP_N+b)(D_{x_0}-iP_N) = -\Delta_{g_+}+M^2+\mathcal{E}_{N+1}^\circ, \tag{3.23}$$

with $\mathcal{E}_{N+1}^\circ \in h^{-2}\Psi_{1-N,\delta}^{0,-2\delta}$, and the proof is complete by induction. We end up with the asymptotic sum

$$P = \sum_{j=0}^\infty \operatorname{Op}_h(a_j),\ \mathcal{E}_0^\circ = ((-\Delta_{g_+}+M^2)-(D_{x_0}+iP+b)(D_{x_0}-iP)), \tag{3.24}$$

as desired. □

Now that we have factored $-\Delta_{g_+}+M^2$, we obtain energy estimates for $(D_{x_0}+iP+b)$. Here we denote $L_{\bar{x}}^2 = L^2(\Omega_+)$, and $L_{x_0}^\infty L_x^2 = L^\infty([0,\infty)_{x_0};L^2(\partial\Omega_x))$.

Lemma 3.3 (Energy estimate for $D_{x_0}-iP$)**.** *Let $\delta\in[0,1)$. Let $N>0$. Let $\epsilon>0$ small. Let $J\geq 0$. There exists $h_0>0$ and $C>0$ such that for all $h\in(0,h_0]$, for all $|s|,|l|,|k|\leq N$, we have*

$$\begin{aligned}\|\langle x_0\rangle^{-\frac12-\epsilon}\Gamma_{k,\delta}^{s+\frac12,l}(x_0P)^Jv\|_{L_{\bar{x}}^2}^2 + \|\Gamma_{k,\delta}^{s,l}(x_0P)^Jv\|_{L_{x_0}^\infty L_x^2}^2 + h^{-1}\|\Gamma_{k+\frac12,\delta}^{s,l-\frac{\delta}{2}}(x_0P)^Jv\|_{L_{\bar{x}}^2}^2&\\ \leq C\sum_{j=0}^J(\|\langle x_0\rangle^{1/2+\epsilon}\Gamma_{k,\delta}^{s-\frac12,l}(x_0P)^j(D_{x_0}-iP)v\|_{L_{\bar{x}}^2}^2 + \|\Gamma_{k,\delta}^{s,l}v(0)\|_{L_x^2}^2),&\quad (3.25)\end{aligned}$$

and

$$\begin{aligned}\|\langle x_0\rangle^{-\frac12-\epsilon}\Gamma_{k,\delta}^{s+\frac12,l}v\|_{L_{\bar{x}}^2}^2 + \|\Gamma_{k,\delta}^{s,l}v\|_{L_{x_0}^\infty L_x^2}^2 + h^{-1}\|\Gamma_{k+\frac12,\delta}^{s,l-\frac{\delta}{2}}v\|_{L_{\bar{x}}^2}^2&\\ \leq C(\|\langle x_0\rangle^{\frac12+\epsilon}\Gamma_{k,\delta}^{s-\frac12,l}(D_{x_0}+iP)v\|_{L^2}^2 + \liminf_{x_0\to\infty}\|\Gamma_{k,\delta}^{s,l}v(x_0)\|_{L_x^2}^2),&\quad (3.26)\end{aligned}$$

uniformly for all $v\in\mathcal{S}'(\Omega_+)$. Here we denoted $x\mapsto v(x_0,x)$ by $v(x_0)$.

Proof. 0. Through the proof, all we assumed is $P\in h^{-1}\Psi_{1,\delta}^{0,-\delta}$ being elliptic, that is for some $c>0$,

$$\Re\sigma_h(P)(\bar{x},\xi) > ch^{-1}\langle x\rangle^{-\delta}\langle\langle x\rangle^\delta\sqrt{|\xi|^2+h^2M^2}\rangle > 0,\ (\bar{x},\xi)\in\Omega_+\times\mathbb{R}^{d-1}. \tag{3.27}$$

Under this assumption, for $h>0$ small, P is invertible with its inverse $P^{-1}\in h\Psi^{0,\delta}_{-1,\delta}$. All results here still hold for any elliptic operators other than P in $h^{-1}\Psi^{0,-\delta}_{1,\delta}$.

1. We first show (3.25) in the case of $(s,l,k)=(0,0,0)$, $J=0$. Consider

$$\begin{aligned}\partial_{x_0}\|v(x_0)\|^2_{L^2_x} &= -2\Im\langle (D_{x_0}-iP)v(x_0),v(x_0)\rangle_{L^2_x} - 2\Re\langle Pv(x_0),v(x_0)\rangle_{L^2_x}\\ &\le 2|\Im\langle (D_{x_0}-iP)v(x_0),v(x_0)\rangle_{L^2_x}| - ch^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v(x_0)\|^2_{L^2_x} \quad (3.28)\end{aligned}$$

Now integrate (3.28) in x_0 from 0 to T to see

$$\begin{aligned}\|v(T)\|^2_{L^2_x} + ch^{-1}\int_0^T \|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v(x_0)\|^2_{L^2_x}\,dx_0 &\le K\int_0^T \|\langle x_0\rangle^{\frac12+\epsilon}\langle D_x\rangle^{-\frac12}(D_{x_0}-iP)v(x_0)\|^2_{L^2_x}\,dx_0\\ &\quad + \frac{1}{4K}\int_0^T \|\langle x_0\rangle^{-\frac12-\epsilon}\langle D_x\rangle^{\frac12}v(x_0)\|^2_{L^2_x}\,dx_0 + \|v(0)\|^2_{L^2_x}. \quad (3.29)\end{aligned}$$

By noting $\mathrm{I}+\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}$ is elliptic over the microsupport of $\langle D_x\rangle^{\frac12}\in\Psi^{0,0}_{\frac12,\delta}$, We can estimate

$$\begin{aligned}\frac{1}{4K}\int_0^T \|\langle x_0\rangle^{-\frac12-\epsilon}\langle D_x\rangle^{\frac12}v\|^2_{L^2_x}\,dx_0 &\le \frac{C}{4K}\|\langle x_0\rangle^{-\frac12-\epsilon}v\|^2_{L^2_{\bar x}} + \frac{C}{4K}\|\langle x_0\rangle^{-\frac12-\epsilon}\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v\|^2_{L^2_{\bar x}}\\ &\le \frac12\|v\|^2_{L^\infty_{x_0}L^2_x} + \frac{c}{2}h^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v\|^2_{L^2_{\bar x}}, \quad (3.30)\end{aligned}$$

by choosing some K large. Then taking the supremum in $T\in[0,\infty)$ in (3.29) we have

$$\frac12\|v\|^2_{L^\infty_{x_0}L^2_x} + \frac12 ch^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v\|^2_{L^2_{\bar x}} \le C\|\langle x_0\rangle^{\frac12+\epsilon}\langle D_x\rangle^{-\frac12}(D_{x_0}-iP)v\|^2_{L^2_{\bar x}} + \|v(0)\|^2_{L^2_x}. \quad (3.31)$$

Note (3.30) allows us to bound $\|\langle x_0\rangle^{-\frac12-\epsilon}\Gamma^{\frac12,0}_{0,\delta}v\|^2_{L^2_{\bar x}}$. This is (3.25) when $(s,l,k)=(0,0,0)$, $J=0$.

2. We now consider $J\ge 1$. Consider

$$\begin{aligned}\partial_{x_0}\|(x_0P)v(x_0)\|^2_{L^2_x} &\le -ch^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}(x_0P)v(x_0)\|^2_{L^2_x} + 2\langle Pv(x_0),(x_0P)v(x_0)\rangle_{L^2_x}\\ &\quad + 2|\Im\langle (x_0P)(D_{x_0}-iP)v(x_0),(x_0P)v(x_0)\rangle_{L^2_x}| + 2|\Re\langle x_0[\partial_{x_0},P]v(x_0),(x_0P)v(x_0)\rangle_{L^2_x}|. \quad (3.32)\end{aligned}$$

We estimate term by term:

$$2\langle Pv(x_0),(x_0P)v(x_0)\rangle_{L^2_x} \le Ch^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v(x_0)\|^2_{L^2_x} + \frac{c}{4h}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}(x_0P)v(x_0)\|^2_{L^2_x}, \quad (3.33)$$

and

$$2|\Re\langle x_0[\partial_{x_0},P]v(x_0),(x_0P)v(x_0)\rangle_{L^2_x}| \le C\|\Gamma^{0,-\frac12}_{0,\delta}(x_0P)v(x_0)\|^2_{L^2_x} \quad (3.34)$$

by that $[\partial_{x_0},P]\in h^{-1}\Psi^{0,-\delta-1}_{1,\delta}$ and $P\in h^{-1}\Psi^{0,-\delta}_{1,\delta}$ is elliptic, and $\Psi^{0,0}_{0,\delta}$ is bounded on L^2_x. So

$$\begin{aligned}\partial_{x_0}\|(x_0P)v(x_0)\|^2_{L^2_x} &\le -\frac{c}{2}h^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}(x_0P)v(x_0)\|^2_{L^2_x} + Ch^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v(x_0)\|^2_{L^2_x}\\ &\quad + K\|\langle x_0\rangle^{\frac12+\epsilon}\langle D_x\rangle^{-\frac12}(x_0P)(D_{x_0}-iP)v(x_0)\|^2_{L^2_x} + \frac{1}{4K}\|\langle x_0\rangle^{-\frac12-\epsilon}\langle D_x\rangle^{\frac12}(x_0P)v(x_0)\|^2_{L^2_x} \quad (3.35)\end{aligned}$$

After integrating and taking supremum in $x_0 \in [0,\infty)$, noting $(x_0P)v(x_0) = 0$ at $x_0 = 0$, we have

$$\|(x_0P)v\|^2_{L^\infty_{x_0}L^2_x} + \frac{c}{2}h^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}(x_0P)v\|^2_{L^2_{\bar x}} \le Ch^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v\|^2_{L^2_{\bar x}} + K\|\langle x_0\rangle^{\frac12+\epsilon}\langle D_x\rangle^{-\frac12}(x_0P)(D_{x_0}-iP)v\|^2_{L^2_{\bar x}} + \frac{1}{4K}\|\langle x_0\rangle^{-\frac12-\epsilon}\langle D_x\rangle^{\frac12}(x_0P)v\|^2_{L^2_{\bar x}}. \quad (3.36)$$

Now similar to (3.30) we have for large $K > 0$,

$$\frac{1}{4K}\int_0^T \|\langle x_0\rangle^{-\frac12-\epsilon}\langle D_x\rangle^{\frac12}(x_0P)v\|^2_{L^2_{\bar x}}\,dx_0 \le \frac12\|(x_0P)v\|^2_{L^\infty_{x_0}L^2_x} + \frac{c}{4}h^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}(x_0P)v\|^2_{L^2_{\bar x}}, \quad (3.37)$$

and using (3.31) to estimate $Ch^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v\|^2_{L^2_{\bar x}}$, we arrive at

$$\|(x_0P)v\|^2_{L^\infty_{x_0}L^2_x} + \frac{c}{4}h^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}(x_0P)v\|^2_{L^2_{\bar x}} \le C\sum_{j=0}^{1}\|\langle x_0\rangle^{\frac12+\epsilon}\langle D_x\rangle^{-\frac12}(x_0P)^j(D_{x_0}-iP)v\|^2_{L^2_{\bar x}} + C\|v(0)\|^2_{L^2_x}. \quad (3.38)$$

Iterate this procedure to observe

$$\frac12\|(x_0P)^Jv\|^2_{L^\infty_{x_0}L^2_x} + \frac{c}{2^{J+1}}h^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}(x_0P)^Jv\|^2_{L^2_{\bar x}} \le C\sum_{j=0}^{J}\|\langle x_0\rangle^{\frac12+\epsilon}\langle D_x\rangle^{-\frac12}(x_0P)^j(D_{x_0}-iP)v\|^2_{L^2_{\bar x}} + C\|v(0)\|^2_{L^2_x}, \quad (3.39)$$

this is (3.25) when $(s,l,k) = (0,0,0)$.

3. We now show (3.26) in the case $(s,l,k) = (0,0,0)$. Replacing P by $-P$ in (3.28) gives

$$-\partial_{x_0}\|v(x_0)\|^2_{L^2_x} \le 2|\Im\langle (D_{x_0}+iP)v(x_0), v(x_0)\rangle_{L^2_x}| - ch^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v(x_0)\|^2_{L^2_x}. \quad (3.40)$$

Let R be a large number to be set. Similarly the integration in x_0 from R back to some $T \in [0,R]$ and taking the supremum of $T \in [0,R]$ gives

$$\sup_{T\in[0,R]}\|v(T)\|^2_{L^2_x} + ch^{-1}\int_0^R \|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v(x_0)\|^2_{L^2_x}\,dx_0 \le C\int_0^R \|\langle x_0\rangle^{1/2+\epsilon}\langle D_x\rangle^{-\frac12}(D_{x_0}+iP)v(x_0)\|^2_{L^2_x}\,dx_0 + \frac12\sup_{T\in[0,R]}\|v(T)\|^2_{L^2_x} + \frac{c}{2}h^{-1}\int_0^R \|\Gamma^{0,-\frac{\delta}{2}}_{\frac12,\delta}v(x_0)\|^2_{L^2_x}\,dx_0 + \|v(R)\|^2_{L^2_x}, \quad (3.41)$$

where C is chosen independent of R. Now taking a sequence of $R \to \infty$ where $\|v(R)\|_{L^2_x} \to \liminf_{x_0\to\infty}\|v(x_0)\|_{L^2_x}$ gives (3.26) similarly when $(s,l,k) = (0,0,0)$, $J = 0$. The procedure for the case $J \ge 1$ is similar to Step 2.

4. For the general case (s,l,k), one notes

$$[\Gamma^{s,l}_{k,\delta}, D_{x_0}]\Gamma^{-s,-l}_{-k,\delta} \in \Psi^{0,-1}_{0,\delta},\ [\Gamma^{s,l}_{k,\delta}, P]\Gamma^{-s,-l}_{-k,\delta} \in \Psi^{0,-1}_{0,\delta} \quad (3.42)$$

are lower order terms compared to $P \in h^{-1}\Psi^{0,-\delta}_{1,\delta}$ elliptic, and thus

$$\Re\sigma_h(P \pm i[\Gamma^{s,l}_{k,\delta}, D_{x_0}]\Gamma^{-s,-l}_{-k,\delta} + [\Gamma^{s,l}_{k,\delta}, P]\Gamma^{-s,-l}_{-k,\delta}) = \Re\sigma_h(P) \quad (3.43)$$

also satisfies the ellipticity assumption specified in Step 0. Noting that

$$\Gamma^{s,l}_{k,\delta}(D_{x_0}-iP)\Gamma^{-s,-l}_{-k,\delta}=D_{x_0}-i(P+i[\Gamma^{s,l}_{k,\delta},D_{x_0}]\Gamma^{-s,-l}_{-k,\delta}+[\Gamma^{s,l}_{k,\delta},P]\Gamma^{-s,-l}_{-k,\delta}), \tag{3.44}$$

we arrive at (3.31) with P replaced with $P+i[\Gamma^{s,l}_{k,\delta},D_{x_0}]\Gamma^{-s,-l}_{-k,\delta}+[\Gamma^{s,l}_{k,\delta},P]\Gamma^{-s,-l}_{-k,\delta}$:

$$\|v\|^2_{L^\infty_{x_0}L^2_x}+\frac{1}{2}ch^{-1}\|\Gamma^{0,-\frac{\delta}{2}}_{\frac{1}{2},\delta}v\|^2_{L^2_{\bar{x}}}\leq C\|\langle x_0\rangle^{\frac{1}{2}+\epsilon}\langle D_x\rangle^{-\frac{1}{2}}\Gamma^{s,l}_{k,\delta}(D_{x_0}-iP)\Gamma^{-s,-l}_{-k,\delta}v\|^2_{L^2_{\bar{x}}}+\|v(0)\|^2_{L^2_x}. \tag{3.45}$$

Replace v by $\Gamma^{s,l}_{k,\delta}v$ to observe (3.25). The proof of cases $J\geq 1$ and (3.26) are similar. □

Next, we construct an extension operator $E_P:\mathcal{S}'(\partial\Omega)\to\mathcal{S}'(\Omega_+)$ that solves $(D_{x_0}-iP)E_P=0$.

Lemma 3.4 (Poisson extension of $D_{x_0}-iP$)**.** *Let $\delta\in[0,1)$. Let $N>0$. Let $\epsilon>0$ small. There exists an extension operator $E_P:\mathcal{S}'(\partial\Omega)\to\mathcal{S}'(\Omega_+)$ such that for any $u_0\in\mathcal{S}'(\partial\Omega)$,*

$$(D_{x_0}-iP)E_Pu_0=0,\ E_Pu_0|_{x_0=0}=u_0,\ E_Pu_0\in C^\infty([0,\infty)_{x_0};\langle x_0\rangle^{\frac{1}{2}+\epsilon}\mathcal{S}'(\partial\Omega)). \tag{3.46}$$

For such extension operator E_P, there exist $h_0>0$ and $C>0$ such that

$$\|\langle x_0\rangle^{-\frac{1}{2}-\epsilon}\Gamma^{-s+\frac{1}{2},-l}_{-k,\delta}E_P\Gamma^{s,l}_{k,\delta}\|_{L^2_x\to L^2_{\bar{x}}}+\|\Gamma^{-s,-l}_{-k,\delta}E_P\Gamma^{s,l}_{k,\delta}\|_{L^2_x\to L^\infty_{x_0}L^2_x}+\|\Gamma^{-s,-l-\frac{\delta}{2}}_{-k+\frac{1}{2},\delta}E_P\Gamma^{s,l}_{k,\delta}\|_{L^2_x\to L^2_{\bar{x}}}<C, \tag{3.47}$$

uniformly for all $h\in(0,h_0]$, for all $|s|,|l|,|k|\leq N$.

Proof. 1. We follow the approach of [Hö85a, Theorem 23.1.2]. Let $u_0\in\Gamma^{-s,-l}_{-k,\delta}L^2(\partial\Omega)$. We want to show that there exists a unique solution u such that $\langle x_0\rangle^{-\frac{1}{2}-\epsilon}\Gamma^{s+\frac{1}{2},l}_{k,\delta}u\in L^2(\Omega_+)$ to

$$(D_{x_0}-iP)u=0,\ u|_{x_0=0}=u_0,\ u=E_Pu_0. \tag{3.48}$$

The uniqueness of u follows from the energy estimate (3.25). We show the existence of u. Observe that u satisfies (3.48), if and only if for all $T>0$ and all $v\in C^\infty_c(\{0\leq x_0\leq T\})$, we have

$$\int_0^\infty\langle u,(D_{x_0}+iP^*)v\rangle_{L^2_x}\,dx_0=-i\langle u_0,v(0)\rangle_{L^2_x}. \tag{3.49}$$

1a. We first find $u\in\langle x_0\rangle^{\frac{1}{2}+\epsilon}\Gamma^{-s-\frac{1}{2},-l}_{-k,\delta}L^2_{\bar{x}}$ such that (3.49) holds. Applying the energy estimate (3.26) to v with P replaced by P^* and noting that $\Re\sigma(P^*)=\Re\sigma(P)$ gives

$$\|\Gamma^{-s,-l}_{-k,\delta}v\|^2_{L^\infty_{x_0}L^2_x}\leq C\|\langle x_0\rangle^{\frac{1}{2}+\epsilon}\Gamma^{-s-\frac{1}{2},-l}_{-k,\delta}(D_{x_0}+iP^*)v\|^2_{L^2_{\bar{x}}} \tag{3.50}$$

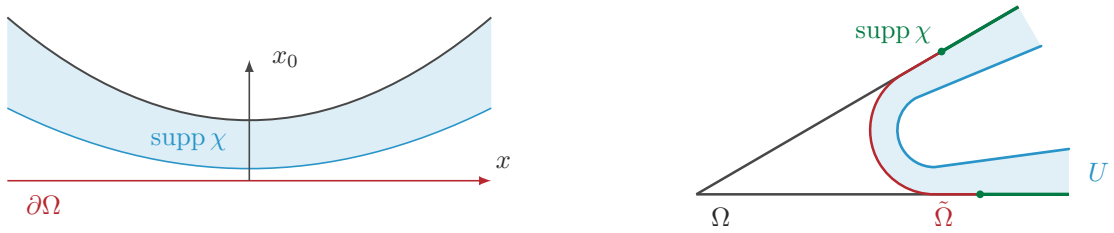
which implies

$$|\langle u_0,v(0)\rangle_{L^2_x}|\leq C\|\langle x_0\rangle^{\frac{1}{2}+\epsilon}\Gamma^{-s-\frac{1}{2},-l}_{-k,\delta}(D_{x_0}+iP^*)v\|_{L^2_{\bar{x}}}\|\Gamma^{s,l}_{k,\delta}u_0\|_{L^2_x}. \tag{3.51}$$

Hence, by the Hahn-Banach theorem, we can extend the functional

$$(D_{x_0}+iP^*)v\mapsto-i\langle u_0,v(0)\rangle_{L^2_x} \tag{3.52}$$

to any $(D_{x_0}+iP^*)v\in\Gamma^{s+\frac{1}{2},l}_{k,\delta}\langle x_0\rangle^{-\frac{1}{2}-\epsilon}L^2_{\bar{x}}$. By the Riesz Theorem there is a $u\in\langle x_0\rangle^{\frac{1}{2}+\epsilon}\Gamma^{-s-\frac{1}{2},-l}_{-k,\delta}L^2_{\bar{x}}$ satisfying (3.49) and thus (3.48), as desired.



(A) Lemma 3.6 (B) Remark 3.11

FIGURE 5. Illustration for Lemma 3.6 and Remark 3.11.

1b. Since $\langle x_0\rangle^{-\frac{1}{2}-\epsilon}\Gamma^{s+\frac{1}{2},l}_{k,\delta}u \in L^2_{\bar{x}}$, we have by the energy estimate (3.25) that

$$\|\langle x_0\rangle^{-\frac{1}{2}-\epsilon}\Gamma^{s+\frac{1}{2},l}_{k,\delta}u\|^2_{L^2_{\bar{x}}} + \|\Gamma^{s,l}_{k,\delta}u\|^2_{L^\infty_{x_0}L^2_x} + h^{-1}\|\Gamma^{s,l-\frac{\delta}{2}}_{k+\frac{1}{2},\delta}u\|^2_{L^2_{\bar{x}}} \le C\|\Gamma^{s,l}_{k,\delta}u_0\|^2_{L^2_x}, \tag{3.53}$$

yielding (3.47) and $u \in L^\infty([0,\infty)_{x_0};\Gamma^{-s,-l}_{-k,\delta}L^2_x)$.

2. We show u is C^∞ in x_0. Note $P \in h^{-1}\Psi^{0,-\delta}_{1,\delta}$ and thus (3.48) implies for almost every $x_0 \in [0,\infty)$.

$$D_{x_0}\Gamma^{s,l}_{k,\delta}u = i\Gamma^{s,l}_{k,\delta}Pu + [D_{x_0},\Gamma^{s,l}_{k,\delta}]u \in \Gamma^{0,-\delta}_{1,\delta}L^2_x, \tag{3.54}$$

where the commutator terms are of lower order. Then $u \in H^1_{\mathrm{loc}}([0,\infty)_{x_0};\Gamma^{-s,-l-\delta}_{-k+1,\delta}L^2_x)$ and is $C^{\frac{1}{2}}$ in x_0. Now note

$$D^2_{x_0}u = -P^2u + i[D_{x_0},P]u,\ [D_{x_0},P] \in h^{-1}\Psi^{0,-1-\delta}_{1,\delta}. \tag{3.55}$$

One can then repeat (3.54) for $D^2_{x_0}$ to conclude $u \in H^2_{\mathrm{loc}}([0,\infty)_{x_0};\Gamma^{-s,-l-2\delta}_{-k+2,\delta}L^2_x)$ embeds into $C^{\frac{3}{2}}([0,\infty)_{x_0};\Gamma^{-s,-l-2\delta}_{-k+2,\delta}L^2_x)$ continuously. Iteration yields the desired regularity in x_0. □

Remark 3.5. From this point onward, we fix some $h > 0$ such that Lemma 3.3 and Lemma 3.4 hold, and disregard all further dependency on h. This also makes $P \in \Psi^{0,-\delta}_{1,\delta}$ invertible with $P^{-1} \in \Psi^{0,\delta}_{-1,\delta}$. The choice of such h only depends on upper bounds C_α of all derivatives of the metric g_+ in (3.11), choice of δ, M_0, and the upper bounds N of indices in Lemma 3.3 and Lemma 3.4.

Lemma 3.6 (Nonlocality of E_P gives decay and regularity)**.** *Consider $\chi(\bar{x}),\varphi(\bar{x}) \in S^0(\Omega_+)$ that $\operatorname{supp}\chi \cap \operatorname{supp}\varphi = \emptyset$. Let $N > 0$. Then, there is $C_N > 0$ that*

$$\|\langle x_0\rangle^N\Gamma^{N,N}_{N,\delta}\chi E_P\varphi f\|_{L^2(\Omega_+)} \le C_N\|\langle x_0\rangle^{-N}\Gamma^{-N,-N}_{-N,\delta}\varphi f\|_{L^2(\partial\Omega)}, \tag{3.56}$$

uniformly for $f \in \mathcal{S}'(\partial\Omega)$. See Figure 5(A).

Proof. It suffices to split the cases into $\operatorname{supp}\chi \subset \{x_0 < 2c\langle x\rangle\}$ and $\operatorname{supp}\chi \cap \{x_0 > c\langle x\rangle\}$ for some $c > 0$. Let $\tilde{\chi} \in S^0(\Omega_+)$ that satisfy all conditions for χ and $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$.

1. Assume $\operatorname{supp}\chi \subset \{x_0 < 2c\langle x\rangle\}$. Note $\chi E_P\varphi|_{x_0=0} \equiv 0$ and

$$(D_{x_0} - iP)\chi E_P\varphi f = [D_{x_0} - iP,\chi]\tilde{\chi}E_P\varphi f + O_\Psi E_P\varphi f,\ [D_{x_0} - iP,\chi] \in \Psi^{0,-1}_{0,\delta}. \tag{3.57}$$

We remind the reader of the notation $O_\Psi = O(\langle \bar{x}\rangle^{-N}\Psi^{-\infty,-\infty}_{-\infty,\delta}(\Omega_+))$. Lemma 3.3 implies

$$\begin{aligned}\|\Gamma^{s,l}_{k,\delta}\chi E_P\varphi f\|_{L^2_{\bar{x}}} &\le C\|\langle x_0\rangle^{1/2+\epsilon}\Gamma^{s-\frac{1}{2},l+\frac{\delta}{2}}_{k-\frac{1}{2},\delta}(D_{x_0}-iP)\chi E_P\varphi f\|_{L^2_{\bar{x}}}\\ &\le C\|\langle x_0\rangle^{1/2+\epsilon}\tilde{\chi}\Gamma^{s-\frac{1}{2},l+\frac{\delta}{2}}_{k-\frac{1}{2},\delta}[D_{x_0}-iP,\chi]\tilde{\chi}E_P\varphi f\|_{L^2_{\bar{x}}} + C\|\langle x_0\rangle^{1/2+\epsilon}O(\Psi^{-\infty,-\infty}_{-\infty,\delta})\tilde{\chi}E_P\varphi f\|_{L^2_{\bar{x}}}\\ &\le C\|\Gamma^{s-\frac{1}{2},l+\frac{\delta}{2}+\epsilon-\frac{1}{2}}_{k-\frac{1}{2},\delta}\tilde{\chi}E_P\varphi f\|_{L^2_{\bar{x}}} + C\|O_\Psi\tilde{\chi}E_P\varphi f\|_{L^2_{\bar{x}}} \le C\|\Gamma^{s-\frac{1}{2},l+\frac{\delta}{2}+\epsilon-\frac{1}{2}}_{k-\frac{1}{2},\delta}\tilde{\chi}E_P\varphi f\|_{L^2_{\bar{x}}}, \quad (3.58)\end{aligned}$$

where we used $x_0 < 2c\langle x\rangle$. Now we choose $\epsilon > 0$ small that $\frac{\delta}{2}+\epsilon-\frac{1}{2} < 0$. Repeating (3.58) by increasing the support of $\tilde{\chi}$ we have

$$\|\Gamma^{s,l}_{k,\delta}\chi E_P\varphi f\|_{L^2_{\bar{x}}} \le C\|\Gamma^{s-\frac{1}{2}N,l+(\frac{\delta}{2}+\epsilon-\frac{1}{2})N}_{k-\frac{1}{2}N,\delta}\tilde{\chi}E_P\varphi f\|_{L^2_{\bar{x}}} \le C\|\Gamma^{s-\frac{1}{2}N,l+\frac{\delta}{2}+(\frac{\delta}{2}+\epsilon-\frac{1}{2})N}_{k-\frac{1}{2}-\frac{1}{2}N,\delta}\varphi f\|_{L^2_x} \quad (3.59)$$

through Lemma 3.4. This concludes the proof for the case $\operatorname{supp}\chi \subset \{x_0 < 2c\langle x\rangle\}$.

2. Assume $\operatorname{supp}\chi \subset \{x_0 > c\langle x\rangle\}$ and $\operatorname{supp}(1-\tilde{\chi}) \subset \{x_0 < 2c\langle x\rangle\}$ without loss of generality. Note $P^{-1} \in \Psi^{0,\delta}_{-1,\delta}$ per Remark 3.5 and

$$\begin{aligned}\Gamma^{s,l}_{k,\delta}\chi E_P\varphi &= \tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi P^{-N}P^N E_P\varphi + (1-\tilde{\chi})O(\Psi^{-\infty,-\infty}_{-\infty,\delta})E_P\varphi\\ &= \tilde{\chi}\langle x\rangle^{N(\frac{1}{2}+\frac{\delta}{2})}O(\Psi^{s,l-\frac{N(1-\delta)}{2}}_{k-N,\delta})P^N E_P\varphi + O_\Psi E_P\varphi. \quad (3.60)\end{aligned}$$

With that $x_0 > c\langle x\rangle$ on $\operatorname{supp}\tilde{\chi}$, we have

$$\begin{aligned}\|x_0^{\frac{N(1-\delta)}{2}}\Gamma^{s,l}_{k,\delta}\chi E_P\varphi f\| &\le C\|\tilde{\chi}\Gamma^{s,l-\frac{N(1-\delta)}{2}}_{k-N,\delta}(x_0P)^N E_P\varphi f\| + C\|O_\Psi E_P\varphi f\|_{L^2_{x_0}H^{-N}_x}\\ &\le C\|\Gamma^{s,l+\frac{\delta}{2}-\frac{N(1-\delta)}{2}}_{k-\frac{1}{2}-N,\delta}\varphi f\|_{L^2_x}, \quad (3.61)\end{aligned}$$

where we used Lemma 3.3 and Lemma 3.4. Renumber N and note the L^2-boundedness of $\Gamma^{\frac{N}{2},0}_{-\frac{N}{2},\delta} \subset \Psi^{0,0}_{0,\delta}$ to conclude the proof. □

3.3. **Boundedness properties on spaces with Hardy-type weights.** Let $\Omega \subset \mathbb{R}^d$ be an AC smooth domain. We now relate the action of the Poisson operators $\mathcal{R}^\Omega_M$ defined on Ω conjugated with operators in $\Gamma^{s,l}_{k,\delta}$ defined on the model space Ω_+, with respect to the spaces $X^l_m(\Omega_+)$ defined in Definition 2.8. Similarly we consider $\dot{Y}^l_m(\Omega_+) = (X^{-l}_{-m}(\Omega_+))^*$ and $Y^l_m(\Omega_+) = (\dot{X}^{-l}_{-m}(\Omega_+))^*$.

Within this whole section, we abuse notation by using $\mathbf{r}$, $\mathbf{d}$ for both weights on $X^l_m(\Omega)$ and the preferred weights $\mathbf{r} = \langle\bar{x}\rangle$, $\mathbf{d} = \langle x_0\rangle$ on $X^l_m(\Omega_+)$. One handy feature about connecting $X^l_m(\Omega_+)$ to $X^l_m(\Omega)$ is that given $\chi(x_0,x) \in S^0(U)$, we have $\mathbf{r} \sim \langle\bar{x}\rangle$ and $\mathbf{d} \sim \langle x_0\rangle$ on U, and thus $\chi X^l_m(\Omega_+) \subset X^l_m(\Omega)$ and $\chi X^l_m(\Omega) \subset X^l_m(\Omega_+)$.

Before stating that result we need to state couple of lemmas that will be relevant in its proof. We first give a decomposition formula for the commutator of the Laplacian with an operator in the $\Psi^{s,l}_{k,\delta}$-calculus.

Lemma 3.7 (Commutator decomposition)**.** *Let $s, l, k \in \mathbb{R}$. Let $\delta \in [0,1)$. Let $A \in \Psi^{s,l}_{k,\delta}(\Omega_+)$. Then there exist*

$$(A_0)_{ij} \in \Psi^{s,l-1+\delta}_{k-1,\delta}(\Omega_+),\ (A_1)_j \in \Psi^{s,l-1}_{k,\delta}(\Omega_+),\ A_2 \in \Psi^{s,l-2}_{k,\delta}(\Omega_+) \tag{3.62}$$

for $0 \le i, j \le d-1$ such that

$$[\Delta_{g_+}, A] = \operatorname{div} A_0 \nabla + \operatorname{div} A_1 + A_2. \tag{3.63}$$

Proof. Locally, the commutator reads

$$\begin{aligned}[\sqrt{g_+}^{-1}\partial_{x_i}\sqrt{g_+}g_+^{ij}\partial_{x_j}, A] = {}& [\sqrt{g_+}^{-1}, A]\partial_{x_i}\sqrt{g_+}g_+^{ij}\partial_{x_j} + \sqrt{g_+}^{-1}[\partial_{x_i}, A]\sqrt{g_+}g_+^{ij}\partial_{x_j} \\ & + \sqrt{g_+}^{-1}\partial_{x_i}[\sqrt{g_+}g_+^{ij}, A]\partial_{x_j} + \sqrt{g_+}^{-1}\partial_{x_i}\sqrt{g_+}g_+^{ij}[\partial_{x_j}, A]\end{aligned} \tag{3.64}$$

We identify the above four terms individually. Note

$$\begin{aligned}[\sqrt{g_+}^{-1}, A]\partial_{x_i}\sqrt{g_+}g_+^{ij}\partial_{x_j} = {}& \partial_{x_i}[\sqrt{g_+}^{-1}, A]\sqrt{g_+}g_+^{ij}\partial_{x_j} + \partial_{x_j}[[\sqrt{g_+}^{-1}, A], \partial_{x_i}]\sqrt{g_+}g_+^{ij} \\ & + [[[\sqrt{g_+}^{-1}, A], \partial_{x_i}]\sqrt{g_+}g_+^{ij}, \partial_{x_j}] \in \partial_{x_i}\Psi^{s,l-1+\delta}_{k-1,\delta}\partial_{x_j} + \partial_{x_j}\Psi^{s,l-2+\delta}_{k-1,\delta} + \Psi^{s,l-3+\delta}_{k-1,\delta},\end{aligned} \tag{3.65}$$

since $[\partial_{x_i}, \Psi^{s,l}_{k,\delta}] \subset \Psi^{s,l-1}_{k,\delta}$. Similarly, we have

$$\begin{aligned}\sqrt{g_+}^{-1}[\partial_{x_i}, A]\sqrt{g_+}g_+^{ij}\partial_{x_j} = {}& \partial_{x_j}\sqrt{g_+}^{-1}[\partial_{x_i}, A]\sqrt{g_+}g_+^{ij} + [\sqrt{g_+}^{-1}[\partial_{x_i}, A]\sqrt{g_+}g_+^{ij}, \partial_{x_j}] \\ & \in \partial_{x_j}\Psi^{s,l-1}_{k,\delta} + \Psi^{s,l-2}_{k,\delta},\end{aligned} \tag{3.66}$$

$$\begin{aligned}\sqrt{g_+}^{-1}\partial_{x_i}\sqrt{g_+}g_+^{ij}[\partial_{x_j}, A] = {}& \partial_{x_i}\sqrt{g_+}^{-1}\sqrt{g_+}g_+^{ij}[\partial_{x_j}, A] + [\sqrt{g_+}^{-1}, \partial_{x_i}]\sqrt{g_+}g_+^{ij}[\partial_{x_j}, A] \\ & \in \partial_{x_i}\Psi^{s,l-1}_{k,\delta} + \Psi^{s,l-2}_{k,\delta},\end{aligned} \tag{3.67}$$

and

$$\begin{aligned}\sqrt{g_+}^{-1}\partial_{x_i}[\sqrt{g_+}g_+^{ij}, A]\partial_{x_j} = {}& \partial_{x_i}\sqrt{g_+}^{-1}[\sqrt{g_+}g_+^{ij}, A]\partial_{x_j} + \partial_{x_j}[\sqrt{g_+}^{-1}, \partial_{x_i}][\sqrt{g_+}g_+^{ij}, A] \\ & + [[\sqrt{g_+}^{-1}, \partial_{x_i}][\sqrt{g_+}g_+^{ij}, A], \partial_{x_j}] \in \partial_{x_i}\Psi^{s,l-1+\delta}_{k-1,\delta}\partial_{x_j} + \partial_{x_j}\Psi^{s,l-2+\delta}_{k-1,\delta} + \Psi^{s,l-3+\delta}_{k-1,\delta}.\end{aligned} \tag{3.68}$$

Thus for each pair i, j, we have

$$[\sqrt{g_+}^{-1}\partial_{x_i}\sqrt{g_+}g_+^{ij}\partial_{x_j}, A] \in \partial_{x_i}\Psi^{s,l-1+\delta}_{k-1,\delta}\partial_{x_j} + \partial_{x_i}\Psi^{s,l-1}_{k,\delta} + \partial_{x_j}\Psi^{s,l-1}_{k,\delta} + \Psi^{s,l-2}_{k,\delta}. \tag{3.69}$$

Summing up indices i, j concludes the proof. □

Another lemma is needed to establish the boundedness property of $\Gamma^{s,l}_{k,\delta}$ with respect to the spaces $X^l_m(\Omega_+)$.

Lemma 3.8 (Boundedness of $\Psi^{s,l}_{k,\delta}$ on $X^l_m(\Omega_+)$)**.** *Let $\delta \in [0,1)$. Let $s, l', k, l, m \in \mathbb{R}$. Let $A \in \Psi^{s,l'}_{k,\delta}$. Let $\chi(x_0, x) \in S^0(U)$. For any $q \geq 0$, there then exists $C > 0$ such that*

$$\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q-1} A\chi u\|_{L^2(\Omega_+)} \leq C\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l-1}\Gamma^{s,l'+q}_{k,\delta}\chi u\|_{L^2(\Omega_+)}, \tag{3.70}$$

$$\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q} A\chi u\|_{L^2(\Omega_+)} \leq C\|\Gamma^{s,l'+1+q}_{k-1,\delta}\chi u\|_{X^l_m(\Omega_+)}, \tag{3.71}$$

$$\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q} \nabla A\chi u\|_{L^2(\Omega_+)} \leq C\|\Gamma^{s,l'+q}_{k,\delta}\chi u\|_{X^l_m(\Omega_+)}, \tag{3.72}$$

$$\|A\chi u\|_{X^l_{m+q}(\Omega_+)} \leq C\|\Gamma^{s,l'+q}_{k,\delta}\chi u\|_{X^l_m(\Omega_+)}, \tag{3.73}$$

for all $u \in \mathcal{S}'(\Omega_+)$. Moreover, for $B \in \Psi^{0,0}_{0,\delta}$, we have for any $N > 0$ large,

$$\|BA\chi u\|_{X^l_m(\Omega_+)} \leq C\|A\chi u\|_{X^l_m(\Omega_+)} + C\|O_\Psi \chi u\|_{X^{-N}_{-N}(\Omega_+)}, \tag{3.74}$$

$$\|Bu\|_{X^N_N(\Omega_+)} \leq C\|u\|_{X^{2N}_{2N+1}(\Omega_+)},\ \|Bu\|_{Y^{-2N}_{-2N-1}(\Omega_+)} \leq C\|u\|_{Y^{-N}_{-N}(\Omega_+)}. \tag{3.75}$$

Proof. 0. We begin with some error estimates. We have for any $N > 0$, any $s, l', k \in \mathbb{R}$,

$$\begin{aligned}\|O_\Psi u\| &\leq C\|\langle \bar{x}\rangle^{-N} O(\Psi^{-\infty,-\infty}_{-\infty,\delta})u\| \leq C\|\langle x_0\rangle^{-N} O(\Psi^{-\infty,-\infty}_{-\infty,\delta})u\| \\ &\leq C\|O(\Psi^{-\infty,-\infty}_{-\infty,\delta})\langle x_0\rangle^{-N}\langle x\rangle^{-N}\Gamma^{s,l'}_{k,\delta}u\| \leq C\|\langle \bar{x}\rangle^{-N}\Gamma^{s,l'}_{k,\delta}u\| \leq C\|\Gamma^{s,l'}_{k,\delta}u\|_{X^{-N}_{-N}}.\end{aligned} \tag{3.76}$$

by noting $\langle \bar{x}\rangle^2 \leq \langle x_0\rangle^2\langle x\rangle^2$. Similarly we have

$$\|\nabla O_\Psi u\| \leq \|O_\Psi \nabla\Gamma^{s,l'}_{k,\delta}u\| + \|O_\Psi\Gamma^{s,l'}_{k,\delta}u\| \leq C\|\Gamma^{s,l'}_{k,\delta}u\|_{X^{-N}_{-N}}. \tag{3.77}$$

since $[\nabla, O_\Psi] = O_\Psi$.

1. Let $\chi_1(\bar{x}), \chi_2(\bar{x}) \in S^0(U)$ that $\chi_1\chi = \chi$, $\chi_2\chi_1 = \chi_1$. For (3.70), note first

$$\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q-1} A\chi u\| \leq C\|\chi_1\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q-1} A\chi u\| + C\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q-1} O_\Psi\chi u\|. \tag{3.78}$$

Noting $\langle \bar{x}\rangle \sim \langle x\rangle$ on U and $\chi_2 A\Gamma^{-s,-l'}_{-k,\delta}\chi_2\langle \bar{x}\rangle^l\langle x_0\rangle^{m-l-1}\Gamma^{s,l'+q}_{k,\delta}$ elliptic over the microsupport of $\chi_1\langle \bar{x}\rangle^{l+q}\langle x_0\rangle^{m-l-1}A$, we have

$$\begin{aligned}\|\chi_1\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q-1} A\chi u\| &\leq C\|\chi_2 A\Gamma^{-s,-l'}_{-k,\delta}\chi_2\langle \bar{x}\rangle^l\langle x_0\rangle^{m-l-1}\Gamma^{s,l'+q}_{k,\delta}\chi u\| + C\|O_\Psi\chi u\| \\ &\leq C\|\langle \bar{x}\rangle^l\langle x_0\rangle^{m-l-1}\Gamma^{s,l'+q}_{k,\delta}\chi u\| + C\|O_\Psi\chi u\| \leq C\|\langle \bar{x}\rangle^l\langle x_0\rangle^{m-l-1}\Gamma^{s,l'+q}_{k,\delta}\chi u\|,\end{aligned} \tag{3.79}$$

yielding (3.70).

2. We claim we have the symbolic decomposition

$$1 = B_0 + \boldsymbol{B}\cdot\nabla_x,\ \boldsymbol{B} = (B_j)_{j=1}^{d-1},\ B_0 \in \Psi^{0,0}_{-1,\delta},\ B_j \in \Psi^{0,\delta}_{-1,\delta} \text{ for } j \geq 1. \tag{3.80}$$

Indeed, consider for $\psi(\xi) \equiv 1$ near 0 and compactly supported,

$$\boldsymbol{B} = (\mathrm{Op}_h(b_j))_{j=1}^{d-1},\ b_j = h\,|\xi|^{-2}\,\xi_j(1 - \psi(\langle x\rangle^\delta\,|\xi|)) \in S^{0,\delta}_{-1,\delta} \tag{3.81}$$

and

$$\tilde{b}_0 = 1 - h^{-1}\sum_{j=1}^{d-1}\xi_j b_j \in S^{0,0}_{-\infty,\delta} \tag{3.82}$$

One can check the vanishing of the $S^{0,0}_{0,\delta}$-principal symbol of

$$F = 1 - \mathrm{Op}_h(\tilde{b}_0) + i\,\mathrm{div}_x \boldsymbol{B} \in \Psi^{0,-1+\delta}_{-1,\delta}. \tag{3.83}$$

Set $B_0 = \mathrm{Op}_h(\tilde{b}_0) + F \in \Psi^{0,0}_{-1,\delta}$. We now have the symbolic decomposition

$$1 = B_0 - i\,\mathrm{div}_x \boldsymbol{B}, \tag{3.84}$$

whose adjoint gives (3.80) after renaming the operators.

3. We show (3.71). Decompose, with $A_0 \in \Psi^{0,-1}_{1,\delta}$ that

$$A = \langle x\rangle^{-q} A_0 \Gamma^{s,l'+1+q}_{k-1,\delta} = \langle x\rangle^{-q} A_0 (B_0 + \boldsymbol{B}\cdot\nabla_x)\Gamma^{s,l'+1+q}_{k-1,\delta}. \tag{3.85}$$

Let $v = \Gamma^{s,l'+1+q}_{k-1,\delta}\chi u$ and we have

$$\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q} A\chi u = \langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q} A_0 B_0 v + \langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q} A_0 \boldsymbol{B}\cdot\nabla_x v, \tag{3.86}$$

and

$$\begin{aligned}\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q} A_0 B_0 \Gamma^{s,l'+1+q}_{k-1,\delta}\chi &= \langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q}\chi_1 A_0 B_0 \Gamma^{s,l'+1+q}_{k-1,\delta}\chi + O_\Psi\chi \\ &= A_0 B_0 \langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q}\chi_1 \Gamma^{s,l'+1+q}_{k-1,\delta}\chi + [\langle \bar{x}\rangle^l \langle x\rangle^{-q}\chi_1, A_0 B_0]\langle x_0\rangle^{m-l+q}\Gamma^{s,l'+1+q}_{k-1,\delta}\chi + O_\Psi\chi.\end{aligned} \tag{3.87}$$

Since $[\langle \bar{x}\rangle^l \langle x\rangle^{-q}\chi_1, A_0 B_0] \in \Psi^{0,l-q-2+\delta}_{-1,\delta}$, we can estimate using $\langle \bar{x}\rangle^{-q} \le \langle x_0\rangle^{-q}$ that

$$\begin{aligned}\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q} A_0 B_0 v\| &\le \|A_0 B_0 \langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q}\chi_1 v\| + \|[\langle \bar{x}\rangle^l \langle x\rangle^{-q}\chi_1, A_0 B_0]\langle x_0\rangle^{m-l+q} v\| \\ &\quad + C\|O_\Psi\chi u\| \le C\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l-1} v\| + C\|O_\Psi\chi u\|.\end{aligned} \tag{3.88}$$

Similarly we have

$$\begin{aligned}&\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q} A_0 \boldsymbol{B}\cdot\nabla_x \Gamma^{s,l'+1+q}_{k-1,\delta}\chi = \langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q}\chi_1 A_0 \boldsymbol{B}\cdot\nabla_x \Gamma^{s,l'+1+q}_{k-1,\delta}\chi + O_\Psi\chi \\ &= A_0 \boldsymbol{B}\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q}\chi_1 \cdot\nabla_x \Gamma^{s,l'+1+q}_{k-1,\delta}\chi + [\langle \bar{x}\rangle^l \langle x\rangle^{-q}\chi_1, A_0\boldsymbol{B}]\cdot\langle x_0\rangle^{m-l+q}\nabla_x \Gamma^{s,l'+1+q}_{k-1,\delta}\chi + O_\Psi\chi,\end{aligned} \tag{3.89}$$

and

$$\begin{aligned}&\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q} A_0 \boldsymbol{B}\cdot\nabla_x v\| \le \|A_0\boldsymbol{B}\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q}\langle x\rangle^{-q}\chi_1\cdot\nabla_x v\| \\ &+ \|[\langle \bar{x}\rangle^l \langle x\rangle^{-q}\chi_1, A_0\boldsymbol{B}]\cdot\langle x_0\rangle^{m-l+q}\nabla_x v\| + C\|O_\Psi\chi v\| \le C\|\langle \bar{x}\rangle^{l-1+\delta}\langle x_0\rangle^{m-l}\nabla_x v\| + C\|O_\Psi\chi u\|,\end{aligned} \tag{3.90}$$

since $[\langle \bar{x}\rangle^l \langle x\rangle^{-q}\chi_1, A_0\boldsymbol{B}] \in \Psi^{0,l-q-2+2\delta}_{-1,\delta}$. Thus we have

$$\begin{aligned}\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l+q} A\chi u\| &\le C\|\langle \bar{x}\rangle^l \langle x_0\rangle^{m-l-1} v\| + C\|\langle \bar{x}\rangle^{l-1+\delta}\langle x_0\rangle^{m-l}\nabla_x v\| + C\|O_\Psi\chi u\| \\ &\le C\|\Gamma^{s,l'+1+q}_{k-1,\delta}\chi u\|_{X^l_m} + C\|O_\Psi\chi u\| \le C\|\Gamma^{s,l'+1+q}_{k-1,\delta}\chi u\|_{X^l_m}.\end{aligned} \tag{3.91}$$

This is (3.71).

4. For (3.72), we employ a decomposition similar to (3.85) to see for some $A_0 \in \Psi^{0,0}_{0,\delta}$,

$$\nabla A\chi = \chi_1\langle x\rangle^{-q}A_0\nabla\Gamma^{s,l'+q}_{k,\delta}\chi + [\nabla, \chi_1\langle x\rangle^{-q}A_0]\Gamma^{s,l'+q}_{k,\delta}\chi + \nabla O_\Psi\chi. \tag{3.92}$$

We estimate for $v = \Gamma^{s,l'+q}_{k,\delta}\chi u$, since $[\langle\bar{x}\rangle^l\langle x\rangle^{-q}\chi_1, A_0] \in \Psi^{0,l-1+\delta-q}_{-1,\delta}$,

$$\begin{aligned}\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l+q}\chi_1\langle x\rangle^{-q}A_0\nabla\Gamma^{s,l'+q}_{k,\delta}\chi u\| &\le C\|A_0\langle\bar{x}\rangle^{l-q}\langle x_0\rangle^{m-l+q}\chi_1\nabla v\| \\ &+ C\|[\langle\bar{x}\rangle^l\langle x\rangle^{-q}\chi_1, A_0]\langle x_0\rangle^{m-l+q}\nabla v\| \le C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\nabla v\| + C\|\langle\bar{x}\rangle^{l-1+\delta}\langle x_0\rangle^{m-l}\nabla v\| \\ &\le C\|\Gamma^{s,l'+q}_{k,\delta}\chi u\|_{X^l_m}. \end{aligned}\tag{3.93}$$

Note that $[\nabla, \chi_1\langle x\rangle^{-q}A_0]\Gamma^{s,l'+q}_{k,\delta} \in \Psi^{s,l'-1}_{k,\delta}$ and applying (3.70) to it gives

$$\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l+q}[\nabla, \chi_1\langle x\rangle^{-q}A_0]\Gamma^{s,l'+q}_{k,\delta}\chi u\| \le C\|\Gamma^{s,l'+q}_{k,\delta}\chi u\|_{X^l_m} + C\|O_\Psi\chi u\|. \tag{3.94}$$

Summing up above two estimates and add $\nabla O_\Psi\chi u$ to obtain (3.72) as desired.

5. So far we have obtained (3.73),

$$\|A\chi u\|_{X^l_{m+q}(\Omega_+)} \le C\|\Gamma^{s,l'+q}_{k,\delta}\chi u\|_{X^l_m(\Omega_+)} + C\|O_\Psi\chi u\|_{L^2(\Omega_+)} + C\|O_\Psi\nabla\chi u\|_{L^2(\Omega_+)}, \tag{3.95}$$

where the two remainder terms are controlled by $C\|\Gamma^{s,l'+q}_{k,\delta}\chi u\|_{X^l_m(\Omega_+)}$ by Step 1. Note

$$BA\chi u = B\chi_1 A\chi u + B(1-\chi_1)A\chi u,\ B(1-\chi_1)A\chi = O(\Psi^{-\infty,-\infty}_{-\infty,\delta})\chi = O_\Psi\chi \tag{3.96}$$

implies (3.74),

$$\|BA\chi u\|_{X^l_m} \le C\|A\chi u\|_{X^l_m} + C\|O_\Psi\chi u\|_{X^{-N}_{-N}}. \tag{3.97}$$

For (3.75), note

$$\|\langle\bar{x}\rangle^N\langle x_0\rangle^{-1}Bu\| \le \|\langle x_0\rangle^{N-1}(\langle x\rangle^N B\langle x\rangle^{-N})\langle x\rangle^N u\| \le C\|\langle x_0\rangle^{N-1}\langle x\rangle^N u\| \le C\|\langle\bar{x}\rangle^{2N}\langle x_0\rangle^{-1}u\| \tag{3.98}$$

and similarly

$$\|\langle\bar{x}\rangle^N\nabla Bu\| \le C\|\langle\bar{x}\rangle^N B\nabla u\| + C\|\langle\bar{x}\rangle^N O(\Psi^{0,-1}_{0,\delta})u\| \le C\|\langle\bar{x}\rangle^{2N}\nabla u\| + C\|\langle\bar{x}\rangle^{2N}u\| \tag{3.99}$$

to observe B is bounded from X^{2N}_{2N+1} to X^N_N and also bounded from $\dot{X}^{2N}_{2N+1}$ to $\dot{X}^N_N$. Duality arguments then give the second half of (3.75). □

Lemma 3.9. *Let $\delta \in [0,1)$. Let $M_0 > 0$. Let $|m| < \mathfrak{r}^{r_0}_\Omega$ and $|m-l| < \frac{1}{2}$. Fix $\chi, \varphi \in S^0(U)$. Then for all $s, k \in \mathbb{R}$,*

$$\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\varphi\Gamma^{-s,0}_{-k,\delta} : Y^l_m(\Omega_+) \to \dot{X}^l_m(\Omega_+) \tag{3.100}$$

is bounded uniformly in $M \in [0, M_0]$.

Proof. 1. Let $\chi_1(\bar{x}), \chi_2(\bar{x}), \chi_3(\bar{x}) \in S^0(U)$ that $\operatorname{supp}\chi \cap \operatorname{supp}(1-\chi_1) = \operatorname{supp}\chi_1 \cap \operatorname{supp}(1-\chi_2) = \operatorname{supp}\chi_2 \cap \operatorname{supp}(1-\chi_3) = \emptyset$. Let $\varphi_1(\bar{x}), \varphi_2(\bar{x}) \in S^0(U)$ that $\operatorname{supp}\varphi \cap \operatorname{supp}(1-\varphi_1) = \operatorname{supp}\varphi_1 \cap \operatorname{supp}(1-\varphi_2) = \emptyset$. We claim that for any $N > 0$ there exists a constant $C > 0$ such that for all $v \in \mathcal{S}'(\Omega_+)$,

$$\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\varphi v\|_{X^l_m} \le C\|\chi_2\Gamma^{s,0}_{k,\delta}v\|_{Y^l_m} + C\|\Gamma^{s,0}_{k,\delta}v\|_{Y^{-N}_{-N}} + C\|\Gamma^{-N,0}_{-N,\delta}\chi_1\mathcal{R}^\Omega_M\varphi v\|_{X^l_m} \tag{3.101}$$

We will prove the claim in this step. Begin with noting that $\chi_1\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\varphi v|_{x_0=0}\equiv 0$ and

$$(-\Delta+M^2)\chi_1\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\varphi v=\chi_1\Gamma^{s,0}_{k,\delta}\chi\varphi v+[-\Delta,\chi_1\Gamma^{s,0}_{k,\delta}\chi]\mathcal{R}^\Omega_M\varphi v. \tag{3.102}$$

The definition of $\mathcal{R}^\Omega_M$ implies

$$\chi_1\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\varphi v=\chi_2\mathcal{R}^\Omega_M\chi_1\Gamma^{s,0}_{k,\delta}\chi\varphi v+\chi_2\mathcal{R}^\Omega_M[-\Delta,\chi_1\Gamma^{s,0}_{k,\delta}\chi]\mathcal{R}^\Omega_M\varphi v, \tag{3.103}$$

the first term of which can be estimated by

$$\|\chi_2\mathcal{R}^\Omega_M\chi_1\Gamma^{s,0}_{k,\delta}\chi\varphi v\|_{X^l_m(\Omega_+)}\le C\|\chi_1\Gamma^{s,0}_{k,\delta}\chi\varphi v\|_{Y^l_m(\Omega)}\le C\|\chi_2\Gamma^{s,0}_{k,\delta}v\|_{Y^l_m(\Omega_+)}+C\|\Gamma^{s,0}_{k,\delta}v\|_{Y^{-N}_{-N}(\Omega_+)}. \tag{3.104}$$

Here we used the boundedness of $\mathcal{R}^\Omega_M: Y^l_m(\Omega)\to\dot X^l_m(\Omega)$ in Proposition 2.16, and for the last inequality, consider

$$\chi_1\Gamma^{s,0}_{k,\delta}\chi\varphi v=\chi_1\Gamma^{s,0}_{k,\delta}\chi\varphi\Gamma^{-s,0}_{-k,\delta}\chi_2\Gamma^{s,0}_{k,\delta}v+\chi_1\Gamma^{s,0}_{k,\delta}\chi\varphi\Gamma^{-s,0}_{-k,\delta}(1-\chi_2)\Gamma^{s,0}_{k,\delta}v, \tag{3.105}$$

which implies for each $\phi\in\dot X^{-l}_{-m}(\Omega)$,

$$\begin{aligned}|\langle\chi_1\Gamma^{s,0}_{k,\delta}\chi\varphi v,\phi\rangle|&\le\|\chi_2\Gamma^{s,0}_{k,\delta}v\|_{Y^l_m(\Omega_+)}\|\Gamma^{-s,0}_{-k,\delta}\varphi\chi\Gamma^{s,0}_{k,\delta}\chi_1\phi\|_{X^{-l}_{-m}(\Omega_+)}\\&\quad+\|\Gamma^{s,0}_{k,\delta}v\|_{Y^{-N}_{-N}(\Omega_+)}\|(1-\chi_2)\Gamma^{-s,0}_{-k,\delta}\varphi\chi\Gamma^{s,0}_{k,\delta}\chi_1\phi\|_{X^N_N(\Omega_+)}\\&\le C\big(\|\chi_2\Gamma^{s,0}_{k,\delta}v\|_{Y^l_m(\Omega_+)}+\|\Gamma^{s,0}_{k,\delta}v\|_{Y^{-N}_{-N}(\Omega_+)}\big)\|\phi\|_{X^{-l}_{-m}(\Omega)},\end{aligned} \tag{3.106}$$

by noting $\Gamma^{-s,0}_{-k,\delta}\varphi\chi\Gamma^{s,0}_{k,\delta}\in\Psi^{0,0}_{0,\delta}$ and $(1-\chi_2)\Gamma^{-s,0}_{-k,\delta}\varphi\chi\Gamma^{s,0}_{k,\delta}\chi_1=O_\Psi\chi_1$ to apply Lemma 3.8. To estimate the second term, Lemma 3.7 gives a decomposition of the commutator:

$$[-\Delta,\chi_1\Gamma^{s,0}_{k,\delta}\chi]=\chi_2\big(\operatorname{div}A_0\nabla+\operatorname{div}A_1+A_2\big)\chi_1 \tag{3.107}$$

with $A_0\in\Psi^{s,-1+\delta}_{k-1,\delta}$, $A_1\in\Psi^{s,-1}_{k,\delta}$, $A_2\in\Psi^{s,-2}_{k,\delta}$. We can then bound

$$\begin{aligned}\|\chi_2\mathcal{R}^\Omega_M[-\Delta,\chi_1\Gamma^{s,0}_{k,\delta}\chi]\mathcal{R}^\Omega_M\varphi v\|_{X^l_m(\Omega_+)}&\le C\|\chi_2\operatorname{div}(A_0\nabla+A_1)\chi_1\mathcal{R}^\Omega_M\varphi v\|_{Y^l_m(\Omega_+)}\\&\quad+C\|\chi_2A_2\chi_1\mathcal{R}^\Omega_M\varphi v\|_{Y^l_m(\Omega_+)}.\end{aligned} \tag{3.108}$$

We first estimate the A_2 term. For each $\phi\in\dot X^{-l}_{-m}(\Omega_+)$, we have $\chi_2\phi\in\dot X^{-l}_{-m}(\Omega_+)$. By Lemma 3.8,

$$\|\langle\bar x\rangle^{-l}\langle x_0\rangle^{-m+l}\Gamma^{0,-1}_{1,\delta}\chi_2\phi\|\le C\|\chi_2\phi\|_{X^{-l}_{-m}(\Omega_+)}\le C\|\phi\|_{X^{-l}_{-m}(\Omega_+)}. \tag{3.109}$$

We then can bound

$$\begin{aligned}|\langle\chi_2A_2\chi_1\mathcal{R}^\Omega_M\varphi v,\phi\rangle|&\le C\|\langle\bar x\rangle^l\langle x_0\rangle^{m-l}\Gamma^{0,1}_{-1,\delta}A_2\chi_1\mathcal{R}^\Omega_M\varphi v\|\|\langle\bar x\rangle^{-l}\langle x_0\rangle^{-m+l}\Gamma^{0,-1}_{1,\delta}\chi_2\phi\|\\&\le C\|\langle\bar x\rangle^l\langle x_0\rangle^{m-l}\Gamma^{0,1}_{-1,\delta}A_2\chi_1\mathcal{R}^\Omega_M\varphi v\|\|\phi\|_{X^{-l}_{-m}(\Omega_+)}.\end{aligned} \tag{3.110}$$

This implies

$$\begin{aligned}\|\chi_2A_2\chi_1\mathcal{R}^\Omega_M\varphi v\|_{Y^l_m(\Omega_+)}&\le C\|\langle\bar x\rangle^l\langle x_0\rangle^{m-l}\Gamma^{0,1}_{-1,\delta}A_2\chi_1\mathcal{R}^\Omega_M\varphi v\|\\&\le C\|\langle\bar x\rangle^l\langle x_0\rangle^{m-l}\chi_2\Gamma^{0,1}_{-1,\delta}A_2\chi_1\mathcal{R}^\Omega_M\varphi v\|+C\|O_\Psi\chi_1\mathcal{R}^\Omega_M\varphi v\|\end{aligned} \tag{3.111}$$

By the ellipticity of $\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\chi_3\Gamma^{s,-1}_{k-1,\delta}$ over the microsupport of $\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\chi_2\Gamma^{0,1}_{-1,\delta}A_2$, we have

$$\begin{aligned}\|\chi_2A_2\chi_1\mathcal{R}^\Omega_M\varphi v\|_{Y^l_m(\Omega_+)}&\leq C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\chi_3\Gamma^{s,-1}_{k-1,\delta}\chi_1\mathcal{R}^\Omega_M\varphi v\|+C\|O_\Psi\chi_1\mathcal{R}^\Omega_M\varphi v\|\\&\leq C\|\Gamma^{s,0}_{k-1,\delta}\chi_1\mathcal{R}^\Omega_M\varphi v\|_{X^l_m(\Omega_+)}+C\|O_\Psi\chi_1\mathcal{R}^\Omega_M\varphi v\|_{X^{-N}_{-N}(\Omega_+)}\leq C\|\Gamma^{s,0}_{k-1,\delta}\chi_1\mathcal{R}^\Omega_M\varphi v\|_{X^l_m(\Omega_+)}.\end{aligned}\tag{3.112}$$

We now deal with the $(A_0\nabla+A_1)$-term. Then

$$\begin{aligned}|\langle\chi_2\operatorname{div}(A_0\nabla+A_1)\chi_1\mathcal{R}^\Omega_M\varphi v,\phi\rangle|&\leq C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}(A_0\nabla+A_1)\chi_1\mathcal{R}^\Omega_M\varphi v\|\|\langle\bar{x}\rangle^{-l}\langle x_0\rangle^{-m+l}\nabla\phi\|\\&\leq C\big(\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}A_0\nabla\chi_1\mathcal{R}^\Omega_M\varphi v\|+\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}A_1\chi_1\mathcal{R}^\Omega_M\varphi v\|\big)\|\phi\|_{X^{-l}_{-m}(\Omega_+)},\end{aligned}\tag{3.113}$$

With $A_0\nabla=\nabla A_0+O(\Psi^{s,-2+\delta}_{k-1,\delta})$, Lemma 3.8 then implies

$$\|\chi_2\operatorname{div}(A_0\nabla+A_1)\chi_1\mathcal{R}^\Omega_M\varphi v\|_{Y^l_m(\Omega_+)}\leq C\|\Gamma^{s,0}_{k-1,\delta}\chi_1\mathcal{R}^\Omega_M\varphi v\|_{X^l_m(\Omega_+)}.\tag{3.114}$$

Summing up the estimates above and noting $\Gamma^{\frac12,0}_{-\frac12,\delta}\in\Psi^{0,0}_{0,\delta}$ is bounded on $X^l_m(\Omega_+)$, we have

$$\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\varphi v\|_{X^l_m}\leq C\|\chi_2\Gamma^{s,0}_{k,\delta}v\|_{Y^l_m}+C\|\Gamma^{s,0}_{k,\delta}v\|_{Y^{-N}_{-N}}+C\|\Gamma^{s-\frac12,0}_{k-\frac12,\delta}\chi_1\mathcal{R}^\Omega_M\varphi v\|_{X^l_m}.\tag{3.115}$$

Repeated use of (3.115) gives (3.101) with cutoff functions renamed.

2. Apply (3.101) with $v=\Gamma^{-s,0}_{-k,\delta}u$, we have

$$\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\varphi\Gamma^{-s,0}_{-k,\delta}u\|_{X^l_m}\leq C\|\chi_2u\|_{Y^l_m}+C\|u\|_{Y^{-N}_{-N}}+C\|\Gamma^{-N,0}_{-N,\delta}\chi_1\mathcal{R}^\Omega_M\varphi\Gamma^{-s,0}_{-k,\delta}u\|_{X^l_m}.\tag{3.116}$$

The first two terms are bounded by $C\|u\|_{Y^l_m}$. We will control the last term. We claim

$$T=\Gamma^{-N,0}_{-N,\delta}\chi_1\mathcal{R}^\Omega_M\varphi\Gamma^{-s,0}_{-k,\delta}:Y^l_m(\Omega_+)\to\dot{X}^l_m(\Omega_+)\tag{3.117}$$

is bounded. This is equivalent to show its adjoint

$$T^*=\Gamma^{-s,0}_{-k,\delta}\varphi\mathcal{R}^\Omega_M\chi_1\Gamma^{-N,0}_{-N,\delta}:Y^{-l}_{-m}(\Omega_+)\to\dot{X}^{-l}_{-m}(\Omega_+)\tag{3.118}$$

is bounded. Here we used the fact that the adjoint of $\mathcal{R}^\Omega_M$ coincides with itself on the spaces of inverted parameters. Now for $w\in Y^{-l}_{-m}(\Omega_+)$, apply (3.101) with inverted parameters and $v=\Gamma^{-N,0}_{-N,\delta}w$ to see for $N>0$ large, we have

$$\begin{aligned}\|T^*w\|_{X^{-l}_{-m}}=\|\Gamma^{-s,0}_{-k,\delta}\varphi\mathcal{R}^\Omega_M\chi_1\Gamma^{-N,0}_{-N,\delta}w\|_{X^{-l}_{-m}(\Omega_+)}&\leq C\|\varphi_2\Gamma^{-s-N,0}_{-k-N,\delta}w\|_{Y^{-l}_{-m}(\Omega_+)}+C\|\Gamma^{-s-N,0}_{-k-N,\delta}w\|_{Y^{-N}_{-N}(\Omega_+)}\\&+C\|\Gamma^{-N,0}_{-N,\delta}\varphi_1\mathcal{R}^\Omega_M\chi_1\Gamma^{-N,0}_{-N,\delta}w\|_{X^{-l}_{-m}(\Omega_+)}\leq C\|w\|_{Y^{-l}_{-m}(\Omega_+)}.\end{aligned}\tag{3.119}$$

Note here we used by Lemma 3.8, for $N>0$ large we have

$$\|\varphi_2\Gamma^{-s-N,0}_{-k-N,\delta}w\|_{Y^{-l}_{-m}(\Omega_+)}\leq C\|w\|_{Y^{-l}_{-m}(\Omega_+)},\ \|\Gamma^{-s-N,0}_{-k-N,\delta}w\|_{Y^{-N}_{-N}(\Omega_+)}\leq C\|w\|_{Y^{-N/2}_{-N/2}(\Omega_+)},\tag{3.120}$$

and since $\mathcal{R}^\Omega_M:Y^l_m(\Omega)\to\dot{X}^l_m(\Omega)$ is bounded, we also have

$$\begin{aligned}\|\Gamma^{-N,0}_{-N,\delta}\varphi_1\mathcal{R}^\Omega_M\chi_1\Gamma^{-N,0}_{-N,\delta}w\|_{X^{-l}_{-m}(\Omega_+)}&\leq C\|\varphi_1\mathcal{R}^\Omega_M\chi_1\Gamma^{-N,0}_{-N,\delta}w\|_{X^{-l}_{-m}(\Omega)}\\&\leq C\|\chi_1\Gamma^{-N,0}_{-N,\delta}w\|_{Y^{-l}_{-m}(\Omega)}\leq C\|w\|_{Y^{-l}_{-m}(\Omega_+)}.\end{aligned}\tag{3.121}$$

Now T and T^* are bounded. Revisit (3.116) to conclude the proof. □

3.4. **Structure of the DtN map on AC Lipschitz domains.**

Definition 3.10. We say two AC Lipschitz domains Ω_1, Ω_2 share the common boundary near a closed set $K \subset \partial\Omega_1 \cap \partial\Omega_2$, if there is $U \subset \Omega_1 \cap \Omega_2$ as an open subset of the manifolds with boundary Ω_1, Ω_2, that U contains K. Furthermore, we say they share the common smooth boundary, if $\partial\Omega_1 \cap \partial\Omega_2$ is further smooth.

Remark 3.11 (Geometric setup of AC Lipschitz domains)**.** Let Ω be an AC Lipschitz domain. We define the regular set $\partial\Omega_{\text{reg}}$ consists of $x \in \partial\Omega$ that $\partial\Omega$ is smooth near x, and the singular set $\partial\Omega_{\text{sing}} = \partial\Omega \setminus \partial\Omega_{\text{reg}}$. Near any closed $K \subset \partial\Omega_{\text{reg}}$, there exists an AC smooth domain $\tilde{\Omega} \subset \Omega$ that $\Omega, \tilde{\Omega}$ share the common smooth boundary near K, associated with its conic collar U, half-space collar $\tilde{\Omega}_+$, P, E_P given in Lemma 3.2 and Lemma 3.4. We call such U a regular conic collar of Ω. See Figure 5(B).

In this setup, we now show the main result in this section: P describes the structure of Λ_M^Ω, the Dirichlet to Neumann map on Ω, with an error that is arbitrarily smooth and decays fast up to a threshold at the zero frequency.

Let $\chi(x), \tilde{\chi}(x) \in S^0(\partial\Omega_{\text{reg}})$ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$. Here we confuse $\chi, \tilde{\chi}$ with any of their extensions in $S^0(U)$ that $\operatorname{supp}\tilde{\chi} \subset \overline{\Omega}$, $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$ and $\partial_\nu\tilde{\chi} = \partial_\nu\chi = 0$.

Proposition 3.12 (Structure of Λ_M^Ω and $\mathcal{H}_M^\Omega$)**.** *Let $\Omega \subset \mathbb{R}^d$ be an AC Lipschitz domain with a regular collar U. Let $M_0 > 0$. Let $\delta \in [0,1)$. Let $\chi, \tilde{\chi}, \tilde{\tilde{\chi}} \in S^0(U)$ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \operatorname{supp}\tilde{\chi} \cap \operatorname{supp}(1-\tilde{\tilde{\chi}}) = \emptyset$. Let $l, m, L \in \mathbb{R}$ such that $|m| < \mathfrak{r}_\Omega^{r_0}$, $|l-m| < \frac{1}{2}$ and*

$$L \geq \max(l, m+1) - \frac{3}{2}\delta. \tag{3.122}$$

For any $N > 0$, we have

$$\|\tilde{\chi}\Gamma_{N,\delta}^{N,l}\chi(\Lambda_M^\Omega - P)\tilde{\chi}f\|_{L^2(\partial\Omega)} \leq C_N\|\tilde{\tilde{\chi}}\Gamma_{-N,\delta}^{-N,L}\tilde{\chi}f\|_{L^2(\partial\Omega)} + C_N\|\tilde{\chi}f\|_{H^{-N,-N}(\partial\Omega)}, \tag{3.123}$$

and

$$\|\tilde{\chi}\Gamma_{N,\delta}^{N,0}\chi(\mathcal{H}_M^\Omega - E_P)\tilde{\chi}f\|_{X_m^l(\Omega)} \leq C_N\|\tilde{\tilde{\chi}}\Gamma_{-N,\delta}^{-N,L}\tilde{\chi}f\|_{L^2(\partial\Omega)} + C_N\|\tilde{\chi}f\|_{H^{-N,-N}(\partial\Omega)}, \tag{3.124}$$

uniformly for all $M \in [0, M_0]$ and $f \in \mathcal{S}'(\partial\Omega)$.

Remark 3.13. Here the error $\mathcal{E}_0$, between Λ_M^Ω and its approximation P, lives on the boundary and consists of the interior error $\mathcal{E}_0^\circ$ induced in the boundary decomposition, and other boundary correction terms.

Proof. 0. It suffices to show the claim $f \in \Gamma_{-N,\delta}^{-N,L}L^2(\partial\Omega) \cap \mathcal{S}(\partial\Omega)$ and extend it to $\mathcal{S}(\partial\Omega)$ using density.

1. Let $\chi_1(x_0, x), \chi_2(x_0, x) \in S^0(U)$ that $\operatorname{supp}\tilde{\chi} \cap \operatorname{supp}(1-\chi_1) = \operatorname{supp}\chi_1 \cap \operatorname{supp}(1-\chi_2) = \emptyset$, and $\operatorname{supp}\chi_2 \subset \overline{\Omega}$. Note $\chi_1 E_P\tilde{\chi}f$ extends $\tilde{\chi}f$ into Ω, and the extension correction formula (2.8)

implies

$$\begin{aligned}\chi\mathcal{H}_M^\Omega\tilde\chi f &= \chi E_P\tilde\chi f-\chi\mathcal{R}_M^\Omega(-\Delta+M^2)\chi_1E_P\tilde\chi f = \chi E_P\tilde\chi f-\chi\mathcal{R}_M^\Omega\chi_1(-\Delta+M^2)\chi_2E_P\tilde\chi f-\chi\mathcal{R}_M^\Omega[-\Delta,\chi_1]E_P\tilde\chi f\\ &= \chi E_P\tilde\chi f-\chi\mathcal{R}_M^\Omega\chi_1(D_{x_0}+iP+b)(D_{x_0}-iP)\chi_2E_P\tilde\chi f-\chi\mathcal{R}_M^\Omega[-\Delta,\chi_1]E_P\tilde\chi f-\chi\mathcal{R}_M^\Omega\chi_1\mathcal{E}_0^\circ\chi_2E_P\tilde\chi f\\ &= \chi E_P\tilde\chi f-\chi\mathcal{R}_M^\Omega[-\Delta,\chi_1]E_P\tilde\chi f-\chi\mathcal{R}_M^\Omega\chi_1\mathcal{E}_0^\circ\chi_2E_P\tilde\chi f+\chi\mathcal{R}_M^\Omega\chi_1(D_{x_0}+iP+b)(D_{x_0}-iP)(1-\chi_2)E_P\tilde\chi f,\end{aligned} \tag{3.125}$$

where we used that $(D_{x_0}-iP)E_P=0$. Now for the last term,

$$\begin{aligned}\chi_1(D_{x_0}+iP+b)(D_{x_0}-iP)(1-\chi_2) &= \chi_1(D_{x_0}^2+b(x_0,x)D_{x_0}+i[P,D_{x_0}]+P^2-ib(x_0,x)P)(1-\chi_2)\\ &= \chi_1O(\Psi_{-\infty,\delta}^{-\infty,-\infty})\end{aligned} \tag{3.126}$$

since χ_1 and $1-\chi_2$ have disjoint support. Note $\mathcal{E}_0^\circ\in\Psi_{-\infty,\delta}^{-\infty,-2\delta}$ and we arrive at

$$\chi(\mathcal{H}_M^\Omega-E_P)\tilde\chi f=\chi\mathcal{R}_M^\Omega\chi_2v,\ v=(-[-\Delta,\chi_1]+\chi_1O(\Psi_{-\infty,\delta}^{-\infty,-2\delta}))E_P\tilde\chi f. \tag{3.127}$$

Here and later we repeatedly used for any $s,l',k\in\mathbb{R}$ that

$$\|\Gamma_{k,\delta}^{s,l'}E_P\tilde\chi f\|\le C\|\Gamma_{k-\frac12,\delta}^{s,l'+\frac\delta2}\tilde\chi f\|_{L^2(\partial\tilde\Omega)},\ \|O(\Psi_{-\infty,\delta}^{-\infty,-\infty})E_P\tilde\chi f\|\le C\|\Gamma_{-N,\delta}^{-N,-N}\tilde\chi f\|_{L^2(\partial\tilde\Omega)} \tag{3.128}$$

from Lemma 3.4. We will focus on estimating the size of v.

2. In this step, we estimate the size of $\Gamma_{k,\delta}^{s,0}v$ in $L^2(\tilde\Omega_+)$ and in $Y_m^l(\tilde\Omega_+)$. We claim for any $N>0$, $s,l',k\in\mathbb{R}$, we have

$$\|\Gamma_{k,\delta}^{s,l'}v\|_{Y_m^l(\tilde\Omega_+)}\le C\|\Gamma_{-N,\delta}^{-N,l'+\max(l,m+1)-\frac32\delta}\tilde\chi f\|_{L^2(\partial\tilde\Omega)}\le C\|\tilde{\tilde\chi}\Gamma_{-N,\delta}^{-N,l'+L}\tilde\chi f\|_{L^2(\partial\Omega)}+C\|\tilde\chi f\|_{H^{-N,-N}(\partial\Omega)}, \tag{3.129}$$

and

$$\|\Gamma_{k,\delta}^{s,l'}v\|_{L^2(\tilde\Omega_+)}\le C\|\Gamma_{-N,\delta}^{-N,l'-\frac32\delta}\tilde\chi f\|_{L^2(\partial\tilde\Omega)}\le C\|\tilde{\tilde\chi}\Gamma_{-N,\delta}^{-N,l'-\frac32\delta}\tilde\chi f\|_{L^2(\partial\Omega)}+C\|\tilde\chi f\|_{H^{-N,-N}(\partial\Omega)}. \tag{3.130}$$

Note that (3.129) implies with $l'=0$, by Lemma 3.9,

$$\|\tilde\chi\Gamma_{k,\delta}^{s,0}\chi(\mathcal{H}_M^\Omega-E_P)\tilde\chi f\|_{X_m^l(\Omega)}\le C\|\Gamma_{k,\delta}^{s,0}\chi\mathcal{R}_M^\Omega\chi_2v\|_{X_m^l(\tilde\Omega_+)}\le C\|\Gamma_{k,\delta}^{s,0}v\|_{Y_m^l(\tilde\Omega_+)}\le C\|\Gamma_{-N,\delta}^{-N,L}\tilde\chi f\|_{L^2(\partial\tilde\Omega)}, \tag{3.131}$$

that is (3.124). The rest of Step 2 is devoted to prove those two estimates.

2a. We first estimate the size of $\Gamma_{k,\delta}^{s,l'}[-\Delta,\chi_1]E_P\tilde\chi f$. Using that $(D_{x_0}-iP)E_P=0$, we have

$$\begin{aligned}[-\Delta,\chi_1]E_P\tilde\chi f &= -(\Delta\chi_1)E_P\tilde\chi f-2i(\partial_{x_0}\chi_1)D_{x_0}E_P\tilde\chi f-2\sum_{j,k=1}^{d-1}(\partial_{x_j}\chi_1)g_+^{jk}\partial_{x_k}E_P\tilde\chi f\\ &= -(\Delta\chi_1)E_P\tilde\chi f+2\big((\partial_{x_0}\chi_1)P-\sum_{j,k=1}^{d-1}(\partial_{x_j}\chi_1)g_+^{jk}\partial_{x_k}\big)E_P\tilde\chi f=BE_P\tilde\chi f,\end{aligned} \tag{3.132}$$

where $B\in\Psi_{1,\delta}^{0,-1-\delta}$ and $B=\varphi_0B\varphi_0+\varphi_0O(\Psi_{-\infty,\delta}^{-\infty,-\infty})$ for some $\varphi_0(x_0,x)\in S^0(U)$ supported in some $\{x_0<c\langle x\rangle\}$ for some $c>0$, and $\varphi_0\nabla\chi_1=\nabla\chi_1$. Let $\varphi\in S^0(U)$ satisfy all conditions for φ_0

with $\operatorname{supp}\varphi_0 \cap \operatorname{supp}(1-\varphi) = \emptyset$. Then $(1-\varphi)\Gamma^{s,l'}_{k,\delta}\varphi_0 \in O(\Psi^{-\infty,-\infty}_{-\infty,\delta})$ implies

$$\|\Gamma^{s,l'}_{k,\delta}[-\Delta,\chi_1]E_P\tilde{\chi}f\| \le C\|\varphi\Gamma^{s,l'}_{k,\delta}B\varphi_0 E_P\tilde{\chi}f\| + \|O(\Psi^{-\infty,-\infty}_{-\infty,\delta})\varphi E_P\tilde{\chi}f\| \le C\|\Gamma^{-N,-N}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}, \tag{3.133}$$

since $\varphi_0 E_P\tilde{\chi} \in O_\Psi$ from Lemma 3.6 and $O(\Psi^{-\infty,-\infty}_{-\infty,\delta})\varphi = O_\Psi$. We have

$$\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m+1-l}\Gamma^{s,l'}_{k,\delta}[-\Delta,\chi_1]E_P\tilde{\chi}f\| \le C\|\varphi\Gamma^{s,l'+m+1}_{k,\delta}BE_P\tilde{\chi}f\| + \|O(\Psi^{-\infty,-\infty}_{-\infty,\delta})\varphi E_P\tilde{\chi}f\| \le C\|\Gamma^{-N,-N}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}. \tag{3.134}$$

and similarly

$$\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m+2-l}\nabla_x\Gamma^{s,l'}_{k,\delta}[-\Delta,\chi_1]E_P\tilde{\chi}f\| \le C\|\varphi\Gamma^{s,l'+m+2}_{k,\delta}\nabla_x BE_P\tilde{\chi}f\| + \|O(\Psi^{-\infty,-\infty}_{-\infty,\delta})\varphi E_P\tilde{\chi}f\| \le C\|\Gamma^{-N,-N}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}. \tag{3.135}$$

Now noting $(D_{x_0} - iP)E_P = 0$ and $[D_{x_0}, \Gamma^{s,l'}_{k,\delta}] = 0$, we have

$$D_{x_0}\Gamma^{s,l'}_{k,\delta}[-\Delta,\chi_1]E_P\tilde{\chi}f = D_{x_0}\Gamma^{s,l'}_{k,\delta}BE_P\tilde{\chi}f = i\Gamma^{s,l'}_{k,\delta}BPE_P\tilde{\chi}f + \Gamma^{s,l'}_{k,\delta}[D_{x_0},B]E_P\tilde{\chi}f, \tag{3.136}$$

where $\Gamma^{s,l'}_{k,\delta}[D_{x_0},B] \in \Psi^{s,l'-2-\delta}_{k+1,\delta}$ has microsupport inside $\{\varphi \equiv 1\}$. Thus we can similarly estimate

$$\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m+2-l}\Gamma^{s,l'}_{k,\delta}BPE_P\tilde{\chi}f\| \le C\|\Gamma^{-N,-N}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}, \tag{3.137}$$

$$\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m+2-l}\Gamma^{s,l'}_{k,\delta}[D_{x_0},B]E_P\tilde{\chi}f\| \le C\|\Gamma^{-N,-N}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}, \tag{3.138}$$

and ultimately we have

$$\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m+2-l}D_{x_0}\Gamma^{s,l'}_{k,\delta}[-\Delta,\chi_1]E_P\tilde{\chi}f\| \le C\|\Gamma^{-N,-N}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}. \tag{3.139}$$

Now we have

$$\|\Gamma^{s,l'}_{k,\delta}[-\Delta,\chi_1]E_P\tilde{\chi}f\|_{Y^l_m(\tilde{\Omega}_+)} \le C\|\Gamma^{s,l'}_{k,\delta}[-\Delta,\chi_1]E_P\tilde{\chi}f\|_{X^l_{m+2}(\tilde{\Omega}_+)}$$
$$\le C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m+2-l}\nabla\Gamma^{s,l'}_{k,\delta}[-\Delta,\chi_1]E_P\tilde{\chi}f\| + C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m+1-l}\Gamma^{s,l'}_{k,\delta}[-\Delta,\chi_1]E_P\tilde{\chi}f\| \le C\|\Gamma^{-N,-N}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}, \tag{3.140}$$

for any $N > 0$.

2b. We estimate the size of $\Gamma^{s,l'}_{k,\delta}\chi_1 O(\Psi^{-\infty,-2\delta}_{-\infty,\delta})E_P\tilde{\chi}f$. Note

$$\|\Gamma^{s,l'}_{k,\delta}\chi_1 O(\Psi^{-\infty,-2\delta}_{-\infty,\delta})E_P\tilde{\chi}f\| \le C\|\Gamma^{-N,l'-2\delta}_{-N,\delta}E_P\tilde{\chi}f\| \le C\|\Gamma^{-N,l'-\frac{3}{2}\delta}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}, \tag{3.141}$$

in the last step of which we used (3.128). Consider that for each $\phi \in \dot{X}^{-l}_{-m}(\tilde{\Omega}_+)$,

$$|\langle\Gamma^{s,l'}_{k,\delta}\chi_1 O(\Psi^{-\infty,-2\delta}_{-\infty,\delta})E_P\tilde{\chi}f,\phi\rangle| = |\langle\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l+1}\Gamma^{s,l'}_{k,\delta}\chi_1 O(\Psi^{-\infty,-2\delta}_{-\infty,\delta})E_P\tilde{\chi}f, \langle\bar{x}\rangle^{-l}\langle x_0\rangle^{-m+l-1}\phi\rangle|$$
$$\le C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l+1}\Gamma^{s,l'}_{k,\delta}\chi_1 O(\Psi^{-\infty,-2\delta}_{-\infty,\delta})E_P\tilde{\chi}f\|\|\phi\|_{X^{-l}_{-m}(\tilde{\Omega}_+)}. \tag{3.142}$$

Noting $\langle\bar{x}\rangle^l\langle x_0\rangle^{m+1-l}\leq\langle x\rangle^{\max(l,m+1)}$ over $\operatorname{supp}\chi_2$, we have by (3.128),

$$\begin{aligned}\|\Gamma^{s,l'}_{k,\delta}\chi_1O(\Psi^{-\infty,-2\delta}_{-\infty,\delta})E_P\tilde{\chi}f\|_{Y^l_m(\tilde{\Omega}_+)}&\leq C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m+1-l}\Gamma^{s,l'}_{k,\delta}\chi_1O(\Psi^{-\infty,-2\delta}_{-\infty,\delta})E_P\tilde{\chi}f\|\\&\leq C\|O(\Psi^{-\infty,l'+\max(l,m+1)-2\delta}_{-\infty,\delta})E_P\tilde{\chi}f\|+C\|O_\Psi E_P\tilde{\chi}f\|\leq C\|\Gamma^{-N,l'+\max(l,m+1)-\frac{3}{2}\delta}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})},\end{aligned}\tag{3.143}$$

where we used $(1-\chi_2)\Gamma^{s,l'}_{k,\delta}\chi_1O(\Psi^{-\infty,-2\delta}_{-\infty,\delta})=O_\Psi$. Sum up (3.140), (3.143) to observe (3.129), and sum up (3.133), (3.141) to observe (3.130). This proves (3.124).

3. It remains to show (3.123). Noting $(\mathcal{H}^\Omega_M-E_P)\tilde{\chi}f$ has zero trace, one can estimate

$$\begin{aligned}\|\tilde{\chi}\Gamma^{s,l}_{k,\delta}D_{x_0}\langle x_0\rangle^{-N}\chi(\mathcal{H}^\Omega_M-E_P)\tilde{\chi}f\|_{H^{-\frac{1}{2}}(\partial\tilde{\Omega})}&=\|D_{x_0}\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi(\mathcal{H}^\Omega_M-E_P)\tilde{\chi}f\|_{H^{-\frac{1}{2}}(\partial\tilde{\Omega})}\\&\leq\|\Delta\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi(\mathcal{H}^\Omega_M-E_P)\tilde{\chi}f\|_{L^2}+\|\nabla\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi(\mathcal{H}^\Omega_M-E_P)\tilde{\chi}f\|_{L^2}\\&=\|\Delta\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|+\|\nabla\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|,\end{aligned}\tag{3.144}$$

where $N>0$ will be chosen such that $-N+l+2\leq m$ to embed spaces into $X^l_m(\Omega_+)$ in this step.

3a. We take care of the term $\nabla\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v$ in this step. Consider

$$\begin{aligned}\|\nabla\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|\leq C\|\nabla\langle\bar{x}\rangle^l\langle x_0\rangle^{-N}\tilde{\chi}O(\Psi^{s,0}_{k,\delta})\chi\mathcal{R}^\Omega_M\chi_2v\|&\leq C\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|_{X^l_{-N+l}(\tilde{\Omega}_+)}\\&\leq C\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|_{X^l_m(\tilde{\Omega}_+)}\end{aligned}\tag{3.145}$$

for large $N>0$, where we used $\nabla\langle\bar{x}\rangle^l\langle x_0\rangle^{-N}=\langle\bar{x}\rangle^l\langle x_0\rangle^{-N}\nabla+O(\langle\bar{x}\rangle^l\langle x_0\rangle^{-N-1})$ and the boundedness of $\nabla O(\Psi^{s,0}_{k,\delta})$ on X^l_m spaces as in Lemma 3.8.

3b. We take care of the term $\Delta\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v$ in this step. Consider

$$\begin{aligned}-\Delta\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v&=\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi(-\Delta)\mathcal{R}^\Omega_M\chi_2v+[-\Delta,\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi]\chi_1\mathcal{R}^\Omega_M\chi_2v\\&=\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi v-M^2\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v+[-\Delta,\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi]\chi_1\mathcal{R}^\Omega_M\chi_2v.\end{aligned}\tag{3.146}$$

For the second term of (3.146), we have from elliptic estimates

$$\begin{aligned}\|\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|&\leq C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{-N}\chi_2\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|+C\|O_\Psi\chi\mathcal{R}^\Omega_M\chi_2v\|\\&\leq C\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|_{X^l_{-N+l+1}}\leq C\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|_{X^l_m}.\end{aligned}\tag{3.147}$$

For the last term of (3.146), note

$$[-\Delta,\langle x_0\rangle^{-N}\tilde{\chi}\Gamma^{s,l}_{k,\delta}\chi]\chi_1\mathcal{R}^\Omega_M\chi_2v=[-\Delta,\langle x_0\rangle^{-N}\tilde{\chi}]\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v+\langle x_0\rangle^{-N}\tilde{\chi}[-\Delta,\Gamma^{s,l}_{k,\delta}\chi]\chi_1\mathcal{R}^\Omega_M\chi_2v\tag{3.148}$$

the first term of which can be bounded by Lemma 3.8: for large $N>0$,

$$\begin{aligned}\|[-\Delta,\langle x_0\rangle^{-N}\tilde{\chi}]\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|&\leq C\|\langle x_0\rangle^{-N-1}\chi_1\nabla\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|+C\|\langle x_0\rangle^{-N-2}\chi_1\Gamma^{s,l}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|\\&\leq C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{-N-1}\chi_1\nabla O(\Psi^{s,0}_{k,\delta})\chi\mathcal{R}^\Omega_M\chi_2v\|+C\|\langle\bar{x}\rangle^{l-1}\langle x_0\rangle^{-N-1}\chi_1O(\Psi^{s,0}_{k,\delta})\chi\mathcal{R}^\Omega_M\chi_2v\|\\&\quad+C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{-N-2}\chi_1O(\Psi^{s,0}_{k,\delta})\chi\mathcal{R}^\Omega_M\chi_2v\|\leq C\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{R}^\Omega_M\chi_2v\|_{X^l_m}.\end{aligned}\tag{3.149}$$

Noting that $-\Delta_{g_+} = D_{x_0}^2 + b(\bar{x})D_{x_0} + O(\Psi_{1,\delta}^{0,-1-\delta}) + O(\Psi_{2,\delta}^{0,-2\delta})$ and $[D_{x_0}, \Gamma_{k,\delta}^{s,l}] = 0$, we have

$$\begin{aligned}\langle x_0\rangle^{-N}\tilde{\chi}[-\Delta, \Gamma_{k,\delta}^{s,l}\chi]\chi_1 &= \langle x_0\rangle^{-N}\tilde{\chi}(O(\Psi_{k,\delta}^{s,l-1})D_{x_0} + O(\Psi_{k+1,\delta}^{s,l-1-\delta}))\chi_1 \\ &= \langle x_0\rangle^{-N}\big(\langle\bar{x}\rangle^{l-1}\tilde{\chi}D_{x_0}O(\Psi_{k,\delta}^{s,0}) + \langle\bar{x}\rangle^{l-2}\tilde{\chi}O(\Psi_{k,\delta}^{s,0}) + \langle\bar{x}\rangle^{l-1-\delta}\tilde{\chi}O(\Psi_{k+1,\delta}^{s,0})\big)\chi_1 \quad (3.150)\end{aligned}$$

and the second term of (3.148) can be estimated by Lemma 3.8: for large $N > 0$,

$$\begin{aligned}\|\langle x_0\rangle^{-N}\tilde{\chi}[-\Delta, \Gamma_{k,\delta}^{s,l}\chi]\chi_1\mathcal{R}_M^{\Omega}\chi_2 v\| &\le C\|\Gamma_{k,\delta}^{s,0}\chi_1\mathcal{R}_M^{\Omega}\chi_2 v\|_{X_{-N+l-1}^{l-1}} + C\|\Gamma_{k+1,\delta}^{s,0}\chi_1\mathcal{R}_M^{\Omega}\chi_2 v\|_{X_{-N+l-\delta}^{l-1-\delta}} \\ &\le C\|\Gamma_{k,\delta}^{s,0}\chi_1\mathcal{R}_M^{\Omega}\chi_2 v\|_{X_m^l} + C\|\Gamma_{k+1,\delta}^{s,0}\chi_1\mathcal{R}_M^{\Omega}\chi_2 v\|_{X_m^l}. \quad (3.151)\end{aligned}$$

Eventually, we have the estimate of (3.146) by summing up those estimates above, and hence back in (3.144) we have

$$\begin{aligned}\|\tilde{\chi}\Gamma_{k,\delta}^{s,l}D_{x_0}\langle x_0\rangle^{-N}\chi(\mathcal{H}_M^{\Omega} - E_P)\tilde{\chi}f\|_{H^{-\frac{1}{2}}(\partial\tilde{\Omega})} &\le C\|\langle x_0\rangle^{-N}\tilde{\chi}\Gamma_{k,\delta}^{s,l}\chi v\| + C(1+M^2)\|\Gamma_{k,\delta}^{s,0}\chi\mathcal{R}_M^{\Omega}\chi_2 v\|_{X_m^l} \\ &\quad + C\|\Gamma_{k,\delta}^{s,0}\chi_1\mathcal{R}_M^{\Omega}\chi_2 v\|_{X_m^l} + C\|\Gamma_{k+1,\delta}^{s,0}\chi_1\mathcal{R}_M^{\Omega}\chi_2 v\|_{X_m^l} \quad (3.152)\end{aligned}$$

4. We now conclude the proof by estimating each term in (3.152). Note per (3.130), we have

$$\|\langle x_0\rangle^{-N}\tilde{\chi}\Gamma_{k,\delta}^{s,l}\chi v\| \le C\|\Gamma_{k,\delta}^{s,l}\chi v\| \le C\|\chi\Gamma_{k,\delta}^{s,l}v\| + C\|O(\Psi_{k-1,\delta}^{s,l-1+\delta})v\| \le C\|\Gamma_{-N,\delta}^{-N,l-\frac{3}{2}\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}. \quad (3.153)$$

For the other terms, we have

$$\|\Gamma_{k,\delta}^{s,0}\chi\mathcal{R}_M^{\Omega}\chi_2 v\|_{X_m^l} \le C\|\Gamma_{k,\delta}^{s,0}v\|_{Y_m^l} \le C\|\Gamma_{-N,\delta}^{-N,\max(l,m+1)-\frac{3}{2}\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} \le C\|\Gamma_{-N,\delta}^{-N,L}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} \quad (3.154)$$

by (3.129) and that $\Gamma_{k,\delta}^{s,0}\chi\mathcal{R}_M^{\Omega}\chi_2\Gamma_{-k,\delta}^{-s,0} : Y_m^l(\tilde{\Omega}_+) \to \dot{X}_m^l(\tilde{\Omega}_+)$ is bounded as in Lemma 3.9. This is the place we used the assumption that $|m| < \mathfrak{r}_{\Omega}^{r_0}$ and $|m-l| < \frac{1}{2}$. All terms of such shape can be estimated similarly. For any $s, k \in \mathbb{R}$, any $N > 0$ large, we end up with

$$\|\tilde{\chi}\Gamma_{k,\delta}^{s,l}D_{x_0}\langle x_0\rangle^{-N}\chi(\mathcal{H}_M^{\Omega} - E_P)\tilde{\chi}f\|_{H^{-\frac{1}{2}}(\partial\tilde{\Omega})} \le C\|\Gamma_{-N,\delta}^{-N,l-\frac{3}{2}\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} + C\|\Gamma_{-N,\delta}^{-N,L}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} \le C\|\Gamma_{-N,\delta}^{-N,L}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}, \quad (3.155)$$

Eventually noting that $D_{x_0}\langle x_0\rangle = 0$ on $\{x_0 = 0\}$, we have

$$\begin{aligned}\|\tilde{\chi}\Gamma_{k,\delta}^{s,l}\chi(\Lambda_M^{\Omega} - P)\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} &\le C\|\tilde{\chi}\Gamma_{k,\delta}^{s+\frac{1}{2},l}D_{x_0}\langle x_0\rangle^{-N}\chi(\mathcal{H}_M^{\Omega} - E_P)\tilde{\chi}f\|_{H^{-\frac{1}{2}}(\partial\tilde{\Omega})} \\ &\quad + C\|\Gamma_{-N,\delta}^{-N,-N}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} \le C\|\Gamma_{-N,\delta}^{-N,L}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} \quad (3.156)\end{aligned}$$

as desired. □

Corollary 3.14 (Bounds for $\mathcal{H}_M^{\Omega}$)**.** *Let $\Omega \subset \mathbb{R}^d$ be an AC Lipschitz domain with a regular collar U. Let $M_0 > 0$. Let $\delta \in [0,1)$. Let $\chi, \tilde{\chi}, \tilde{\tilde{\chi}} \in S^0(U)$ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \operatorname{supp}\tilde{\chi} \cap \operatorname{supp}(1-\tilde{\tilde{\chi}}) = \emptyset$. Let $l, m \in \mathbb{R}$ such that $|m| < \mathfrak{r}_{\Omega}^{r_0}$, $|l-m| < \frac{1}{2}$. Let $s, k \in \mathbb{R}$. For each $N > 0$, we have*

$$\begin{aligned}\|\tilde{\chi}\Gamma_{k,\delta}^{s,0}\chi\mathcal{H}_M^{\Omega}\tilde{\chi}f\|_{X_m^l(\Omega)} &\le C\|\tilde{\tilde{\chi}}\Gamma_{k,\delta}^{s-\frac{1}{2},\max(l,m-\frac{1}{2}+0)}\tilde{\chi}f\|_{L^2(\partial\Omega)} + C\|\tilde{\tilde{\chi}}\Gamma_{k+\frac{1}{2},\delta}^{s,\max(l,m)-\frac{\delta}{2}}\tilde{\chi}f\|_{L^2(\partial\Omega)} \\ &\quad + C\|\tilde{\tilde{\chi}}\Gamma_{-N,\delta}^{-N,\max(l,m+1)-\frac{3}{2}\delta}\tilde{\chi}f\|_{L^2(\partial\Omega)} + C_N\|\tilde{\chi}f\|_{H^{-N,-N}(\partial\Omega)}, \quad (3.157)\end{aligned}$$

uniformly for all $M \in [0, M_0]$ and $f \in \mathcal{S}'(\partial\Omega)$.

Proof. Note that Proposition 3.12 implies

$$\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{H}^{\Omega}_{M}\tilde{\chi}f\|_{X^l_m(\tilde{\Omega}_+)} \leq \|\Gamma^{s,0}_{k,\delta}\chi E_P\tilde{\chi}f\|_{X^l_m(\tilde{\Omega}_+)} + C\|\Gamma^{-N,\max(l,m+1)-\frac{3}{2}\delta}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}. \tag{3.158}$$

We will estimate the $E_P\tilde{\chi}f$ term. Consider $D_{x_0}E_P = iPE_P$ with $P \in \Psi^{0,-\delta}_{1,\delta}$ and

$$\begin{aligned}\|\Gamma^{s,0}_{k,\delta}\chi E_P\tilde{\chi}f\|_{X^l_m(\tilde{\Omega}_+)} \leq{}& C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l-1}\Gamma^{s,0}_{k,\delta}\chi E_P\tilde{\chi}f\| + C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\nabla_x\Gamma^{s,0}_{k,\delta}\chi E_P\tilde{\chi}f\| \\ &+ C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\Gamma^{s,0}_{k,\delta}\chi PE_P\tilde{\chi}f\| + C\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\Gamma^{s,0}_{k,\delta}[D_{x_0},\chi]E_P\tilde{\chi}f\|. \end{aligned}\tag{3.159}$$

Since $\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l-\frac{1}{2}+0} \leq C\langle x\rangle^{\max(l,m-\frac{1}{2}+0)}$ on U, the first term of (3.159) can be estimated by

$$\begin{aligned}\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l-1}\Gamma^{s,0}_{k,\delta}\chi E_P\tilde{\chi}f\| \leq{}& C\|\tilde{\chi}\langle x_0\rangle^{-\frac{1}{2}-0}O(\Psi^{s,\max(l,m-\frac{1}{2}+0)}_{k,\delta})\chi E_P\tilde{\chi}f\| + C\|O_\Psi\chi E_P\tilde{\chi}f\| \\ &\leq C\|\Gamma^{s-\frac{1}{2},\max(l,m-\frac{1}{2}+0)}_{k,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} \end{aligned}\tag{3.160}$$

where we used the boundedness of E_P as in Lemma 3.4. Similarly we have

$$\begin{aligned}\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\nabla_x\Gamma^{s,0}_{k,\delta}\chi E_P\tilde{\chi}f\| + \|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\Gamma^{s,0}_{k,\delta}\chi PE_P\tilde{\chi}f\| \leq{}& C\|\tilde{\chi}\Gamma^{s,\max(l,m)-\delta}_{k+1,\delta}E_P\tilde{\chi}f\| \\ &+ C\|O_\Psi E_P\tilde{\chi}f\| \leq C\|\Gamma^{s,\max(l,m)-\frac{\delta}{2}}_{k+\frac{1}{2},\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}, \end{aligned}\tag{3.161}$$

and we have

$$\|\langle\bar{x}\rangle^l\langle x_0\rangle^{m-l}\Gamma^{s,0}_{k,\delta}[D_{x_0},\chi]E_P\tilde{\chi}f\| \leq C\|\Gamma^{s,\max(l,m)+\frac{\delta}{2}-1}_{k-\frac{1}{2},\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}. \tag{3.162}$$

Thus we obtain

$$\|\Gamma^{s,0}_{k,\delta}\chi E_P\tilde{\chi}f\|_{X^l_m(\tilde{\Omega}_+)} \leq C\|\Gamma^{s-\frac{1}{2},\max(l,m-\frac{1}{2}+0)}_{k,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} + C\|\Gamma^{s,\max(l,m)-\frac{\delta}{2}}_{k+\frac{1}{2},\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}. \tag{3.163}$$

Revisit (3.158) to see

$$\begin{aligned}\|\Gamma^{s,0}_{k,\delta}\chi\mathcal{H}^{\Omega}_{M}\tilde{\chi}f\|_{X^l_m(\tilde{\Omega}_+)} \leq{}& C\|\Gamma^{s-\frac{1}{2},\max(l,m-\frac{1}{2}+0)}_{k,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} + C\|\Gamma^{s,\max(l,m)-\frac{\delta}{2}}_{k+\frac{1}{2},\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})} \\ &+ C\|\Gamma^{-N,\max(l,m+1)-\frac{3}{2}\delta}_{-N,\delta}\tilde{\chi}f\|_{L^2(\partial\tilde{\Omega})}. \end{aligned}\tag{3.164}$$

Localize with $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$ to conclude the proof. □

Remark 3.15. All constants in this sections can depend on Ω in the following way: the upper bound on the choice of $r_0 \geq 0$, and the lower bound on the radial Hardy constant $\mathfrak{r}^{r_0}_{\Omega}$, the upper bound on the diameter of $\partial\Omega \setminus \partial\tilde{\Omega}$ in Remark 3.11, for each multi-index α, the upper bounds on C_α of derivatives of the metric g_+ in (3.11), and lower bounds on $|\det g_+|$.

4. Refined estimates near infinity on AC Lipschitz domains

The goal of this section is to study the solution operators $\mathcal{R}_M^\Omega$ and $\mathcal{H}_M^\Omega$ in more detail. Specifically, we show that they are pseudolocal when acting on functions with non-trivial oscillation near infinity. We then use this to understand the structure of the DtN map on AC Lipschitz domains– showing that it is almost local at high frequencies and giving precise estimates on how varying the domain changes the DtN map.

Outline of the strategy

As we saw in Section 2, $\mathcal{R}_M^\Omega : Y_m^l \to X_m^l$ is well defined only for

$$|m-l| < \frac{1}{2}, \qquad |m| < \mathfrak{r}_\Omega^{r_0}. \tag{4.1}$$

The main idea in this section is to commute certain (b)- vector fields that are tangent to $\partial\Omega$ and have linearly growing coefficients through $\mathcal{R}_M^\Omega$ to improve these thresholds when acting on functions that oscillate.

Roughly speaking, we will use the fact that

$$[-\Delta, \langle x\rangle] = \langle x\langle x\rangle^{-1}, \nabla\rangle + O(\langle x\rangle^{-1}) \tag{4.2}$$

to see that

$$xD_x\mathcal{R}_M^\Omega = \mathcal{R}_M^\Omega xD_x + \text{bounded} \tag{4.3}$$

and hence that, if $f = D_x\tilde{f} + \textit{decaying}$, then we can apply $\mathcal{R}_M^\Omega$ to f even when f lies in Y_{m-1}^{l-1} with (m,l) satisfying (4.1). Similarly, if f oscillates in the sense above and $f \in Y_{m+1}^{l+1}$, with (m,l) satisfying (4.1), then $\mathcal{R}_M^\Omega f \in X_{m+1}^{l+1}$; i.e. the requirements on (m,l) relax wehn acting on oscillating functions.

Functions that can be written as $f = D_x^N\tilde{f}_N + \text{decaying}$ have non-trivial oscillations at $|x| = \infty$ and hence by repeating the commutator argument, we can show that $\mathcal{R}_M^\Omega$ acts on functions with any amount of polynomial growth/decay, so long as these functions are oscillating.

By further localizing this argument we, in fact, show that $\mathcal{R}_M^\Omega$ and $\mathcal{H}_M^\Omega$ are pseudolocal when acting on oscillating functions. We finally use this in Section 4.4 to show that, on oscillatory functions and near infinity, Λ_M^Ω depends only on the local geometry of Ω.

Remark 4.1. Throughout this section, if not stated otherwise, we assume $r_0 \geq 0$, $m, l \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^d$ be any AC Lipschitz domain with $M_0 > 0$.

4.1. **Spaces with b-regularity.** In this section, we will study spaces on an AC Lipschitz domain $\Omega \subset \mathbb{R}^d$. We first discuss the idea of b-regularity. Let z be the Euclidean coordinates in $\Omega \subset \mathbb{R}^d$. We say $\chi \in S^l(\Omega)$ if $\chi \in C^\infty(\Omega)$ and $|\partial_z^\alpha \chi| \leq C_\alpha \langle z\rangle^{l-|\alpha|}$. We say $\gamma \in S^l(\partial\Omega)$ if it extends to a function in $S^l(\Omega)$. Denote the order $(1,l)$ vector fields by

$$^\bullet\mathrm{Diff}^{1,l}(\Omega) := \Big\{ \sum_{|\alpha|=1} a_\alpha(z) D_z^\alpha,\ a_\alpha \in S^l(\Omega)\Big\},\ \mathrm{Diff}^{1,l}(\Omega) := \Big\{ \sum_{|\alpha|\leq 1} a_\alpha(z) D_z^\alpha,\ a_\alpha \in S^{l-1+|\alpha|}(\Omega)\Big\}. \tag{4.4}$$

In order to understand the boundary regularities, we define the order $(1,l)$ b-vector fields:

$$ {}^{\bullet}\mathrm{Diff}_b^{1,l}(\Omega) := \Big\{ \mathcal{V} \in {}^{\bullet}\mathrm{Diff}^{1,l}(\Omega) : \mathcal{V}|_z \in T_z\partial\Omega \text{ for all } z \in \partial\Omega \Big\}, \tag{4.5} $$

$$ \mathrm{Diff}_b^{1,l}(\Omega) := \Big\{ \mathcal{V} + f : f \in S^{l-1}(\Omega), \mathcal{V} \in {}^{\bullet}\mathrm{Diff}_b^{1,l}(\Omega), \mathcal{V}|_z \in T_z\partial\Omega \text{ for all } z \in \partial\Omega \Big\}. \tag{4.6} $$

Note by definition, $\mathcal{V} \in {}^{\bullet}\mathrm{Diff}_b^{1,l}(\Omega)$ must be 0 on $\partial\Omega_{\mathrm{sing}}$. Those b-vector fields are tangential to $\partial\Omega$. We define $\mathrm{Diff}_b^{1,l}(\partial\Omega)$ by the restriction of $\mathrm{Diff}_b^{1,l}(\Omega)$ onto $\partial\Omega$, and ${}^{\bullet}\mathrm{Diff}_b^{1,l}(\partial\Omega)$ similarly. We then move on to define for $s \in \mathbb{N}$ that

$$ \mathrm{Diff}^{s,sl}(\Omega) = \mathrm{span}\Big\{ \mathcal{V}_1 \cdots \mathcal{V}_k : k \le s, \mathcal{V}_1, \ldots, \mathcal{V}_k \in \mathrm{Diff}^{1,l}(\Omega) \Big\}, \tag{4.7} $$

$$ \mathrm{Diff}_b^{s,sl}(\Omega) = \mathrm{span}\Big\{ \mathcal{V}_1 \cdots \mathcal{V}_k : k \le s, \mathcal{V}_1, \ldots, \mathcal{V}_k \in \mathrm{Diff}_b^{1,l}(\Omega) \Big\}. \tag{4.8} $$

We now define X_m^l-spaces with order k $\mathrm{Diff}_b^{1,1}$-regularity and order s $\mathrm{Diff}_b^{1,0}$-regularity.

Definition 4.2 ($X_{m,s}^{l,k}$)**.** For $k \in \mathbb{N}_0$, we define the (m,l,k)-spaces by

$$ X_m^{l,k}(\Omega) = \big\{ u \in X_m^l(\Omega) : \mathcal{V}_1 \cdots \mathcal{V}_{k'} u \in X_m^l(\Omega), \mathcal{V}_j \in {}^{\bullet}\mathrm{Diff}_b^{1,1}(\Omega), 1 \le k' \le k \big\}, \tag{4.9} $$

$$ Y_m^{l,k}(\Omega) = \big\{ u \in Y_m^l(\Omega) : \mathcal{V}_1 \cdots \mathcal{V}_{k'} u \in Y_m^l(\Omega), \mathcal{V}_j \in {}^{\bullet}\mathrm{Diff}_b^{1,1}(\Omega), 1 \le k' \le k \big\}, \tag{4.10} $$

$$ \dot{X}_m^{l,k}(\Omega) = \big\{ u \in \dot{X}_m^l(\Omega) : \mathcal{V}_1 \cdots \mathcal{V}_{k'} u \in \dot{X}_m^l(\Omega), \mathcal{V}_j \in {}^{\bullet}\mathrm{Diff}_b^{1,1}(\Omega), 1 \le k' \le k \big\}, \tag{4.11} $$

$$ \dot{Y}_m^{l,k}(\Omega) = \big\{ u \in \dot{Y}_m^l(\Omega) : \mathcal{V}_1 \cdots \mathcal{V}_{k'} u \in \dot{Y}_m^l(\Omega), \mathcal{V}_j \in {}^{\bullet}\mathrm{Diff}_b^{1,1}(\Omega), 1 \le k' \le k \big\}. \tag{4.12} $$

To allow the negative indices, we let

$$ X_m^{l,-k}(\Omega) = (\dot{Y}_{-m}^{-l,k}(\Omega))^*, \ Y_m^{l,-k}(\Omega) = (\dot{X}_{-m}^{-l,k}(\Omega))^*, \tag{4.13} $$

and $\dot{X}_m^{l,-k}$, $\dot{Y}_m^{l,-k}$ as the dual to $Y_{-m}^{-l,k}$, $X_{-m}^{-l,k}$ correspondingly. The above allows the definition of $X_m^{l,k}(\Omega)$, $Y_m^{l,k}(\Omega)$ for any $k \in \mathbb{Z}$. We use $X_m^{l,k}(\Omega)$ as the pivot space to define (m,l,k,s)-spaces. Define for $s \in \mathbb{N}_0$,

$$ X_{m,s}^{l,k}(\Omega) = \big\{ u \in X_m^{l,k}(\Omega) : \mathcal{U}_1 \cdots \mathcal{U}_{s'} u \in X_m^{l,k}(\Omega), \mathcal{U}_j \in {}^{\bullet}\mathrm{Diff}_b^{1,0}(\Omega), 1 \le s' \le s \big\}, \tag{4.14} $$

$$ Y_{m,s}^{l,k}(\Omega) = \big\{ u \in Y_m^{l,k}(\Omega) : \mathcal{U}_1 \cdots \mathcal{U}_{s'} u \in Y_m^{l,k}(\Omega), \mathcal{U}_j \in {}^{\bullet}\mathrm{Diff}_b^{1,0}(\Omega), 1 \le s' \le s \big\}, \tag{4.15} $$

$$ \dot{X}_{m,s}^{l,k}(\Omega) = \big\{ u \in \dot{X}_m^{l,k}(\Omega) : \mathcal{U}_1 \cdots \mathcal{U}_{s'} u \in \dot{X}_m^{l,k}(\Omega), \mathcal{U}_j \in {}^{\bullet}\mathrm{Diff}_b^{1,0}(\Omega), 1 \le s' \le s \big\}, \tag{4.16} $$

$$ \dot{Y}_{m,s}^{l,k}(\Omega) = \big\{ u \in \dot{Y}_m^{l,k}(\Omega) : \mathcal{U}_1 \cdots \mathcal{U}_{s'} u \in \dot{Y}_m^{l,k}(\Omega), \mathcal{U}_j \in {}^{\bullet}\mathrm{Diff}_b^{1,0}(\Omega), 1 \le s' \le s \big\}. \tag{4.17} $$

$$ X_{m,-s}^{l,k}(\Omega) = (\dot{Y}_{-m,s}^{-l,-k}(\Omega))^*, \ Y_{m,-s}^{l,k}(\Omega) = (\dot{X}_{-m,s}^{-l,-k}(\Omega))^*. \tag{4.18} $$

The above defines $X_{m,s}^{l,k}(\Omega)$ for all $k, s \in \mathbb{Z}$.

We identify a basis $\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_{d-1} \in {}^{\bullet}\mathrm{Diff}_b^{1,0}(\Omega)$, that $\{\mathcal{W}_j|_z\}_{j=0}^{d-1}$ spans $T_z\Omega$ for each z in the interior of Ω. We further ask, for each $z \in \partial\Omega_{\mathrm{reg}}$, locally near z inside $\overline{\Omega}$ in the Fermi normal coordinates (x_0, x) where x_0 is the variable normal to $\partial\Omega$, we have

$$ \mathcal{W}_0 = a_0(x_0,x) x_0 D_{x_0}, \ \mathcal{W}_j = a_j(x_0,x) D_{x_j} \text{ for } 1 \le j \le d-1, \tag{4.19} $$

for some $a_0, a_1, \dots, a_{d-1}$ locally C^∞ near z inside $\overline{\Omega}$. Given multi-index $\alpha = (\alpha_0, \dots, \alpha_{d-1})$ with $\alpha_j \in \mathbb{N}_0$, we define the multi-index operator $\mathcal{W}^\alpha$ by

$$\mathcal{W}^\alpha = \mathcal{W}_0^{\alpha_0} \mathcal{W}_1^{\alpha_1} \cdots \mathcal{W}_{d-1}^{\alpha_{d-1}}. \tag{4.20}$$

Given a word $\alpha = (\alpha_1, \dots, \alpha_k)$ of length k, with $\alpha_j \in \{0, \dots, d-1\}$ for each $1 \leq j \leq k$, we define the operator word $\mathcal{W}_\alpha$ by

$$\mathcal{W}_\alpha = \mathcal{W}_{\alpha_1} \cdots \mathcal{W}_{\alpha_k}. \tag{4.21}$$

In this section, γ cutoffs live on $\partial\Omega$ and χ cutoffs live in Ω.

4.2. **Regularity of the resolvent near infinity.** Lemma 2.5 allows us to obtain higher regularity inside a compact set. Our next estimates provide increased regularity near infinity. Before, proceeding, we record two lemmas on the interaction between commutators with the Laplacian and various vector fields.

We define the remainder class of divergence form operators,

$$O_{\text{div}}^{X \to Y} = O(\text{div}_z B_1 \nabla_z + B_2 \nabla_z + B_3), \ (B_1)_{ij} \in S^0(\Omega), (B_2)_j \in S^{-1}(\Omega), B_3 \in S^{-2}(\Omega). \tag{4.22}$$

We show any such remainder in $O_{\text{div}}^{X \to Y}$ is bounded from $X_m^l(\Omega)$ to $Y_m^l(\Omega)$, where we remind the reader that $Y_m^l = (\dot{X}_{-m}^{-l})^*$.

Lemma 4.3. *Let Ω be a uniformly Lipschitz domain. Let $B_1 = \sum_{i,j=1}^d \partial_{z_i} b_{1,i}^j(z) \partial_{z_j}$ and $B_2 = \sum_{j=1}^d b_2^j(z) \partial_{z_j}$, $B_3 = b_3(z)$, where the $L^\infty(\Omega)$-coefficients satisfy*

$$|b_{1,i}^j| \leq C, \ |b_2^j| \leq C\mathbf{d}^{-1}, \ |b_3| \leq C\mathbf{d}^{-2}. \tag{4.23}$$

Then, there is $C > 0$ that uniformly for all $u \in X_m^l(\Omega)$ we have

$$\|B_1 u\|_{Y_m^l(\Omega)} + \|B_2 u\|_{Y_m^l(\Omega)} + \|B_3 u\|_{Y_m^l(\Omega)} \leq C \|u\|_{X_m^l(\Omega)}. \tag{4.24}$$

Proof. Note for $\phi \in \dot{X}_{-m}^{-l}$ we have

$$|\langle b_3(z) u, \phi \rangle| \leq C \|\mathbf{r}^l \mathbf{d}^{m-l-1} u\| \|\mathbf{r}^{-l} \mathbf{d}^{-m+l-1} \phi\| \leq C \|u\|_{X_m^l} \|\phi\|_{X_{-m}^{-l}}. \tag{4.25}$$

Observe that if $f = v + \text{div}\, w$, we have

$$\|f\|_{Y_m^l} \leq \|\mathbf{r}^l \mathbf{d}^{m-l+1} v\|_{L^2} + \|\mathbf{r}^l \mathbf{d}^{m-l} w\|_{L^2}. \tag{4.26}$$

Therefore,

$$\|B_2 u\|_{Y_m^l} \leq \|\mathbf{r}^l \mathbf{d}^{m-l+1} B_2 u\|_{L^2} \leq C \|\mathbf{r}^l \mathbf{d}^{m-l} \nabla u\|_{L^2} \leq C \|u\|_{X_m^l}, \tag{4.27}$$

and

$$\|B_1 u\|_{Y_m^l} \leq \sum_{i,j=1}^d \|\mathbf{r}^l \mathbf{d}^{m-l} b_{1,i}^j \partial_{z_j} u\|_{L^2} \leq C \|\mathbf{r}^l \mathbf{d}^{m-l} \nabla u\|_{L^2} \leq C \|u\|_{X_m^l}, \tag{4.28}$$

as desired. □

We show commutators between $-\Delta$, $\text{Diff}^{1,1}(\Omega)$ are remainders in $O_{\text{div}}^{X \to Y}$.

Lemma 4.4. *Let $V \in {}^{\bullet}\mathrm{Diff}^{1,1}(\Omega)$, $\varphi \in S^0(\Omega)$. Separately we have*

$$[-\Delta, V] = O_{\mathrm{div}}^{X\to Y},\ [-\Delta,\varphi] = O_{\mathrm{div}}^{X\to Y},\ [-\Delta, V\varphi] = O_{\mathrm{div}}^{X\to Y},\ [-\Delta,\varphi]V = O_{\mathrm{div}}^{X\to Y}, \tag{4.29}$$

and

$$[V, O_{\mathrm{div}}^{X\to Y}] = O_{\mathrm{div}}^{X\to Y},\ [\varphi, O_{\mathrm{div}}^{X\to Y}] = O_{\mathrm{div}}^{X\to Y}. \tag{4.30}$$

Proof. Given $V \in {}^{\bullet}\mathrm{Diff}^{1,1}(\Omega)$, we have

$$V = \sum_{k=1}^{d} v^k(z)\partial_{z_k},\ v^k \in S^1(\Omega) \text{ for } 1 \le k \le d. \tag{4.31}$$

To obtain (4.29), we compute

$$[\partial_{z_i}\partial_{z_j}, v^k\partial_{z_k}] = \partial_{z_i}[\partial_{z_j}, v^k]\partial_{z_k} + \partial_{z_j}[\partial_{z_i}, v^k]\partial_{z_k} + [[\partial_{z_i}, v^k]\partial_{z_k}, \partial_{z_j}], \tag{4.32}$$

which is in $\mathrm{div}_z(S^0(\Omega))_{d\times d}\nabla_z + (S^{-1}(\Omega))_d\nabla_z$. Summing up in i,j to observe $[-\Delta, V] = O_{\mathrm{div}}^{X\to Y}$. The rest is similar. □

We show any such remainder in $O_{\mathrm{div}}^{X\to Y}$ is bounded from $X_{m,s}^{l,k}(\Omega)$ to $Y_{m,s}^{l,k}(\Omega)$ for all $k, s \in \mathbb{Z}$.

Lemma 4.5. *Let $l, m \in \mathbb{R}$, $k, s \in \mathbb{Z}$. Let $\varphi \in S^0(\Omega)$. Then we have uniformly for all $u \in X_{m,s}^{l,k}(\Omega)$,*

$$\|O_{\mathrm{div}}^{X\to Y}u\|_{Y_{m,s}^{l,k}(\Omega)} \le C\|u\|_{X_{m,s}^{l,k}(\Omega)}, \tag{4.33}$$

$$\|[-\Delta,\varphi]u\|_{Y_{m,s+1}^{l+1,k}(\Omega)} \le C\|u\|_{X_{m,s}^{l,k}(\Omega)}. \tag{4.34}$$

Proof. 1. Let $s = 0$. We claim for $k \ge 0$ that

$$\|\mathcal{V}_1\cdots\mathcal{V}_k O_{\mathrm{div}}^{X\to Y}u\|_{Y_m^l} \le C\sum_{|\alpha|\le k}\|\mathcal{V}_\alpha u\|_{X_m^l}. \tag{4.35}$$

The case $k = 0$ follows from Lemma 4.3. Assume (4.35) holds for cases up to $k-1$, we have

$$\|\mathcal{V}_1\cdots\mathcal{V}_k O_{\mathrm{div}}^{X\to Y}u\|_{Y_m^l} \le \|O_{\mathrm{div}}^{X\to Y}\mathcal{V}_1\cdots\mathcal{V}_k u\|_{Y_m^l} + \|[\mathcal{V}_1\cdots\mathcal{V}_k, O_{\mathrm{div}}^{X\to Y}]u\|_{Y_m^l} \le C\sum_{|\alpha|\le k}\|\mathcal{V}_\alpha u\|_{X_m^l}, \tag{4.36}$$

as desired in (4.35). Note here we used the iterative assumption and Lemma 4.4 for

$$[\mathcal{V}_1\cdots\mathcal{V}_k, O_{\mathrm{div}}^{X\to Y}] = \sum_{|\gamma|<k}(\mathrm{ad}_{\mathcal{V}}^{\gamma^c}(O_{\mathrm{div}}^{X\to Y}))\mathcal{V}_\gamma = \sum_{|\gamma|<k} O_{\mathrm{div}}^{X\to Y}\mathcal{V}_\gamma, \tag{4.37}$$

as a summation over all proper subwords γ of $(1,\dots,k)$ where $\gamma^c = (1,\dots,k)\setminus\gamma$. Here we used the notation

$$\mathrm{ad}_{\mathcal{V}}^{\gamma}(\cdot) = [\mathcal{V}_{\gamma_1}, [\mathcal{V}_{\gamma_2}, \dots, [\mathcal{V}_{\gamma_j}, \cdot]]],\ \gamma = (\gamma_1,\dots,\gamma_j). \tag{4.38}$$

The $k < 0$ cases follow from the duality and $(O_{\mathrm{div}}^{X\to Y})^* = O_{\mathrm{div}}^{X\to Y}$.

2. Now let $s > 0$. Assume the claim holds for cases up to $s-1$ and all k. We have

$$\|\mathcal{U}_1\cdots\mathcal{U}_s O_{\mathrm{div}}^{X\to Y}u\|_{Y_m^{l,k}} \le \|O_{\mathrm{div}}^{X\to Y}\mathcal{U}_1\cdots\mathcal{U}_s u\|_{Y_m^{l,k}} + \|[\mathcal{U}_1\cdots\mathcal{U}_s, O_{\mathrm{div}}^{X\to Y}]u\|_{Y_m^{l,k}} \le C\sum_{|\alpha|\le s}\|\mathcal{U}_\alpha u\|_{X_m^{l,k}}, \tag{4.39}$$

similarly as in Step 1. We here used Lemma 4.4 and ${}^{\bullet}\mathrm{Diff}_b^{1,0} \subset {}^{\bullet}\mathrm{Diff}_b^{1,1}$. The duality gives the cases for $s < 0$ and all k.

3. Note for any $m, l \in \mathbb{R}$, we have $[-\Delta, \varphi] : X_m^l(\Omega) \to \dot{Y}_{m,1}^{l+1,0}(\Omega)$. Indeed, note

$$[-\Delta, \varphi] = \mathbf{r}^{-1} O(S^0)\nabla + \mathbf{r}^{-2} O(S^0). \tag{4.40}$$

For each $u \in X_m^l(\Omega)$ and $\phi \in X_{-m}^{-l-1}(\Omega)$, we have

$$|\langle \mathbf{r}^{-1} O(S^0)\nabla u, \phi\rangle| \le C\|\mathbf{r}^l \mathbf{d}^{-l+m}\nabla u\|\|\mathbf{r}^{-l-1}\mathbf{d}^{l-m}\phi\| \le C\|u\|_{X_m^l}\|\phi\|_{X_{-m}^{-l-1}}, \tag{4.41}$$

$$|\langle \mathcal{U}_1 \mathbf{r}^{-1} O(S^0)\nabla u, \phi\rangle| \le C\|\mathbf{r}^l \mathbf{d}^{-l+m}\nabla u\|\|\mathbf{r}^{-l-1}\mathbf{d}^{l-m}\mathcal{U}_1^*\phi\| \le C\|u\|_{X_m^l}\|\phi\|_{X_{-m-1}^{-l-1}} \tag{4.42}$$

showing $\mathbf{r}^{-1}O(S^0)\nabla : X_m^l(\Omega) \to \dot{Y}_{m,1}^{l+1,0}(\Omega)$. Similarly one can show $\mathbf{r}^{-2}O(S^0) : X_m^l(\Omega) \to \dot{Y}_{m,1}^{l+2,0}(\Omega)$. This concludes the claim. Now address the k, s indices similarly like in Step 1 and 2. □

We can now improve the mapping properties of $\mathcal{R}_M^\Omega$ on AC Lipschitz domains.

Lemma 4.6 ($\mathcal{R}_M^\Omega$ conserves positive b-regularity)**.** *Let $k, s \in \mathbb{N}_0$. Let $|m| < \mathfrak{r}_\Omega^{r_0}$, $|m - l| < \frac{1}{2}$. Let vector fields $\mathcal{V}_1, \dots \mathcal{V}_k \in {}^{\bullet}\mathrm{Diff}_b^{1,1}(\Omega)$, $\mathcal{U}_1, \dots \mathcal{U}_s \in {}^{\bullet}\mathrm{Diff}_b^{1,0}(\Omega)$. Consider $\tilde{\chi} \in S^0(\Omega)$ with $\operatorname{supp}\tilde{\chi} \cap \partial\Omega$ is smooth, and $\tilde{\chi} \equiv 1$ near $(\cap_{k'=1}^k \operatorname{supp}\mathcal{V}_{k'}) \cap (\cap_{s'=1}^s \operatorname{supp}\mathcal{U}_{s'})$. Then we have*

$$\|\mathcal{U}_1 \dots \mathcal{U}_s \mathcal{V}_1 \dots \mathcal{V}_k \mathcal{R}_M^\Omega f\|_{X_m^l(\Omega)} \le C\|f\|_{Y_m^l(\Omega)} + C \sum_{|\alpha|\le k, |\beta| \le s} \|\tilde{\chi}\mathcal{W}^\beta (\mathbf{r}\mathcal{W})^\alpha f\|_{Y_m^l(\Omega)}, \tag{4.43}$$

and

$$\|(-\Delta + M^2)\mathcal{U}_1 \dots \mathcal{U}_s \mathcal{V}_1 \dots \mathcal{V}_k \mathcal{R}_M^\Omega f\|_{Y_m^l(\Omega)} \le C\|f\|_{Y_m^l(\Omega)} + C \sum_{|\alpha|\le k, |\beta| \le s} \|\tilde{\chi}\mathcal{W}^\beta (\mathbf{r}\mathcal{W})^\alpha f\|_{Y_m^l(\Omega)}, \tag{4.44}$$

uniformly for all $f \in Y_{m,s}^{l,k}(\Omega)$ and $M \in [0, M_0]$. This specifically implies

$$\mathcal{R}_M^\Omega : Y_{m,s}^{l,k}(\Omega) \to \dot{X}_{m,s}^{l,k}(\Omega) \tag{4.45}$$

is uniformly bounded in $M \in [0, M_0]$.

Proof. 1. Let $\chi \in S^0(\Omega)$ satisfying all requirements for $\tilde{\chi}$ and $\operatorname{supp}\chi \cap \operatorname{supp}(1 - \tilde{\chi}) = \emptyset$. We start by showing for each $k, s \ge 0$, we have the following decomposition:

$$(-\Delta + M^2)\chi\mathcal{U}_1 \cdots \mathcal{U}_s \mathcal{V}_1 \cdots \mathcal{V}_k \mathcal{R}_M^\Omega = \chi\mathcal{U}_1 \dots \mathcal{U}_s \mathcal{V}_1 \cdots \mathcal{V}_k + \sum_{|\alpha|+|\beta|<k+s} O_{\mathrm{div}}^{X\to Y} \tilde{\chi}\mathcal{U}^\beta \mathcal{V}^\alpha \mathcal{R}_M^\Omega, \tag{4.46}$$

as a summation over all possibly improper subwords α of $(1, \dots, k)$ and β of $(1, \dots, s)$. This is done by noting

$$[-\Delta, \chi\mathcal{U}_1 \cdots \mathcal{U}_s \mathcal{V}_1 \cdots \mathcal{V}_k] = -\sum_{|\alpha|+|\beta|<k+s} \chi(\mathrm{ad}_{\mathcal{U}}^{\beta^c} \mathrm{ad}_{\mathcal{V}}^{\alpha^c}(-\Delta))\tilde{\chi}\mathcal{U}^\beta \mathcal{V}^\alpha, \tag{4.47}$$

as a summation in subwords described above. Note $\mathrm{ad}_{\mathcal{U}}^{\beta^c} \mathrm{ad}_{\mathcal{V}}^{\alpha^c}(-\Delta) = O_{\mathrm{div}}^{X\to Y}$ per Lemma 4.4 to conclude (4.46).

2. We claim for each $(k,s)\in\mathbb{N}_0\times\mathbb{N}_0$, we have for all $f\in Y^{l,k}_{m,s}$,

$$\|(-\Delta+M^2)\chi\mathcal{U}_1\dots\mathcal{U}_s\mathcal{V}_1\dots\mathcal{V}_k\mathcal{R}^\Omega_M f\|_{Y^l_m(\Omega)}\leq C\|f\|_{Y^l_m(\Omega)}+C\sum_{|\alpha|\leq k,|\beta|\leq s}\|\tilde\chi\mathcal{W}^\beta(\mathbf{r}\mathcal{W})^\alpha f\|_{Y^l_m(\Omega)},\tag{4.48}$$

and

$$\|\chi\mathcal{U}_1\dots\mathcal{U}_s\mathcal{V}_1\dots\mathcal{V}_k\mathcal{R}^\Omega_M f\|_{X^l_m(\Omega)}\leq C\|f\|_{Y^l_m(\Omega)}+C\sum_{|\alpha|\leq k,|\beta|\leq s}\|\tilde\chi\mathcal{W}^\beta(\mathbf{r}\mathcal{W})^\alpha f\|_{Y^l_m(\Omega)},\tag{4.49}$$

and we prove it iteratively. Note $-[\Delta,\chi]=O^{X\to Y}_{\mathrm{div}}$ and we have

$$(-\Delta+M^2)\chi\mathcal{R}^\Omega_M f=\chi f-[\Delta,\chi]\mathcal{R}^\Omega_M f\in Y^l_m.\tag{4.50}$$

The case $(k,s)=(0,0)$ here follows from Lemma 4.3, the boundedness of $\mathcal{R}^\Omega_M: Y^l_m\to\dot X^l_m$ per Proposition 2.16, and that $\chi\mathcal{R}^\Omega_M f$ has zero trace over the boundary. Now assume (4.48) and (4.49) hold for all cases (k,s) where $k+s<N$. Consider some case where $k+s=N$. Note $f\in Y^{l,k}_{m,s}$, and (4.46) implies

$$(-\Delta+M^2)\chi\mathcal{U}_1\cdots\mathcal{U}_s\mathcal{V}_1\cdots\mathcal{V}_k\mathcal{R}^\Omega_M f=\chi\mathcal{U}_1\cdots\mathcal{U}_s\mathcal{V}_1\cdots\mathcal{V}_k f+\sum_{|\alpha|+|\beta|<N}O^{X\to Y}_{\mathrm{div}}\tilde\chi\mathcal{U}^\beta\mathcal{V}^\alpha\mathcal{R}^\Omega_M f,\tag{4.51}$$

as a summation over all possibly improper subwords α of $(1,\dots,k)$ and β of $(1,\dots,s)$. The iterative assumption (4.49) implies

$$\|\sum_{|\alpha|+|\beta|<N}O^{X\to Y}_{\mathrm{div}}\tilde\chi\mathcal{U}^\beta\mathcal{V}^\alpha\mathcal{R}^\Omega_M f\|_{Y^l_m}\leq C\|f\|_{Y^l_m}+C\sum_{|\alpha|+|\beta|<N}\|\tilde\chi\mathcal{W}^\beta(\mathbf{r}\mathcal{W})^\alpha f\|_{Y^l_m}\tag{4.52}$$

over subwords α,β specified above. One can estimate the principal term by

$$\|\chi\mathcal{U}_1\cdots\mathcal{U}_s\mathcal{V}_1\cdots\mathcal{V}_k f\|_{Y^l_m}\leq C\sum_{|\alpha|\leq k,|\beta|\leq s}\|\tilde\chi\mathcal{W}^\beta(\mathbf{r}\mathcal{W})^\alpha f\|_{Y^l_m}.\tag{4.53}$$

Those two estimates justify (4.48) for $k+s=N$ and the zero trace of $\chi\mathcal{U}_1\dots\mathcal{U}_s\mathcal{V}_1\dots\mathcal{V}_k\mathcal{R}^\Omega_M f$ then gives (4.49). Iteration concludes the proof. □

Using self-adjointness of $\mathcal{R}^\Omega_M$, we are able to work in spaces with very low b-regularity.

Lemma 4.7 ($\mathcal{R}^\Omega_M$ conserves b-regularity)**.** *Let $|m|<\mathfrak{r}^{r_0}_\Omega$, $|m-l|<\frac12$. For all $k,s\in\mathbb{Z}$, we have*

$$\mathcal{R}^\Omega_M: Y^{l,k}_{m,s}(\Omega)\to\dot X^{l,k}_{m,s}(\Omega)\tag{4.54}$$

is uniformly bounded for $M\in[0,M_0]$.

Proof. Let $k\leq 0, s=0$. By Lemma 4.6, $\mathcal{R}^\Omega_M: Y^{-l,-k}_{-m}\to\dot X^{-l,-k}_{-m}$ is bounded. Note for $f\in Y^{l,k}_m=(\dot X^{-l,-k}_{-m})^*$, we have

$$\langle(-\Delta+M^2)(\mathcal{R}^\Omega_M)^*f,\phi\rangle=\langle f,\mathcal{R}^\Omega_M(-\Delta+M^2)\phi\rangle=\langle f,\phi\rangle\tag{4.55}$$

for all $\phi\in\dot X^{-l,-k}_{-m}$, where we used that $\mathcal{R}^\Omega_M(-\Delta+M^2)$ is identity on $\dot X^{-l}_{-m}$. This implies $(\mathcal{R}^\Omega_M)^*f=\mathcal{R}^\Omega_M f$ for $f\in Y^{l,k}_m$ and thus $\mathcal{R}^\Omega_M=(\mathcal{R}^\Omega_M)^*: Y^{l,k}_m\to\dot X^{l,k}_m$ is bounded. The inclusion of negative s is similar. □

Finally, when the data is away from the measurement, we obtain a smoothing operator.

Lemma 4.8 (Nonlocal effect of $\mathcal{R}^\Omega_M$ on smooth part is b-smoothing)**.** *Let $|m| < \mathfrak{r}^{r_0}_\Omega$, $|m-l| < \frac{1}{2}$. Let $\chi_1, \chi_2 \in S^0(\Omega)$ with disjoint support. Then for all $N \in \mathbb{N}_0$,*

$$\chi_1 \mathcal{R}^\Omega_M \chi_2 : Y^{l,-N}_{m,-N}(\Omega) \to \dot{X}^{l,N}_{m,N}(\Omega) \tag{4.56}$$

is uniformly bounded for $M \in [0, M_0]$.

Proof. By Lemma 4.6, for any such χ_1, χ_2 with disjoint support, we have $\chi_1 \mathcal{R}^\Omega_M \chi_2 : Y^l_m \to \dot{X}^{l,N}_{m,N}$. Since $(\mathcal{R}^\Omega_M)^* = \mathcal{R}^\Omega_M$ on $Y^{-l,-N}_{-m,-N}$, we also have $\chi_1 \mathcal{R}^\Omega_M \chi_2 : Y^{-l,-N}_{-m,-N} \to \dot{X}^{-l}_{-m}$. Now, let $\tilde{\chi}_1 \in S^0(\Omega)$, $\operatorname{supp}\tilde{\chi}_1 \cap \operatorname{supp}\chi_2 = \emptyset$, and $\operatorname{supp}\chi_1 \cap \operatorname{supp}(1-\tilde{\chi}_1) = \emptyset$. Then we have

$$\tilde{\chi}_1 \mathcal{R}^\Omega_M = \mathcal{R}^\Omega_M \tilde{\chi}_1 + \mathcal{R}^\Omega_M [-\Delta, \tilde{\chi}_1] \mathcal{R}^\Omega_M. \tag{4.57}$$

Letting $\varphi \in S^0(\Omega)$, $\operatorname{supp}\varphi$ disjoint from the support of χ_1 and χ_2, and $\varphi \nabla \tilde{\chi}_1 = \nabla \tilde{\chi}_1$, we have

$$\chi_1 \mathcal{R}^\Omega_M \chi_2 = \chi_1 \mathcal{R}^\Omega_M \varphi [-\Delta, \tilde{\chi}_1] \varphi \mathcal{R}^\Omega_M \chi_2. \tag{4.58}$$

It suffices to note

$$\varphi \mathcal{R}^\Omega_M \chi_2 : Y^{l,-N}_{m,-N} \to \dot{X}^l_m, \ [-\Delta, \tilde{\chi}_1] : \dot{X}^l_m \to Y^l_m, \ \chi_1 \mathcal{R}^\Omega_M \varphi : Y^l_m \to \dot{X}^{l,N}_{m,N} \tag{4.59}$$

are bounded to conclude the proof. □

Corollary 4.9. *Let $|m| < \mathfrak{r}^{r_0}_\Omega$, $|m-l| < \frac{1}{2}$. For all $m_2 \in \mathbb{R}$, $s \in \mathbb{Z}$, we have*

$$\langle M\mathbf{d}\rangle^{m_2} \mathcal{R}^\Omega_M \langle M\mathbf{d}\rangle^{-m_2} : Y^{l,0}_{m,s}(\Omega) \to \dot{X}^{l,0}_{m,s}(\Omega) \tag{4.60}$$

is uniformly bounded for $M \in [0, M_0]$. Let $\chi_1, \chi_2 \in S^0(\Omega)$ with disjoint support. Then for all $N \in \mathbb{N}_0$,

$$\chi_1 \langle M\mathbf{d}\rangle^{m_2} \mathcal{R}^\Omega_M \langle M\mathbf{d}\rangle^{-m_2} \chi_2 : Y^{l,0}_{m,-N}(\Omega) \to \dot{X}^{l,0}_{m,N}(\Omega) \tag{4.61}$$

is uniformly bounded for $M \in [0, M_0]$.

Proof. 1. We briefly show (4.60). We follow the strategy of Lemma 4.6 and show the case when $s = 1$. Given $f \in Y^{l,0}_{m,1}(\Omega)$ and $\mathcal{U}_1 \in {}^\bullet\mathrm{Diff}^{1,0}_b(\Omega)$, consider

$$(-\Delta + M^2)\mathcal{U}_1 \langle M\mathbf{d}\rangle^{m_2} \mathcal{R}^\Omega_M \langle M\mathbf{d}\rangle^{-m_2} f = \mathcal{U}_1 f + [-\Delta, \mathcal{U}_1 \langle M\mathbf{d}\rangle^{m_2}] \langle M\mathbf{d}\rangle^{-m_2} \langle M\mathbf{d}\rangle^{m_2} \mathcal{R}^\Omega_M \langle M\mathbf{d}\rangle^{-m_2} f. \tag{4.62}$$

Note $\langle M\mathbf{d}\rangle^{m_2} \mathcal{R}^\Omega_M \langle M\mathbf{d}\rangle^{-m_2} : Y^l_m \to \dot{X}^l_m$ is bounded per Proposition 2.16 and

$$[-\Delta, \mathcal{U}_1 \langle M\mathbf{d}\rangle^{m_2}] \langle M\mathbf{d}\rangle^{-m_2} = O(\mathbf{d}^{-1})\nabla^2 + O(\mathbf{d}^{-2})\nabla + O(\mathbf{d}^{-3}) : X^l_m \to Y^l_m \tag{4.63}$$

is also bounded. This implies

$$(-\Delta + M^2)\mathcal{U}_1 \langle M\mathbf{d}\rangle^{m_2} \mathcal{R}^\Omega_M \langle M\mathbf{d}\rangle^{-m_2} f \in Y^l_m(\Omega), \ \mathcal{U}_1 \langle M\mathbf{d}\rangle^{m_2} \mathcal{R}^\Omega_M \langle M\mathbf{d}\rangle^{-m_2} f \in X^l_m(\Omega) \tag{4.64}$$

as desired. The cases $s \geq 0$ follow from induction, and the cases $s \leq 0$ follow from taking the adjoint.

2. The proof of (4.61) is similar to Lemma 4.8. □

4.3. **Regularity of the harmonic extension near infinity.** In this subsection, we use the results obtained early to obtain improved the regularity of $\mathcal{H}^\Omega_M$ near infinity.

We show that $\mathcal{H}^\Omega_M$ maps functions on $\partial\Omega$ with k $\mathrm{Diff}^{1,1}_b$-regularity to $X^{l,k}_m(\Omega)$ for $k \in \mathbb{N}_0$.

Lemma 4.10 ($\mathcal{H}^\Omega_M$ conserves negative b-regularity)**.** *Let $k, s \in \mathbb{N}_0$. Let $m, l \in \mathbb{R}$ that $|m| < \mathfrak{r}^{r_0}_\Omega$, $|m - l| < \frac{1}{2}$. Let $\gamma, \tilde\gamma \in S^0(\partial\Omega)$ with $\operatorname{supp}\tilde\gamma \subset \partial\Omega_{\mathrm{reg}}$ and $\operatorname{supp}\gamma \cap \operatorname{supp}(1-\tilde\gamma) = \emptyset$. Then for all vector fields $\mathcal{V}_1, \dots, \mathcal{V}_k \in \mathrm{Diff}^{1,1}_b(\partial\Omega)$, we have*

$$\|\mathcal{H}^\Omega_M \gamma \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X^{l,-k}_{m,-s}(\Omega)} \le C \|\tilde\gamma f\|_{H^{\frac{1}{2}-s,l}(\partial\Omega)}, \tag{4.65}$$

uniformly for all $f \in \mathcal{S}'(\partial\Omega)$ and $M \in [0, M_0]$. See Figure 6(A).

Proof. 0. Let $U \subset \Omega$ be a conic neighbourhood whose closure contains $\operatorname{supp}\tilde\gamma \subset \partial\Omega_{\mathrm{reg}}$, on which we have the Fermi normal coordinates. We confuse $\gamma \in S^0(\partial\Omega)$ with one of its extensions $\gamma \in S^0(U)$ that $\partial_\nu \gamma = 0$. This similarly applies to $\tilde\gamma$. We confuse $\mathcal{V}_j \in \mathrm{Diff}^{1,1}_b(\partial\Omega)$ with their extension $\mathcal{V}_j \in \mathrm{Diff}^{1,1}_b(\Omega)$ with $\operatorname{supp}\mathcal{V}_j \subset U$. Assume without loss of generality none of $\mathcal{V}_j$'s contains a component in $x_0 D_{x_0}$: if it does, then $x_0 D_{x_0}$ is effectively 0 on $\partial\Omega$.

1. We claim

$$\mathcal{V}_1 \cdots \mathcal{V}_k \partial_\nu = \partial_\nu \mathcal{V}_1 \cdots \mathcal{V}_k + \sum_{|\alpha| \le k, \alpha_0 = 0} O(S^{-1}(U))(\langle \bar x \rangle D)^\alpha. \tag{4.66}$$

The case $k = 0$ is trivial. For the case $k = 1$, one can check in local coordinates in U that

$$\mathcal{V}_1 \partial_\nu = \partial_\nu \mathcal{V}_1 + O(S^{-1}(U)) + \sum_{j=1}^{d-1} O(S^{-1}(U)) \langle \bar x \rangle D_{x_j}, \tag{4.67}$$

by identifying U as $\tilde\Omega_+ = \{(x_0, x) : x_0 > 0\}$ as in Remark 3.11 and ∂_ν by $-\partial_{x_0}$. Assume the case $k-1$ holds, and

$$\begin{aligned} \mathcal{V}_1 \cdots \mathcal{V}_k \partial_\nu &= \mathcal{V}_1 \big(\partial_\nu \mathcal{V}_2 \cdots \mathcal{V}_k + \sum_{|\alpha| \le k-1, \alpha_0 = 0} O(S^{-1}(U))(\langle \bar x \rangle D)^\alpha \big) = \partial_\nu \mathcal{V}_1 \cdots \mathcal{V}_k \\ &\quad + \sum_{|\alpha| \le 1, \alpha_0 = 0} O(S^{-1}(U))(\langle \bar x \rangle D)^\alpha \mathcal{V}_2 \cdots \mathcal{V}_k + \sum_{|\alpha| \le k-1, \alpha_0 = 0} \mathcal{V}_1 O(S^{-1}(U))(\langle \bar x \rangle D)^\alpha \end{aligned} \tag{4.68}$$

justifies the iterative claim.

2. Let multi-index $|\beta| \le s$ with $\beta_0 = 0$. We claim uniformly for all $\phi \in \dot Y^{-l,k}_{-m,s}(\Omega)$,

$$\|D^\beta \mathcal{V}_1 \cdots \mathcal{V}_k \gamma \partial_\nu \mathcal{R}^\Omega_M \phi\|_{H^{-\frac{1}{2},-l}(\partial\Omega)} \le C \|\phi\|_{Y^{-l,k}_{-m,s}} \tag{4.69}$$

in this step. First note D^β commutes with ∂_ν in U, and hence Step 1 implies

$$D^\beta \mathcal{V}_1 \cdots \mathcal{V}_k \gamma \partial_\nu \mathcal{R}^\Omega_M \phi = \partial_\nu D^\beta \mathcal{V}_1 \cdots \mathcal{V}_k \gamma \mathcal{R}^\Omega_M \phi + \sum_{|\alpha| \le k, \alpha_0 = 0} O(S^{-1}) D^\beta (\langle \bar x \rangle D)^\alpha \gamma \mathcal{R}^\Omega_M \phi, \tag{4.70}$$

the second term of which can be estimated via Lemma 4.6 by

$$\|O(S^{-1})D^\beta(\langle\bar{x}\rangle D)^\alpha\gamma\mathcal{R}_M^\Omega\phi\|_{H_\partial^{-\frac12,-l}} \le C\sum_{|\alpha|\le k,\alpha_0=0}\|D^\beta(\langle\bar{x}\rangle D)^\alpha\gamma\mathcal{R}_M^\Omega\phi\|_{X_{-m}^{-l}}$$
$$\le C\|\phi\|_{Y_{-m}^{-l}} + C\sum_{|\alpha|\le k,\alpha_0=0}\|\tilde{\gamma}D^\beta(\langle\bar{x}\rangle D)^\alpha\phi\|_{Y_{-m}^{-l}} \le C\|\phi\|_{Y_{-m,s}^{-l,k}}, \quad (4.71)$$

For the first term of (4.70), we have

$$\|\partial_\nu D^\beta\mathcal{V}_1\cdots\mathcal{V}_k\gamma\mathcal{R}_M^\Omega\phi\|_{H_\partial^{-\frac12,-l}} \le C\|\partial_\nu\langle x_0\rangle^{-N}\psi(x_0)D^\beta\mathcal{V}_1\cdots\mathcal{V}_k\gamma\mathcal{R}_M^\Omega\phi\|_{H_\partial^{-\frac12,-l}}$$
$$\le C\|\langle x_0\rangle^{-N}(-\Delta+M^2)D^\beta\mathcal{V}_1\cdots\mathcal{V}_k\gamma\mathcal{R}_M^\Omega\phi\|_{H^{-1,-l}} + C\|\psi(x_0)\nabla D^\beta\mathcal{V}_1\cdots\mathcal{V}_k\gamma\mathcal{R}_M^\Omega\phi\|_{H^{0,-l}}$$
$$+ C\|\langle x_0\rangle^{-N}D^\beta\mathcal{V}_1\cdots\mathcal{V}_k\gamma\mathcal{R}_M^\Omega\phi\|_{H^{-1,-l}}, \quad (4.72)$$

for some $\psi\in C_c^\infty(\mathbb{R})$ that $\psi(x_0)\equiv 1$ near $x_0=0$. By Lemma 4.6 we have

$$\|\langle x_0\rangle^{-N}(-\Delta+M^2)D^\beta\mathcal{V}_1\cdots\mathcal{V}_k\gamma\mathcal{R}_M^\Omega\phi\|_{H^{-1,-l}} \le C\|(-\Delta+M^2)D^\beta\mathcal{V}_1\cdots\mathcal{V}_k\gamma\mathcal{R}_M^\Omega\phi\|_{Y_{-m}^{-l}}$$
$$\le C\|\phi\|_{Y_{-m}^{-l}} + C\sum_{|\alpha|\le k}\|\tilde{\gamma}D^\beta(\langle\bar{x}\rangle D)^\alpha\phi\|_{Y_{-m}^{-l}} \le C\|\phi\|_{Y_{-m,s}^{-l,k}} \quad (4.73)$$

and

$$\|\psi(x_0)\nabla D^\beta\mathcal{V}_1\cdots\mathcal{V}_k\gamma\mathcal{R}_M^\Omega\phi\|_{H^{0,-l}} \le C\|D^\beta\mathcal{V}_1\cdots\mathcal{V}_k\gamma\mathcal{R}_M^\Omega\phi\|_{X_{-m}^{-l}} \le C\|\phi\|_{Y_{-m,s}^{-l,k}}. \quad (4.74)$$

This shows the claim.

3. Noting $(\mathcal{H}_M^\Omega)^* = -\partial_\nu\mathcal{R}_M^\Omega$ we have for any $\phi\in\dot{Y}_{-m,s}^{-l,k}(\Omega)$,

$$|\langle\mathcal{H}_M^\Omega\gamma\mathcal{V}_1\cdots\mathcal{V}_k f,\phi\rangle| = |\langle\tilde{\gamma}f,\mathcal{V}_k^*\cdots\mathcal{V}_1^*\gamma\partial_\nu\mathcal{R}_M^\Omega\phi\rangle_{\partial\Omega}|$$
$$\le C\|\tilde{\gamma}f\|_{H_\partial^{\frac12-s,l}}\sum_{|\beta|\le s,\beta_0=0}\|D^\beta\mathcal{V}_k^*\cdots\mathcal{V}_1^*\gamma\partial_\nu\mathcal{R}_M^\Omega\phi\|_{H^{-\frac12,-l}} \le C\|\tilde{\gamma}f\|_{H_\partial^{\frac12-s,l}}\|\phi\|_{Y_{-m,s}^{-l,k}}. \quad (4.75)$$

This demonstrates the desired bounds over $\mathcal{H}_M^\Omega\gamma\mathcal{V}_1\cdots\mathcal{V}_k f$. □

Corollary 4.11. *Let $m,l\in\mathbb{R}$ that $|m|<\mathfrak{r}_\Omega^{r_0}$, $|m-l|<\frac12$. Let $m_2\in\mathbb{R}$. We have*

$$\|\langle M\mathbf{d}\rangle^{m_2}\mathcal{H}_M^\Omega f\|_{Y_{m-2,-1}^{l,0}(\Omega)} \le C\|f\|_{H^{0,l}(\partial\Omega)}, \quad (4.76)$$

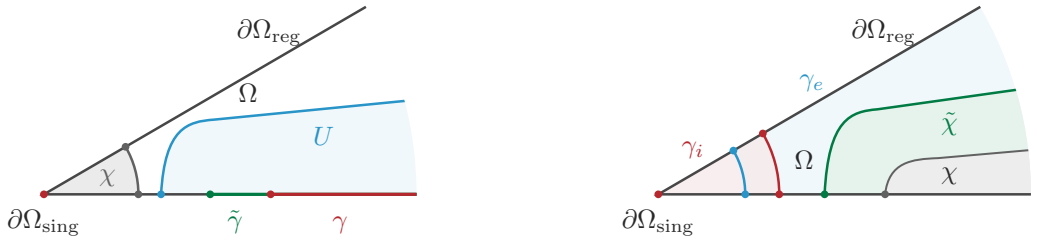
uniformly for all $f\in\mathcal{S}'(\partial\Omega)$ and $M\in[0,M_0]$.

Proof. 1. Let $\gamma,\tilde{\gamma}\in S^0(\partial\Omega)$ with $\operatorname{supp}\tilde{\gamma}\subset\partial\Omega_{\mathrm{reg}}$ and $\operatorname{supp}\gamma\cap\operatorname{supp}(1-\tilde{\gamma})=\emptyset$. We show

$$\|\langle M\mathbf{d}\rangle^{m_2}\mathcal{H}_M^\Omega\gamma f\|_{X_{m,-1}^{l,0}(\Omega)} \le C\|\tilde{\gamma}f\|_{H^{-\frac12,l}(\partial\Omega)}. \quad (4.77)$$

The proof is similar to that of Lemma 4.10. Note $[D_{x_j},\partial_\nu]=0$ for the boundedness of

$$D_{x_j}\gamma\partial_\nu\langle M\mathbf{d}\rangle^{-m_2}\mathcal{R}_M^\Omega\langle M\mathbf{d}\rangle^{m_2} = \partial_\nu D_{x_j}\gamma\langle M\mathbf{d}\rangle^{-m_2}\mathcal{R}_M^\Omega\langle M\mathbf{d}\rangle^{m_2} : Y_{-m,1}^{-l,0}\to H_\partial^{-\frac12,-l}, \quad (4.78)$$



(A) Lemma 4.10, Lemma 4.13 (B) Lemma 4.12

FIGURE 6. Illustration for Lemma 4.10, Lemma 4.13 and Lemma 4.12.

where we used Corollary 4.9. Like in Step 3 of Lemma 4.10, for each $\phi \in \dot{Y}^{-l,0}_{-m,1}(\Omega)$ we have

$$|\langle\langle M\mathbf{d}\rangle^{m_2}\mathcal{H}^{\Omega}_M\gamma f,\phi\rangle| = |\langle\langle M\mathbf{d}\rangle^{m_2}\tilde{\gamma}f,\gamma\partial_\nu\langle M\mathbf{d}\rangle^{-m_2}\mathcal{R}^{\Omega}_M\langle M\mathbf{d}\rangle^{m_2}\phi\rangle_{\partial\Omega}|$$
$$\leq C\|\tilde{\gamma}f\|_{H^{-\frac{1}{2},l}_{\partial}}\big(\|\gamma\partial_\nu\langle M\mathbf{d}\rangle^{-m_2}\mathcal{R}^{\Omega}_M\langle M\mathbf{d}\rangle^{m_2}\phi\|_{H^{-\frac{1}{2},-l}_{\partial}}+\sum_{j=1}^{d-1}\|D_{x_j}\gamma\partial_\nu\langle M\mathbf{d}\rangle^{-m_2}\mathcal{R}^{\Omega}_M\langle M\mathbf{d}\rangle^{m_2}\phi\|_{H^{-\frac{1}{2},-l}_{\partial}}\big)$$
$$\leq C\|\tilde{\gamma}f\|_{H^{-\frac{1}{2},l}_{\partial}}\|\phi\|_{Y^{-l,0}_{-m,1}} \tag{4.79}$$

yielding the desired bounds.

2. Let $\kappa,\kappa_1,\tilde{\kappa}\in C^\infty_c(\overline{\Omega})$ be that $\operatorname{supp}(1-\kappa)\cap\partial\Omega\subset\partial\Omega_{\text{reg}}$, and $\operatorname{supp}\kappa\cap\operatorname{supp}(1-\kappa_1)=\operatorname{supp}\kappa_1\cap\operatorname{supp}(1-\tilde{\kappa})=\emptyset$. Let $\varphi\in C^\infty_c(\overline{\Omega})$ that $\operatorname{supp}\nabla\kappa_1\cap\operatorname{supp}(1-\varphi)=\emptyset$. We decompose

$$\mathcal{H}^{\Omega}_M f=\tilde{\kappa}\mathcal{H}^{\Omega}_M f+(1-\tilde{\kappa})\mathcal{H}^{\Omega}_M(1-\kappa)f+(1-\tilde{\kappa})\mathcal{H}^{\Omega}_M\kappa f \tag{4.80}$$

and estimate it term by term. Lemma 2.24 implies

$$\|\langle M\mathbf{d}\rangle^{m_2}\tilde{\kappa}\mathcal{H}^{\Omega}_M f\|_{Y^{N,0}_{N,0}}\leq C\|\langle M\mathbf{d}\rangle^{m_2}\tilde{\kappa}\mathcal{H}^{\Omega}_M f\|_{H^{-1}}\leq C\|f\|_{H^{0,l}_{\partial}}. \tag{4.81}$$

For the second term, Step 1 implies

$$\|\langle M\mathbf{d}\rangle^{m_2}(1-\tilde{\kappa})\mathcal{H}^{\Omega}_M(1-\kappa)f\|_{X^{l,0}_{m,-1}}\leq C\|(1-\kappa)f\|_{H^{-\frac{1}{2},l}_{\partial}}. \tag{4.82}$$

For the last term, utilizing $(1-\tilde{\kappa})\mathcal{H}^{\Omega}_M\kappa=(1-\tilde{\kappa})\mathcal{R}^{\Omega}_M[\Delta,\kappa_1]\mathcal{H}^{\Omega}_M\kappa$ we have

$$\|\langle M\mathbf{d}\rangle^{m_2}(1-\tilde{\kappa})\mathcal{R}^{\Omega}_M[\Delta,\kappa_1]\mathcal{H}^{\Omega}_M\kappa f\|_{X^{l,0}_{m,-1}}\leq C\|\langle M\mathbf{d}\rangle^{m_2}[\Delta,\kappa_1]\mathcal{H}^{\Omega}_M\kappa f\|_{Y^{l,0}_{m,-1}}$$
$$\leq C\|\varphi\mathcal{H}^{\Omega}_M\kappa f\|_{H^{-1}}\leq C\|\kappa f\|_{H^{0,l}_{\partial}} \tag{4.83}$$

from Corollary 4.9 and Lemma 2.24. Summing up above gives

$$\|\langle M\mathbf{d}\rangle^{m_2}(1-\tilde{\kappa})\mathcal{H}^{\Omega}_M(1-\kappa)f+\langle M\mathbf{d}\rangle^{m_2}(1-\tilde{\kappa})\mathcal{H}^{\Omega}_M\kappa f\|_{Y^{l,0}_{m-2,-1}}$$
$$\leq C\|\langle M\mathbf{d}\rangle^{m_2}(1-\tilde{\kappa})\mathcal{H}^{\Omega}_M(1-\kappa)f+\langle M\mathbf{d}\rangle^{m_2}(1-\tilde{\kappa})\mathcal{H}^{\Omega}_M\kappa f\|_{X^{l,0}_{m,-1}}\leq C\|f\|_{H^{0,l}_{\partial}}, \tag{4.84}$$

which concludes the proof. □

Lemma 4.12 ($\mathcal{H}^\Omega_M$ conserves positive b-regularity)**.** *Let Ω be an AC Lipschitz domain. Let $k, s \in \mathbb{N}_0$. Let $m, l \in \mathbb{R}$ that $|m| < \mathfrak{r}^{r_0}_\Omega$, $|m - l| < \frac{1}{2}$. Let vector fields $\mathcal{V}_1, \dots \mathcal{V}_k \in {}^\bullet\mathrm{Diff}^{1,1}_b(\Omega)$, $\mathcal{U}_1, \dots \mathcal{U}_s \in {}^\bullet\mathrm{Diff}^{1,0}_b(\Omega)$. Let $\chi, \tilde{\chi} \in S^0(\Omega)$ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$. Let $\gamma_i \in C^\infty_c(\overline{\Omega})$, $\gamma_e \in S^0(\Omega)$ that $\gamma_i + \gamma_e = 1$ on Ω, $\operatorname{supp}\tilde{\chi} \cap \operatorname{supp}\gamma_i = \emptyset$, $(\operatorname{supp}\gamma_e \cap \partial\Omega) \cap \partial\Omega_{\mathrm{sing}} = \emptyset$. For each $N \geq 1$ large, we then have*

$$\|\chi \mathcal{U}_1 \cdots \mathcal{U}_s \mathcal{V}_1 \cdots \mathcal{V}_k \mathcal{H}^\Omega_M f\|_{X^l_m(\Omega)} \leq C \sum_{|\alpha|\leq k, |\beta| \leq s} \|\tilde{\chi} D^\beta (\langle x\rangle D)^\alpha f\|_{H^{\frac{1}{2},l}(\partial\Omega)} + C\|\gamma_i f\|_{H^{0,-N}(\partial\Omega)} + C\|\gamma_e f\|_{H^{-N,m-\frac{1}{2}+0}(\partial\Omega)}, \quad (4.85)$$

uniformly for all $f \in \mathcal{S}'(\partial\Omega)$ and $M \in [0, M_0]$. See Figure 6(B).

Proof. 0. Let $\tilde{\gamma}_i \in C^\infty_c(\overline{\Omega})$ that $\operatorname{supp}\gamma_i \cap \operatorname{supp}(1-\tilde{\gamma}_i) = \emptyset$, $\operatorname{supp}\tilde{\gamma}_i \cap \operatorname{supp}\chi = \emptyset$ and $\tilde{\gamma}_e \in S^0$ that $\operatorname{supp}\gamma_e \cap \operatorname{supp}(1-\tilde{\gamma}_e) = \emptyset$, $(\operatorname{supp}\tilde{\gamma}_e \cap \partial\Omega) \cap \partial\Omega_{\mathrm{sing}} = \emptyset$. Let $\kappa \in C^\infty_c(\bar{\Omega})$ that $\operatorname{supp}\nabla\tilde{\gamma}_i \cap \operatorname{supp}(1-\kappa) = \emptyset$ and $\operatorname{supp}\kappa \cap \operatorname{supp}\tilde{\chi} = \emptyset$. Let $\chi_1, \chi_2 \in S^0(\Omega)$ that satisfy all requirements for χ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\chi_1) = \operatorname{supp}\chi_1 \cap \operatorname{supp}(1-\chi_2) = \operatorname{supp}\chi_2 \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$. We decompose

$$\chi\mathcal{V}_1 \cdots \mathcal{V}_k \mathcal{H}^\Omega_M f = \chi\mathcal{V}_1 \cdots \mathcal{V}_k \mathcal{H}^\Omega_M \gamma_e \tilde{\chi} f + \chi\mathcal{V}_1 \cdots \mathcal{V}_k \mathcal{H}^\Omega_M \gamma_e (1-\tilde{\chi}) f + \chi\mathcal{V}_1 \cdots \mathcal{V}_k \mathcal{H}^\Omega_M \gamma_i f \quad (4.86)$$

We will estimate (4.86) term by term in the rest of the proof to conclude

$$\|\chi\mathcal{V}_1 \cdots \mathcal{V}_k \mathcal{H}^\Omega_M f\|_{X^l_m(\Omega)} \leq C \sum_{|\alpha|\leq k} \|\tilde{\chi}(\langle x\rangle D)^\alpha f\|_{H^{\frac{1}{2},l}(\partial\Omega)} + C\|\gamma_i f\|_{H^{0,-N}(\partial\Omega)} + C\|\gamma_e f\|_{H^{-N,m-\frac{1}{2}+0}(\partial\Omega)}, \quad (4.87)$$

for all $k \in \mathbb{N}_0$. Adding $\mathcal{U}_1 \cdots \mathcal{U}_s$ is similar: see Lemma 4.6. This would close the proof.

1. For the first term in (4.86), we iteratively show for $k \in \mathbb{N}_0$ that

$$\|\chi\mathcal{V}_1 \cdots \mathcal{V}_k \tilde{\gamma}_e \mathcal{H}^\Omega_M \gamma_e \tilde{\chi} f\|_{X^l_m(\Omega)} \leq C \sum_{|\alpha|\leq k} \|\tilde{\chi}(\langle x\rangle D)^\alpha f\|_{H^{\frac{1}{2},l}(\partial\Omega)}. \quad (4.88)$$

The case $k = 0$ follows from Proposition 2.19. Assume (4.88) holds for $k-1$. Noting $\chi_1\mathcal{V}_k\mathcal{H}^\Omega_M\gamma_e\tilde{\chi}f$ extends $\chi_1\mathcal{V}_k f$, we have by the extension correction formula (2.8),

$$\tilde{\gamma}_e \mathcal{H}^\Omega_M \chi_1 \mathcal{V}_k f = \chi_1 \mathcal{V}_k \tilde{\gamma}_e \mathcal{H}^\Omega_M \gamma_e \tilde{\chi} f - \tilde{\gamma}_e \mathcal{R}^\Omega_M [-\Delta, \chi_1 \mathcal{V}_k] \tilde{\gamma}_e \mathcal{H}^\Omega_M \gamma_e \tilde{\chi} f \quad (4.89)$$

and thus by $[-\Delta, \chi_1\mathcal{V}_k] = O^{X\to Y}_{\mathrm{div}}\chi_2$ from Lemma 4.4,

$$\chi\mathcal{V}_1 \cdots \mathcal{V}_k \tilde{\gamma}_e \mathcal{H}^\Omega_M \gamma_e \tilde{\chi} f = \chi\mathcal{V}_1 \cdots \mathcal{V}_{k-1} \tilde{\gamma}_e \mathcal{H}^\Omega_M \chi_1 \mathcal{V}_k f + \chi\mathcal{V}_1 \cdots \mathcal{V}_{k-1} \tilde{\gamma}_e \mathcal{R}^\Omega_M O^{X\to Y}_{\mathrm{div}} \chi_2 \mathcal{H}^\Omega_M \gamma_e \tilde{\chi} f. \quad (4.90)$$

By the iterative assumption, the first term can be bounded by

$$\|\chi\mathcal{V}_1 \cdots \mathcal{V}_{k-1} \tilde{\gamma}_e \mathcal{H}^\Omega_M \chi_1 \mathcal{V}_k \gamma_e f\|_{X^l_m} \leq C \sum_{|\alpha|\leq k-1} \|\chi_1(\langle x\rangle D)^\alpha \mathcal{V}_k f\|_{H^{\frac{1}{2},l}_\partial} \leq C \sum_{|\alpha|\leq k} \|\tilde{\chi}(\langle x\rangle D)^\alpha f\|_{H^{\frac{1}{2},l}_\partial}. \quad (4.91)$$

With the iterative assumption with χ_2 in place of χ, the second term can be bounded by Lemma 4.6,

$$\|\chi\mathcal{V}_1\cdots\mathcal{V}_{k-1}\tilde{\gamma}_e\mathcal{R}_M^\Omega O_{\mathrm{div}}^{X\to Y}\chi_2\mathcal{H}_M^\Omega\gamma_e\tilde{\chi}f\|_{X_m^l}\leq C\|\chi_2\mathcal{H}_M^\Omega\gamma_e\tilde{\chi}f\|_{X_m^{l,k-1}}\leq C\sum_{|\alpha|\leq k-1}\|\tilde{\chi}(\langle x\rangle D)^\alpha f\|_{H_\partial^{\frac{1}{2},l}}. \tag{4.92}$$

This yields (4.88) in the case k as desired.

2. We claim in this step the second term of (4.86) can be bounded by

$$\|\chi\mathcal{V}_1\cdots\mathcal{V}_k\chi_1\mathcal{H}_M^\Omega\gamma_e(1-\tilde{\chi})f\|_{X_m^l(\Omega)}\leq C\|\gamma_e f\|_{H_\partial^{\frac{1}{2}-N,m-\frac{1}{2}+0}}. \tag{4.93}$$

Let $\varphi\in S^0(\Omega)$ such that $\operatorname{supp}\gamma_e(1-\tilde{\chi})\cap\operatorname{supp}(1-\varphi)=\operatorname{supp}\varphi\cap\operatorname{supp}\chi_1=\emptyset$. Let $\varphi_1,\varphi_2,\dots$ satisfy the same requirements for φ while $\operatorname{supp}\varphi_n\cap\operatorname{supp}(1-\varphi_{n+1})=\emptyset$ for all n, $\operatorname{supp}\varphi\cap\operatorname{supp}(1-\varphi_1)=\emptyset$. We first note

$$\chi_1\mathcal{H}_M^\Omega\varphi=\chi_1\mathcal{R}_M^\Omega[\Delta,\varphi_{N+1}]\mathcal{R}_M^\Omega[\Delta,\varphi_N]\cdots\mathcal{R}_M^\Omega[\Delta,\varphi_1]\varphi_2\mathcal{H}_M^\Omega\varphi. \tag{4.94}$$

Lemma 4.8 implies $\chi_1\mathcal{R}_M^\Omega[\Delta,\varphi_{N+1}]:X_m^{l,-N}\to X_m^{l,N}$ for m,l within $|m|<\mathfrak{r}_\Omega^{r_0}$, $|m-l|<\frac{1}{2}$. Lemma 4.5 implies for each n, $\mathcal{R}_M^\Omega[\Delta,\varphi_n]:X_m^{l-1,-N}\to X_m^{l,-N}$ for any m,l within the same range. Then by choosing a large N, we have

$$\|\chi_1\mathcal{H}_M^\Omega\varphi f\|_{X_m^{l,N}}\leq C\|\varphi_2\mathcal{H}_M^\Omega\varphi f\|_{X_m^{m-\frac{1}{2}+0,-N}}\leq C\|\varphi_2\mathcal{H}_M^\Omega\varphi f\|_{X_{m,-N}^{m-\frac{1}{2}+0}} \tag{4.95}$$

for any f. Thus for N large we have

$$\|\chi\mathcal{V}_1\cdots\mathcal{V}_k\chi_1\mathcal{H}_M^\Omega\gamma_e(1-\tilde{\chi})f\|_{X_m^l}\leq C\|\varphi_2\mathcal{H}_M^\Omega\gamma_e(1-\tilde{\chi})f\|_{X_{m,-N}^{m-\frac{1}{2}+0}}\leq C\|\gamma_e f\|_{H_\partial^{\frac{1}{2}-N,m-\frac{1}{2}+0}} \tag{4.96}$$

from Lemma 4.10.

3. For the last term in (4.86), we have

$$\chi\mathcal{V}_1\cdots\mathcal{V}_k\mathcal{H}_M^\Omega\gamma_i f=\chi\mathcal{V}_1\cdots\mathcal{V}_k\mathcal{R}_M^\Omega[\Delta,\tilde{\gamma}_i]\mathcal{H}_M^\Omega\gamma_i f. \tag{4.97}$$

Lemma 4.8 and Lemma 2.24 then imply

$$\|\chi\mathcal{V}_1\cdots\mathcal{V}_k\mathcal{R}_M^\Omega[\Delta,\tilde{\gamma}_i]\mathcal{H}_M^\Omega\gamma_i f\|_{X_m^l(\Omega)}\leq C\|\kappa\mathcal{H}_M^\Omega\gamma_i f\|_{L^2(\Omega)}\leq C\|\gamma_i f\|_{H^{0,-N}(\partial\Omega)}. \tag{4.98}$$

since $\operatorname{supp}\chi\cap\operatorname{supp}\kappa=\emptyset$ and κ is compactly supported. □

Lemma 4.13 (Nonlocal effect of $\mathcal{H}_M^\Omega$ on smooth part is b-smoothing)**.** *Let Ω be an AC Lipschitz domain. Let $m,l\in\mathbb{R}$ that $|m|<\mathfrak{r}_\Omega^{r_0}$, $|m-l|<\frac{1}{2}$. Let $\gamma\in S^0(\partial\Omega)$ with $\operatorname{supp}\gamma\subset\partial\Omega_{\mathrm{reg}}$. Let $\chi\in S^0(\Omega)$ with $\operatorname{supp}\gamma\cap(\operatorname{supp}\chi\cap\partial\Omega)=\emptyset$. Then for all $k\in\mathbb{N}_0$, for all $\mathcal{V}_1,\dots,\mathcal{V}_k\in\mathrm{Diff}_b^{1,1}(\Omega)$, for any $N>0$ large, we have*

$$\chi\mathcal{H}_M^\Omega\gamma\mathcal{V}_1\cdots\mathcal{V}_k:H^{-N,m-\frac{1}{2}+0}(\partial\Omega)\to\dot{X}_m^{l,N}(\Omega) \tag{4.99}$$

uniformly in $M\in[0,M_0]$. See Figure 6(A).

Proof. 1. We confuse $\gamma\in S^0(\partial\Omega)$ with one of its extensions $\gamma\in S^0(\Omega)$ such that $\operatorname{supp}\gamma\cap\operatorname{supp}\chi=\emptyset$. Let $\tilde{\gamma}\in S^0(\Omega)$ that $(\operatorname{supp}\tilde{\gamma}\cap\partial\Omega)\subset\partial\Omega_{\mathrm{reg}}$, $\operatorname{supp}\gamma\cap\operatorname{supp}(1-\tilde{\gamma})=\emptyset$ while $\operatorname{supp}\tilde{\gamma}\cap\operatorname{supp}\chi=\emptyset$. Let $\gamma_1,\gamma_2,\dots$ satisfy the same conditions for $\tilde{\gamma}$ with $\operatorname{supp}\gamma_j\cap\operatorname{supp}(1-\gamma_{j+1})=\operatorname{supp}\gamma_j\cap\operatorname{supp}(1-$

$\tilde{\gamma}) = \emptyset$ for each j, and $\operatorname{supp}\gamma \cap \operatorname{supp}(1-\gamma_1) = \emptyset$. Without loss of generality picking N large that $\frac{1}{2} + k - 2N < -N$, we have from Lemma 4.10 that

$$\|\mathcal{H}_M^\Omega \gamma \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_m^{m-\frac{1}{2}+0,-2N}} \le C\|\mathcal{H}_M^\Omega \gamma \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_{m,k-2N}^{m-\frac{1}{2}+0,-k}} \le C\|\tilde{\gamma} f\|_{H_\partial^{\frac{1}{2}+k-2N,m-\frac{1}{2}+0}} \le C\|\tilde{\gamma} f\|_{H_\partial^{-N,m-\frac{1}{2}+0}}, \quad (4.100)$$

since $X_{m,-s}^{m-\frac{1}{2}+0,-k} \subset X_m^{m-\frac{1}{2}+0,-k-s}$ for all $s, k \in \mathbb{N}_0$.

2. The extension correction formula (2.8) implies

$$\chi \mathcal{H}_M^\Omega \gamma \mathcal{V}_1 \cdots \mathcal{V}_k f = \chi \mathcal{R}_M^\Omega \tilde{\gamma}[\Delta, \gamma_{N+1}] \mathcal{R}_M^\Omega[\Delta, \gamma_N] \cdots \mathcal{R}_M^\Omega[\Delta, \gamma_1] \gamma_2 \mathcal{H}_M^\Omega \gamma \mathcal{V}_1 \cdots \mathcal{V}_k f. \quad (4.101)$$

Since $\chi \mathcal{R}_M^\Omega \tilde{\gamma}$ is bounded per Lemma 4.8, Lemma 4.5 now implies

$$\|\chi \mathcal{H}_M^\Omega \gamma \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_m^{l,N}} \le C\|\mathcal{R}_M^\Omega[\Delta, \gamma_N] \cdots \mathcal{R}_M^\Omega[\Delta, \gamma_1] \gamma_2 \mathcal{H}_M^\Omega \gamma \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_m^{l,-N}} \le C\|\mathcal{H}_M^\Omega \gamma \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_m^{m-\frac{1}{2}+0,-N}}, \quad (4.102)$$

similar to Step 2 of Lemma 4.12. Revisit (4.100) to conclude the proof. □

Lemma 4.14. *Let Ω be an AC Lipschitz domain. Let $m, l \in \mathbb{R}$ that $|m| < \mathfrak{r}_\Omega^{r_0}$, $|m - l| < \frac{1}{2}$. Let $\gamma \in S^0(\partial\Omega)$ with $\operatorname{supp}\gamma \subset \partial\Omega_{\mathrm{reg}}$, $\chi \in S^0(\Omega)$ with $(\operatorname{supp}\chi \cap \partial\Omega) \subset \partial\Omega_{\mathrm{reg}}$. Let $\tilde{\gamma} \in S^0(\partial\Omega)$ that $\operatorname{supp}\tilde{\gamma} \subset \partial\Omega_{\mathrm{reg}}$ and $\operatorname{supp}\gamma \cap \operatorname{supp}(1-\tilde{\gamma}) = \emptyset$. Let $\varphi \in S^0(\partial\Omega)$ that $\operatorname{supp}\gamma\chi \cap \operatorname{supp}(1-\varphi) = \emptyset$, $\operatorname{supp}\varphi \subset \partial\Omega_{\mathrm{reg}}$. Then for all $k, k' \in \mathbb{N}_0$, vector fields $\mathcal{V}_1, \ldots, \mathcal{V}_k, \mathcal{V}_1', \ldots, \mathcal{V}_{k'}' \in \mathrm{Diff}_b^{1,1}(\Omega)$, for each $N > 0$, we have*

$$\|\chi \mathcal{V}_1' \ldots \mathcal{V}_{k'}' \mathcal{H}_M^\Omega \gamma \mathcal{V}_1 \ldots \mathcal{V}_k f\|_{X_m^l(\Omega)} \le C\|\tilde{\gamma} f\|_{H^{-N,m-\frac{1}{2}+0}(\partial\Omega)} + C \sum_{|\alpha| \le k+k'} \|\varphi(\langle x\rangle D)^\alpha f\|_{H^{\frac{1}{2},l}(\partial\Omega)}, \quad (4.103)$$

uniformly for all $f \in \mathcal{S}'(\partial\Omega)$ and $M \in [0, M_0]$. See Figure 7(A).

Proof. We do not distinguish $\gamma \in S^0(\partial\Omega)$ from its extension $\gamma \in S^0(\Omega)$. Let $\chi_1, \chi_2, \chi_3, \chi_4 \in S^0(\Omega)$ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\chi_1) = \emptyset$, $\operatorname{supp}\chi_j \cap \operatorname{supp}(1-\chi_{j+1}) = \emptyset$ for $j = 1, 2, 3$. Without loss of generality assume $\operatorname{supp}\gamma\chi_2 \cap \operatorname{supp}(1-\varphi) = \emptyset$. By decomposing $\mathcal{H}_M^\Omega = \mathcal{H}_M^\Omega \chi_2 + \mathcal{H}_M^\Omega(1-\chi_2)$, we have

$$\|\chi \mathcal{V}_1' \cdots \mathcal{V}_{k'}' \mathcal{H}_M^\Omega \gamma \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_m^l} \le C\|\chi_1 \mathcal{H}_M^\Omega \gamma \chi_2 \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_m^{l,k'}} + C\|\chi_1 \mathcal{H}_M^\Omega \gamma (1-\chi_2) \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_m^{l,k'}} \quad (4.104)$$

With that the support of χ_1 and $1-\chi_2$ are disjoint, Lemma 4.13 implies

$$\|\chi_1 \mathcal{H}_M^\Omega \gamma (1-\chi_2) \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_m^{l,k'}} \le C\|\tilde{\gamma} f\|_{H_\partial^{-N,m-\frac{1}{2}+0}}, \quad (4.105)$$

that the second term of (4.104) is an error term.

2. We iterate in $k \in \mathbb{N}_0$ to show for all $k' \in \mathbb{N}_0$,

$$\|\chi_1 \mathcal{H}_M^\Omega \gamma \chi_2 \mathcal{V}_1 \cdots \mathcal{V}_k f\|_{X_m^{l,k'}} \le C\|\gamma f\|_{H_\partial^{-N,m-\frac{1}{2}+0}} + C \sum_{|\alpha| \le k'+k} \|\varphi(\langle x\rangle D)^\alpha f\|_{H_\partial^{\frac{1}{2},l}}. \quad (4.106)$$



(A) Lemma 4.14 (B) Lemma 4.17, Lemma 4.18

FIGURE 7. Illustration for Lemma 4.12, Lemma 4.17 and Lemma 4.18.

Lemma 4.12 and Lemma 4.13 give case $k = 0$ and all $k' \in \mathbb{N}_0$. Let (4.106) holds for $k-1$ and all $k' \in \mathbb{N}_0$. Note

$$(-\Delta + M^2)\gamma\chi_2\mathcal{V}_1\chi_3\mathcal{H}^{\Omega}_M\tilde{\gamma}\chi_4\mathcal{V}_2\cdots\mathcal{V}_k f = \chi_3[-\Delta,\gamma\chi_2\mathcal{V}_1]\chi_3\mathcal{H}^{\Omega}_M\tilde{\gamma}\chi_4\mathcal{V}_2\cdots\mathcal{V}_k f, \tag{4.107}$$

where $\gamma\chi_2\mathcal{V}_1\chi_3\mathcal{H}^{\Omega}_M\tilde{\gamma}\chi_4\mathcal{V}_2\cdots\mathcal{V}_k f$ has trace $\gamma\chi_2\mathcal{V}_1\cdots\mathcal{V}_k f$ on $\partial\Omega$. This implies

$$\chi_1\mathcal{H}^{\Omega}_M\gamma\chi_2\mathcal{V}_1\ldots\mathcal{V}_k f = \gamma\chi_1\mathcal{V}_1\chi_3\mathcal{H}^{\Omega}_M\tilde{\gamma}\chi_4\mathcal{V}_2\cdots\mathcal{V}_k f - \chi_1\mathcal{R}^{\Omega}_M\chi_3[-\Delta,\gamma\chi_2\mathcal{V}_1]\chi_3\mathcal{H}^{\Omega}_M\tilde{\gamma}\chi_4\mathcal{V}_2\cdots\mathcal{V}_k f. \tag{4.108}$$

Now note $\chi_1\mathcal{R}^{\Omega}_M\chi_3 : Y^{l,k'}_m \to \dot{X}^{l,k'}_m$ and $[-\Delta,\gamma\chi_2\mathcal{V}_1] = O^{X\to Y}_{\text{div}} : \dot{X}^{l,k'}_m \to Y^{l,k'}_m$ are bounded per Lemma 4.7, Lemma 4.5 and Lemma 4.4. Apply the iterative assumption to χ_3,χ_4 instead of χ_1,χ_2 to see

$$\begin{aligned}\|\chi_1\mathcal{H}^{\Omega}_M\gamma\chi_2\mathcal{V}_1\cdots\mathcal{V}_k f\|_{X^{l,k'}_m} &\le C\|\chi_3\mathcal{H}^{\Omega}_M\tilde{\gamma}\chi_4\mathcal{V}_2\cdots\mathcal{V}_k f\|_{X^{l,k'+1}_m} + C\|\chi_3\mathcal{H}^{\Omega}_M\tilde{\gamma}\chi_4\mathcal{V}_2\cdots\mathcal{V}_k f\|_{X^{l,k'}_m}\\ &\le C\|\tilde{\gamma}f\|_{H^{-N,m-\frac{1}{2}+0}_{\partial}} + C\sum_{|\alpha|\le(k'+1)+(k-1)}\|\tilde{\varphi}(\langle x\rangle D)^{\alpha}f\|_{H^{\frac{1}{2},l}_{\partial}},\end{aligned} \tag{4.109}$$

for some $\tilde{\varphi}$ with slightly larger support than φ. Iterate and rename $\tilde{\varphi}$ to φ to get (4.106) for all $k,k' \in \mathbb{N}_0$. Revisit (4.104) to conclude the proof. □

Lemma 4.15. *Let Ω be an AC Lipschitz domain. Let $m,l \in \mathbb{R}$ that $|m| < \mathfrak{r}^{r_0}_{\Omega}$, $|m-l| < \frac{1}{2}$. Let $k,s \in \mathbb{N}_0$. Let $N \in \mathbb{N}_0$ be large. Let α be a multi-index. Let $\chi \in S^0(\Omega)$. Let $\gamma,\tilde{\gamma} \in S^0(\partial\Omega)$ with support inside $\partial\Omega_{\text{reg}}$, and $\operatorname{supp}\gamma \cap \operatorname{supp}(1-\tilde{\gamma}) = \emptyset$. Let $\psi(\xi) \in S^0(\mathbb{R}^{d-1})$ with $0 \notin \operatorname{supp}\psi$. Let $\varphi \in S^0(\Omega)$, with properties to be specified. Let $\mathcal{U}_1,\ldots,\mathcal{U}_k \in \operatorname{Diff}^{1,0}_b(\Omega)$. Then the following are true uniformly for all $f \in \mathcal{S}'(\Omega)$, $f_{\partial} \in \mathcal{S}'(\partial\Omega)$:*

(1) If $\operatorname{supp}\varphi \cap \operatorname{supp}\chi = \emptyset$, then

$$\|\chi\mathbf{r}^{l+k}\mathbf{d}^{m-l}\mathcal{U}_1\cdots\mathcal{U}_k\nabla\mathcal{R}^{\Omega}_M\varphi f\|_{L^2(\Omega)} \le C\|f\|_{Y^{l,-N}_m(\Omega)}. \tag{4.110}$$

(2) We have

$$\|\chi\mathbf{r}^{l+k}\mathbf{d}^{m-l}\mathcal{U}_1\cdots\mathcal{U}_k\nabla\mathcal{R}^{\Omega}_M f\|_{L^2(\Omega)} \le C\sum_{|\alpha|\le k}\|(\mathbf{r}\mathcal{W})^{\alpha}f\|_{Y^{l}_m(\Omega)}. \tag{4.111}$$

(3) If $(\operatorname{supp}\varphi \cap \partial\Omega) \cap \operatorname{supp}\gamma = \emptyset$, then

$$\|\varphi\mathbf{r}^{l+k}\mathbf{d}^{m-l}\mathcal{U}_1\cdots\mathcal{U}_k\nabla\mathcal{H}^{\Omega}_M\gamma f_{\partial}\|_{L^2(\Omega)} \le C\|\gamma f_{\partial}\|_{H^{-N,m-\frac{1}{2}+0}(\partial\Omega)}. \tag{4.112}$$

(4) We have

$$\|\chi\mathbf{r}^{l+k}\mathbf{d}^{m-l}\mathcal{U}_1\dots\mathcal{U}_k\nabla\mathcal{H}_M^\Omega\gamma f_\partial\|_{L^2(\Omega)}\le C\sum_{|\alpha|\le k}\|(\langle x\rangle D)^\alpha\gamma f_\partial\|_{H^{\frac12,l}(\partial\Omega)}+C\|\gamma f_\partial\|_{H^{-N,m-\frac12+0}}. \tag{4.113}$$

(5) We have

$$\|\mathcal{H}_M^\Omega\gamma D^\alpha f_\partial\|_{X_{m,s}^{l,-|\alpha|}(\Omega)}\le C\|\tilde\gamma f_\partial\|_{H^{s+\frac12,l-|\alpha|}(\partial\Omega)}. \tag{4.114}$$

(6) If $(\operatorname{supp}\varphi\cap\partial\Omega)\cap\operatorname{supp}\gamma=\emptyset$*, then*

$$\|\varphi\mathbf{r}^k\mathcal{U}_1\cdots\mathcal{U}_k\mathcal{H}_M^\Omega\gamma D^\alpha f_\partial\|_{X_m^l(\Omega)}\le C\|\tilde\gamma f_\partial\|_{H^{-N,m-\frac12-|\alpha|+0}(\partial\Omega)}. \tag{4.115}$$

(7) We have

$$\|\mathcal{H}_M^\Omega\gamma\psi(D)\tilde\gamma f_\partial\|_{X_{m-2k,s}^{l-2k,0}}\le C\|\tilde\gamma f_\partial\|_{H^{s+\frac12,l-2k}(\partial\Omega)}. \tag{4.116}$$

(8) If $(\operatorname{supp}\varphi\cap\partial\Omega)\cap\operatorname{supp}\gamma=\emptyset$*, then*

$$\|\varphi\mathcal{H}_M^\Omega\gamma\psi(D)\tilde\gamma f_\partial\|_{X_{m,N}^{l,N}(\Omega)}\le C\|\tilde\gamma f_\partial\|_{H^{-N,-N}(\partial\Omega)}. \tag{4.117}$$

(9) If $\operatorname{supp}\varphi$ *is compact with* $(\operatorname{supp}\varphi\cap\partial\Omega)\cap\operatorname{supp}\gamma=\emptyset$*, then*

$$\|\varphi\mathcal{H}_M^\Omega\gamma\psi(D)\tilde\gamma f_\partial\|_{H^{\frac32-0,N}(\Omega)}\le C\|\tilde\gamma f_\partial\|_{H^{-N,-N}(\partial\Omega)}. \tag{4.118}$$

(10) If $(\operatorname{supp}\varphi\cap\partial\Omega)\cap\operatorname{supp}\gamma=\emptyset$*, then*

$$\|\gamma\psi(D)\gamma\partial_\nu\mathcal{R}_M^\Omega\varphi f\|_{H^{N,N}(\partial\Omega)}\le C\|\varphi f\|_{Y_{-m,-N}^{-l,-N}(\Omega)}. \tag{4.119}$$

Proof. We set up some cutoffs. Let $\tilde\chi$, $\tilde\varphi$, $\tilde{\tilde\gamma}$ satisfy the same conditions for χ, φ, $\tilde\gamma$ with $\operatorname{supp}\chi\cap\operatorname{supp}(1-\tilde\chi)=\operatorname{supp}\varphi\cap\operatorname{supp}(1-\tilde\varphi)=\operatorname{supp}\tilde\gamma\cap\operatorname{supp}(1-\tilde{\tilde\gamma})=\emptyset$. We confuse $\gamma,\tilde\gamma,\tilde{\tilde\gamma}\in S^0(\partial\Omega)$ with their extensions in $S^0(\Omega)$ that $\operatorname{supp}\gamma\cap\operatorname{supp}(1-\tilde\gamma)=\operatorname{supp}\tilde\gamma\cap\operatorname{supp}(1-\tilde{\tilde\gamma})=\emptyset$.

(1) Note $\tilde\chi$ and $\tilde\varphi$ have disjoint support. Lemma 4.8 implies

$$\|\chi\mathbf{r}^{l+k}\mathbf{d}^{m-l}\mathcal{U}_1\cdots\mathcal{U}_k\nabla\mathcal{R}_M^\Omega\varphi f\|\le C\|\tilde\chi\mathcal{R}_M^\Omega\tilde\varphi\varphi f\|_{X_m^{l,k}}\le C\|\varphi f\|_{Y_m^{l,-N}}. \tag{4.120}$$

(2) Lemma 4.6 implies

$$\|\chi\mathbf{r}^{l+k}\mathbf{d}^{m-l}\mathcal{U}_1\cdots\mathcal{U}_k\nabla\mathcal{R}_M^\Omega f\|\le C\|f\|_{Y_m^{l,k}}\le C\sum_{|\alpha|\le k}\|(\mathbf{r}\mathcal{W})^\alpha f\|_{Y_m^l}. \tag{4.121}$$

(3) Lemma 4.13 implies

$$\|\varphi\mathbf{r}^{l+k}\mathbf{d}^{m-l}\mathcal{U}_1\cdots\mathcal{U}_k\nabla\mathcal{H}_M^\Omega\gamma f_\partial\|\le\|\varphi\mathbf{r}^k\mathcal{U}_1\cdots\mathcal{U}_k\mathcal{H}_M^\Omega\tilde\gamma\gamma f_\partial\|_{X_m^l}\le C\|\gamma f_\partial\|_{H_\partial^{-N,m-\frac12+0}}. \tag{4.122}$$

(4) Lemma 4.14 implies

$$\|\chi\mathbf{r}^{l+k}\mathbf{d}^{m-l}\mathcal{U}_1\cdots\mathcal{U}_k\nabla\mathcal{H}_M^\Omega\gamma f_\partial\|\le\|\chi\mathbf{r}^k\mathcal{U}_1\cdots\mathcal{U}_k\mathcal{H}_M^\Omega\tilde\gamma\gamma f_\partial\|_{X_m^l}\le C\sum_{|\alpha|\le k}\|(\langle x\rangle D)^\alpha\gamma f_\partial\|_{H_\partial^{\frac12,l}}+C\|\gamma f_\partial\|_{H^{-N,m-\frac12+0}}. \tag{4.123}$$

(5) Let $s \in \mathbb{N}_0$. We claim that by induction in $k \in \mathbb{N}_0$,

$$\mathcal{H}^\Omega_M \gamma \mathcal{U}_1 \cdots \mathcal{U}_k : H^{s+\frac{1}{2},l-k}_\partial \to X^{l,-k}_{m,s} \tag{4.124}$$

is bounded. Lemma 4.12 and Proposition 2.19 implies

$$\|\mathcal{H}^\Omega_M \gamma f_\partial\|_{X^l_{m,s}} \leq C \sum_{|\beta|\leq s} \|D^\beta \gamma f_\partial\|_{H^{\frac{1}{2},l}_\partial} \leq C \|\gamma f_\partial\|_{H^{s+\frac{1}{2},l}_\partial}, \tag{4.125}$$

and that the case $k = 0$ holds. Assume that the cases up to $k-1$ hold, then consider that

$$\mathcal{H}^\Omega_M \gamma \mathcal{U}_1 \cdots \mathcal{U}_k = \mathcal{H}^\Omega_M \gamma \mathbf{r} \mathcal{U}_1 \mathcal{U}_2 \cdots \mathcal{U}_k \mathbf{r}^{-1} - \mathcal{H}^\Omega_M \gamma [\mathbf{r}, \mathcal{U}_1 \cdots \mathcal{U}_k] \mathbf{r}^{-1}, \tag{4.126}$$

the second term of which is desirably bounded from $H^{s+\frac{1}{2},l-k}_\partial$ to $X^{l,-k+1}_{m,s}$ because of the inductive assumption and that $[\mathbf{r}, \mathcal{U}_1 \ldots \mathcal{U}_k] \in \mathrm{Diff}^{k-1,0}_b$. For the first term, since $\mathcal{U}_1$ is tangential to $\partial\Omega$,

$$\gamma \mathbf{r} \mathcal{U}_1 \mathcal{H}^\Omega_M \tilde{\gamma} \mathcal{U}_2 \cdots \mathcal{U}_k \mathbf{r}^{-1} f_\partial = \mathcal{H}^\Omega_M \gamma \mathbf{r} \mathcal{U}_1 \mathcal{U}_2 \cdots \mathcal{U}_k \mathbf{r}^{-1} f_\partial \ \text{ on } \partial\Omega \tag{4.127}$$

and thus the extension correction formula (2.8) gives

$$\mathcal{H}^\Omega_M \gamma \mathbf{r} \mathcal{U}_1 \tilde{\gamma} \mathcal{U}_2 \cdots \mathcal{U}_k \mathbf{r}^{-1} = \gamma \mathbf{r} \mathcal{U}_1 \mathcal{H}^\Omega_M \tilde{\gamma} \mathcal{U}_2 \cdots \mathcal{U}_k \mathbf{r}^{-1} - \mathcal{R}^\Omega_M [-\Delta, \gamma \mathbf{r} \mathcal{U}_1] \mathcal{H}^\Omega_M \tilde{\gamma} \mathcal{U}_2 \cdots \mathcal{U}_k \mathbf{r}^{-1}. \tag{4.128}$$

Now note that by the inductive assumption, the first term here $\gamma \mathbf{r} \mathcal{U}_1 \mathcal{H}^\Omega_M \tilde{\gamma} \mathcal{U}_2 \cdots \mathcal{U}_k \mathbf{r}^{-1}$ is bounded from $H^{s+\frac{1}{2},l-k}_\partial$ to $X^{l,-k}_{m,s}$. Now note $[-\Delta, \gamma \mathbf{r} \mathcal{U}_1] = O^{X \to Y}_{\mathrm{div}}$ maps $X^{l,-k}_{m,s}$ to $Y^{l,-k}_{m,s}$ and $\mathcal{R}^\Omega_M : Y^{l,-k}_{m,s} \to \dot{X}^{l,-k}_{m,s}$ per Lemma 4.7, we have the second term of (4.128) being bounded from $H^{s+\frac{1}{2},l-k}_\partial$ to $\dot{X}^{l,-k}_{m,s}$. Revisit (4.126) to observe the case k holds and this concludes the induction.

(6) Note $\varphi \gamma = 0$ and

$$\tilde{\varphi} \mathcal{H}^\Omega_M \gamma D^\alpha f_\partial = \tilde{\varphi} \mathcal{H}^\Omega_M \gamma (\mathbf{r} D)^\alpha \mathbf{r}^{-|\alpha|} f_\partial + \tilde{\varphi} \mathcal{H}^\Omega_M \gamma O(\mathrm{Diff}^{|\alpha|-1,|\alpha|-1}_b) \mathbf{r}^{-|\alpha|} f_\partial. \tag{4.129}$$

Applying Lemma 4.13 on each term then gives

$$\|\varphi \mathbf{r}^k \mathcal{U}_1 \cdots \mathcal{U}_k \mathcal{H}^\Omega_M \gamma D^\alpha f_\partial\|_{X^l_m} \leq C \|\tilde{\gamma} \mathbf{r}^{-|\alpha|} f_\partial\|_{H^{-N,m-\frac{1}{2}+0}_\partial}. \tag{4.130}$$

(7) Consider that

$$\mathcal{H}^\Omega_M \gamma \psi(D) \tilde{\tilde{\gamma}} = \mathcal{H}^\Omega_M \gamma (-\Delta)^k \tilde{\gamma} (-\Delta)^{-k} \psi(D) \tilde{\tilde{\gamma}}. \tag{4.131}$$

Here $\tilde{\gamma}(-\Delta)^{-k}\psi(D)\tilde{\tilde{\gamma}}$ is bounded from $H^{s+\frac{1}{2},l-2k}(\partial\Omega)$ to $H^{s+\frac{1}{2}+2k,l-2k}(\partial\Omega)$. Now invoke part (5) to see $\mathcal{H}^\Omega_M \gamma \psi(D) \tilde{\gamma} f_\partial \in X^{l,-2k}_{m,s+2k}(\Omega) \subset X^{l-2k,0}_{m-2k,s}(\Omega)$ provided $\tilde{\gamma} f_\partial \in H^{s+\frac{1}{2},l-2k}(\partial\Omega)$.

(8) Consider that

$$\varphi \mathcal{H}^\Omega_M \gamma \psi(D) \tilde{\tilde{\gamma}} = \varphi \mathcal{H}^\Omega_M \gamma (-\Delta)^k \tilde{\gamma} (-\Delta)^{-k} \psi(D) \tilde{\tilde{\gamma}}. \tag{4.132}$$

Given $\tilde{\gamma} f_\partial \in H^{-N,-N}(\partial\Omega)$, we have $\tilde{\gamma}(-\Delta)^{-k}\psi(D)\tilde{\gamma} f_\partial \in H^{2k-N,-N}_\partial$. Take k sufficiently large that $l - 2k \leq -N$. Part (6) implies that $\varphi \mathcal{H}^\Omega_M \gamma \psi(D) \tilde{\gamma} f_\partial \in X^{l,2N}_{m,0} \subset X^{l,N}_{m,N}$.

(9) Note $\varphi \mathcal{H}^\Omega_M \gamma \psi(D) \tilde{\gamma} f_\partial|_{\partial\Omega} = 0$. This implies

$$\varphi \mathcal{H}^\Omega_M \gamma \psi(D) \tilde{\gamma} f_\partial = \mathcal{R}^\Omega_M [-\Delta, \varphi] \tilde{\varphi} \mathcal{H}^\Omega_M \gamma \psi(D) \tilde{\gamma} f_\partial \tag{4.133}$$

and thus part (8) implies

$$\|\tilde{\varphi} \mathcal{H}^\Omega_M \gamma \psi(D) \tilde{\gamma} f_\partial\|_{H^{\frac{1}{2}}} \leq \|\tilde{\varphi} \mathcal{H}^\Omega_M \gamma \psi(D) \tilde{\gamma} f_\partial\|_{X^0_0} \leq \|\tilde{\gamma} f_\partial\|_{H^{-N,-N}_\partial}, \tag{4.134}$$

with $[-\Delta,\varphi]\tilde{\varphi}\mathcal{H}^\Omega_M\gamma\psi(D)\tilde{\gamma}f_\partial \in Y^0_0$. Note $\mathcal{R}^\Omega_M : Y^0_0 \to \dot{X}^0_0$ and thus Lemma 2.22 implies

$$\begin{aligned}\|\varphi\mathcal{H}^\Omega_M\gamma\psi(D)\tilde{\gamma}f_\partial\|_{H^{\frac{3}{2}-0}} &\le C\|\mathbf{r}^{-N}\psi_{\mathbf{d}}(\mathbf{d})\mathcal{R}^\Omega_M[-\Delta,\varphi]\tilde{\varphi}\mathcal{H}^\Omega_M\gamma\psi(D)\tilde{\gamma}f_\partial\|_{H^{\frac{3}{2}-0}}\\ &\le C\|[-\Delta,\varphi]\tilde{\varphi}\mathcal{H}^\Omega_M\gamma\psi(D)\tilde{\gamma}f_\partial\|_{H^{-\frac{1}{2}-0}} + C\|\mathcal{R}^\Omega_M[-\Delta,\varphi]\tilde{\varphi}\mathcal{H}^\Omega_M\gamma\psi(D)\tilde{\gamma}f_\partial\|_{X^0_0} \le C\|\tilde{\gamma}f_\partial\|_{H^{-N,-N}_\partial}\end{aligned} \tag{4.135}$$

as desired, for some $\psi_{\mathbf{d}} \in C^\infty_c(\mathbb{R})$ that $\psi_{\mathbf{d}}(\mathbf{d})\varphi = \varphi$.

(10) Part (8) implies for any $\phi_\partial \in \mathcal{S}'(\partial\Omega)$, we have $\tilde{\varphi}\mathcal{H}^\Omega_M\gamma\psi(D)\gamma\phi_\partial \in \dot{X}^{l,N}_{m,N}$ and

$$\|\tilde{\varphi}\mathcal{H}^\Omega_M\gamma\psi(D)\gamma\phi_\partial\|_{X^{l,N}_{m,N}} \le C\|\gamma\phi_\partial\|_{H^{-N,-N}_\partial}. \tag{4.136}$$

By noting the duality $(\mathcal{H}^\Omega_M)^* = -\partial_\nu\mathcal{R}^\Omega_M$, we have

$$|\langle\gamma\psi(D)\gamma\partial_\nu\mathcal{R}^\Omega_M\varphi f, \phi_\partial\rangle_{\partial\Omega}| = |\langle\varphi f, \tilde{\varphi}\mathcal{H}^\Omega_M\gamma\psi(D)^*\gamma\phi_\partial\rangle| \le C\|\varphi f\|_{Y^{-l,-N}_{-m,-N}}\|\phi_\partial\|_{H^{-N,-N}_\partial}, \tag{4.137}$$

yielding the desired bounds. □

4.4. **The DtN map on AC Lipschitz domains.** Next, we record local elliptic estimates for Λ^Ω_M.

Lemma 4.16 (Elliptic estimate for Λ^Ω_M near the Lipschitz boundary)**.** *Let Ω be an AC Lipschitz domain. Let $l \in \mathbb{R}$ that $|l| < \mathfrak{r}^{r_0}_\Omega + \frac{1}{2}$. Let $\omega > 0$. Let $\chi, \tilde{\chi}, \tilde{\tilde{\chi}} \in C^\infty_c(\partial\Omega)$ that $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \operatorname{supp}\tilde{\chi} \cap \operatorname{supp}(1-\tilde{\tilde{\chi}}) = \emptyset$, $\operatorname{supp}(1-\chi) \subset \partial\Omega_{\mathrm{reg}}$. Let $\psi_0(\xi) \in C^\infty_c(\mathbb{R}^{d-1})$ that $0 \notin \operatorname{supp}(1-\psi_0)$. For each $0 \le s \le 1$, $|l| < \mathfrak{r}^{r_0}_\Omega + \frac{1}{2}$, and $N \in \mathbb{R}$, we have*

$$\begin{aligned}\|\chi u\|_{H^s(\partial\Omega)} \le C\|\tilde{\chi}(\Lambda^\Omega_M - \sqrt{\omega^2+M^2})u\|_{H^{s-1}(\partial\Omega)} + C\|\tilde{\tilde{\chi}}u\|_{H^0(\partial\Omega)} + C\|(1-\tilde{\chi})u\|_{H^{-N,-N}(\partial\Omega)}\\ + C\|(1-\tilde{\tilde{\chi}})\psi_0(D)(1-\tilde{\chi})u\|_{H^{-N,l}(\partial\Omega)}.\end{aligned} \tag{4.138}$$

uniformly for all $u \in \mathcal{S}'(\partial\Omega)$ and $M \in [0, M_0]$.

Proof. We confuse the cutoffs with their extension onto Ω with 0 Neumann data. By (2.122) from Lemma 2.24, we have

$$\begin{aligned}\|\chi\mathcal{H}^\Omega_M u\|_{H^{s+\frac{1}{2}}} \le C\|\tilde{\chi}(\Lambda^\Omega_M - \sqrt{\omega^2+M^2})u\|_{H^{s-1}_\partial} + C\|\tilde{\chi}\mathcal{H}^\Omega_M\big(\tilde{\tilde{\chi}} + (1-\tilde{\tilde{\chi}})\psi_0(D)(1-\tilde{\chi})\big)u\|_{H^{s-\frac{1}{2}}}\\ + C\|\tilde{\chi}u\|_{H^{s-1}_\partial} + C\|\tilde{\chi}\mathcal{H}^\Omega_M(1-\tilde{\tilde{\chi}})(1-\psi_0(D))(1-\tilde{\chi})u\|_{H^{s-\frac{1}{2}}}.\end{aligned} \tag{4.139}$$

It remains to estimate the last two terms. Lemma 2.24 implies

$$\|\tilde{\chi}\mathcal{H}^\Omega_M\big(\tilde{\tilde{\chi}}+(1-\tilde{\tilde{\chi}})\psi_0(D)(1-\tilde{\chi})\big)u\|_{H^{s-\frac{1}{2}}} \le C\|\tilde{\tilde{\chi}}u\|_{H^0_\partial} + C\|(1-\tilde{\tilde{\chi}})\psi_0(D)(1-\tilde{\chi})u\|_{H^{-N,l}_\partial} + C\|(1-\tilde{\chi})u\|_{H^{-N,-N}_\partial}. \tag{4.140}$$

On another hand, (4.118) implies

$$\|\tilde{\chi}\mathcal{H}^\Omega_M(1-\tilde{\tilde{\chi}})(1-\psi_0(D))(1-\tilde{\chi})u\|_{H^{s-\frac{1}{2}}} \le C\|(1-\tilde{\chi})u\|_{H^{-N,-N}_\partial}. \tag{4.141}$$

This concludes the proof. □

Lemma 4.17 (Change of domain for Λ_M^Ω)**.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be AC Lipschitz domains. Let $\chi \in C^\infty(\Omega_1)$ that $\operatorname{supp}\chi \subset \Omega_1 \cap \Omega_2$, Ω_1, Ω_2 share the common boundary near $\operatorname{supp}\chi$ and $\partial_\nu \chi \equiv 0$. Then we have*

$$\chi \Lambda_M^{\Omega_1} = \Lambda_M^{\Omega_2} \chi + \tilde{\mathcal{E}}_M,\ \tilde{\mathcal{E}}_M = \partial_\nu \mathcal{R}_M^{\Omega_2}[-\Delta, \chi]\mathcal{H}_M^{\Omega_1}, \tag{4.142}$$

uniformly for all $M \in [0, M_0]$. See Figure 7(B).

Proof. Noting $\chi \mathcal{H}_M^{\Omega_1} u$ extends χu into Ω_2, we have

$$\mathcal{H}_M^{\Omega_2} \chi u = \chi \mathcal{H}_M^{\Omega_1} u - \mathcal{R}_M^{\Omega_2}(-\Delta + M^2)\chi \mathcal{H}_M^{\Omega_1} u = \chi \mathcal{H}_M^{\Omega_1} u - \mathcal{R}_M^{\Omega_2}[-\Delta, \chi]\mathcal{H}_M^{\Omega_1} u, \tag{4.143}$$

by the extension correction formula (2.8). Thus

$$\Lambda_M^{\Omega_2} \chi u = \partial_\nu \mathcal{H}_M^{\Omega_2} \chi u = \chi \Lambda_M^{\Omega_1} u - \partial_\nu \mathcal{R}_M^{\Omega_2}[-\Delta, \chi]\mathcal{H}_M^{\Omega_1} u, \tag{4.144}$$

as desired. □

We can now prove our main result on the structure of the DtN map near infinity on an AC Lipschitz domain. Given $\psi(\xi) \in S^0(\mathbb{R}^{d-1})$, we interpret $\psi(D)$ as a Fourier multiplier in $\Psi^{0,0}(\mathbb{R}^{d-1})$. Given $\chi, \tilde{\chi} \in S^0(\partial\Omega_{\mathrm{reg}})$, we have $\chi\psi(D)\tilde{\chi} \in \Psi^{0,0}(\partial\Omega_{\mathrm{reg}})$.

Lemma 4.18 (Comparing Λ_M^Ω near infinity)**.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be AC Lipschitz domains. Let $\chi_1, \chi_2 \in S^0(\partial\Omega_1 \cap \partial\Omega_2)$ with $\operatorname{supp}\chi_1 \cap \operatorname{supp}(1 - \chi_2) = \emptyset$. Assume Ω_1, Ω_2 share the common smooth boundary near $\operatorname{supp}\chi_2$. Then we have*

$$\chi_1 \Lambda_M^{\Omega_1} = \chi_1 \Lambda_M^{\Omega_2} \chi_2 + \chi_1 \tilde{\mathcal{E}}_M, \tag{4.145}$$

where we confuse χ_1, χ_2 with extensions specified in the proof to write

$$\chi_1 \tilde{\mathcal{E}}_M = \chi_1 \partial_\nu \mathcal{R}_M^{\Omega_2}[-\Delta, \chi_2]\mathcal{H}_M^{\Omega_1}. \tag{4.146}$$

Moreover, let $\psi(\xi) \in S^0(\mathbb{R}^{d-1})$ with $0 \notin \operatorname{supp}\psi$, $N \geq 0$, $l_j \in (-\mathfrak{r}_{\Omega_j}^{r_0} - \frac{1}{2}, \mathfrak{r}_{\Omega_j}^{r_0} + \frac{1}{2})$ for $j = 1, 2$ such that

$$l_2 < l_1 + 1,\ l_2 < \mathfrak{r}_{\Omega_1}^{r_0} + \frac{1}{2},\ l_1 > -\mathfrak{r}_{\Omega_2}^{r_0} - \frac{1}{2}. \tag{4.147}$$

We can estimate the error $\tilde{\mathcal{E}}_M$ by

$$\|\chi_1\psi(D)\chi_1\tilde{\mathcal{E}}_M\chi_1\psi(D)\chi_1 f\|_{H^{N,N}(\partial\Omega_1)} \leq C\|f\|_{H^{-N,-N}(\partial\Omega_1)} \tag{4.148}$$

$$\|\chi_1\psi(D)\chi_1\tilde{\mathcal{E}}_M f\|_{H^{N,N}(\partial\Omega_1)} \leq C\|f\|_{H^{0,l_1}(\partial\Omega_1)}, \tag{4.149}$$

$$\|\chi_1\tilde{\mathcal{E}}_M\chi_1\psi(D)\chi_1 f\|_{H^{N,l_2}(\partial\Omega_1)} \leq C\|f\|_{H^{-N,-N}(\partial\Omega_1)}, \tag{4.150}$$

$$\|\chi_1\tilde{\mathcal{E}}_M f\|_{H^{N,l_2}(\partial\Omega_1)} \leq C\|f\|_{H^{0,l_1}(\partial\Omega_1)}, \tag{4.151}$$

uniformly for all $f \in \mathcal{S}'(\partial\Omega_1)$ and $M \in [0, M_0]$. See Figure 7(B).

Proof. 0. Throughout the proof, we let m_j that $|m_j| < \mathfrak{r}_{\Omega_j}^{r_0}$, $|m_j - l_j| < \frac{1}{2}$ for $j = 1, 2$. We further assume without loss of generality that $m_2 \leq m_1$, whose existence is guaranteed by (4.147). Note $\Omega_1 \cap \Omega_2$ admits a conic subset U that $\operatorname{supp}\chi_2 \subset U$. We confuse χ_2 with its extension in $S^0(U)$ that $\partial_\nu \chi_2 = 0$.

1. We have by (4.142)

$$\chi_2 \Lambda_M^{\Omega_1} = \Lambda_M^{\Omega_2} \chi_2 + \partial_\nu \mathcal{R}_M^{\Omega_2}[-\Delta, \chi_2]\mathcal{H}_M^{\Omega_1}. \tag{4.152}$$

Hence

$$\chi_1\Lambda_M^{\Omega_1}=\chi_1\Lambda_M^{\Omega_2}\chi_2+\chi_1\partial_\nu\mathcal{R}_M^{\Omega_2}[-\Delta,\chi_2]\mathcal{H}_M^{\Omega_1}=\chi_1\Lambda_M^{\Omega_2}\chi_2+\chi_1\tilde{\mathcal{E}}_M. \tag{4.153}$$

It remains to establish the four desired estimates for

$$\chi_1\tilde{\mathcal{E}}_M=\chi_1\partial_\nu\mathcal{R}_M^{\Omega_2}\varphi[-\Delta,\chi_2]\varphi\mathcal{H}_M^{\Omega_1}, \tag{4.154}$$

where $\varphi\in S^0(U)$ with $\varphi\equiv 1$ near $\operatorname{supp}\nabla\chi_2$, $\operatorname{supp}\varphi\cap\operatorname{supp}\chi_1=\emptyset$.

2. We show (4.148). By (4.117) and (4.119), we have

$$\varphi\mathcal{H}_M^{\Omega_1}\chi_1\psi(D)\chi_1: H_\partial^{-N,-N}\to X_{m_1}^{l_1}(\Omega_1),\quad \chi_1\psi(D)\chi_1\partial_\nu\mathcal{R}_M^{\Omega_2}\varphi:\dot{Y}_{m_2}^{l_2}(\Omega_2)\to H_\partial^{N,N}, \tag{4.155}$$

bounded uniformly in M. Lemma 4.5 implies $[-\Delta,\chi_2]: X_{m_1}^{l_1}(\Omega_1)\to\varphi\dot{Y}_{m_1}^{l_1+1}(\Omega_1)\subset\dot{Y}_{m_2}^{l_2}(\Omega_2)$ and this concludes (4.148). Note here we used $l_2\le l_1+1$ and $m_2\le m_1$.

3. For (4.149), it suffices to show $[-\Delta,\chi_2]\varphi\mathcal{H}_M^{\Omega_1}: H_\partial^{0,l_1}\to\dot{Y}_{m_1}^{l_1+1}(\Omega_1)$ is bounded. Take $\kappa\in C_c^\infty(\overline{\Omega_1})$ that $\partial\Omega_1$ is smooth over $\operatorname{supp}(1-\kappa)$. Let $\kappa_1,\tilde{\kappa}$ satisfy the same requirement for κ while $\operatorname{supp}\kappa\cap\operatorname{supp}(1-\kappa_1)=\operatorname{supp}\kappa_1\cap\operatorname{supp}(1-\tilde{\kappa})=\emptyset$. For $f\in H_\partial^{0,l_1}$, Lemma 2.24 implies

$$\|[-\Delta,\chi_2]\varphi\tilde{\kappa}\mathcal{H}_M^{\Omega_1}f\|_{H^{-1}(\Omega_1)}\le C\|\varphi\tilde{\kappa}\mathcal{H}_M^{\Omega_1}f\|_{H^{\frac12}(\Omega_1)}\le C\|f\|_{H_\partial^{0,l_1}}. \tag{4.156}$$

To estimate $[-\Delta,\chi_2]\varphi(1-\tilde{\kappa})\mathcal{H}_M^{\Omega_1}f$, we decompose it into

$$[-\Delta,\chi_2]\varphi(1-\tilde{\kappa})\mathcal{H}_M^{\Omega_1}f=[-\Delta,\chi_2]\varphi(1-\tilde{\kappa})\mathcal{H}_M^{\Omega_1}(1-\kappa)f+[-\Delta,\chi_2]\varphi(1-\tilde{\kappa})\mathcal{H}_M^{\Omega_1}\kappa f. \tag{4.157}$$

Lemma 4.5 and Lemma 4.10 allow us to bound the first term by

$$\|[-\Delta,\chi_2]\varphi(1-\tilde{\kappa})\mathcal{H}_M^{\Omega_1}(1-\kappa)f\|_{Y_{m_1}^{l_1+1}}\le C\|\mathcal{H}_M^{\Omega_1}(1-\kappa)f\|_{X_{m_1,-1}^{l_1,0}}\le C\|f\|_{H_\partial^{-\frac12,l_1}}. \tag{4.158}$$

Noting $(1-\tilde{\kappa})\mathcal{H}_M^{\Omega_1}\kappa=(1-\tilde{\kappa})\mathcal{R}_M^{\Omega_1}[\Delta,\kappa_1]\mathcal{H}_M^{\Omega_1}\kappa$ we have

$$\begin{aligned}\|[-\Delta,\chi_2]\varphi(1-\tilde{\kappa})\mathcal{H}_M^{\Omega_1}\kappa f\|_{Y_{m_1}^{l_1+1}}&\le C\|(1-\tilde{\kappa})\mathcal{R}_M^{\Omega_1}[\Delta,\kappa_1]\mathcal{H}_M^{\Omega_1}\kappa f\|_{X_{m_1,-1}^{l_1,0}}\\&\le C\|[\Delta,\kappa_1]\mathcal{H}_M^{\Omega_1}\kappa f\|_{H^{-1}}\le C\|f\|_{H_\partial^{0,l_1}}\end{aligned} \tag{4.159}$$

again by Lemma 2.24. The above give $[-\Delta,\chi_2]\varphi\mathcal{H}_M^{\Omega_1}f\in\dot{Y}_{m_1}^{l_1+1}(\Omega_1)$ as desired. One concludes (4.149) like in Step 2.

4. For (4.150), Lemma 4.13 implies $\varphi\mathcal{H}_M^{\Omega_2}\chi_1: H_\partial^{-N,-l_2}\to X_{-m_2}^{-l_2}(\Omega_2)$ is bounded, and thus its adjoint $-\chi_1\partial_\nu\mathcal{R}_M^{\Omega_2}\varphi:\dot{Y}_{m_2}^{l_2}(\Omega_2)\to H_\partial^{N,l_2}$ is bounded. Equation (4.151) follows similarly. □

Lemma 4.19 (M-dependency of error)**.** *Denote* $\mathfrak{r}=\min(\mathfrak{r}_{\Omega_1}^{r_0},\mathfrak{r}_{\Omega_2}^{r_0})$. *Let* $|l|<\mathfrak{r}+\frac12$, $|l'|<\mathfrak{r}+\frac12$ *with* $l'\ge l$. *In the setting of Lemma 4.18, we can estimate the error* $\tilde{\mathcal{E}}_M$ *by*

$$\|\chi_1\psi(D)\chi_1(\tilde{\mathcal{E}}_M-\tilde{\mathcal{E}}_0)\chi_1\psi(D)\chi_1f\|_{H^{N,N}(\partial\Omega_1)}\le CM^{\min(2\mathfrak{r}-0,2)}\|f\|_{H^{-N,-N}(\partial\Omega_1)} \tag{4.160}$$

$$\|\chi_1\psi(D)\chi_1(\tilde{\mathcal{E}}_M-\tilde{\mathcal{E}}_0)f\|_{H^{N,N}(\partial\Omega_1)}\le CM^{\min(2\mathfrak{r}-0,\mathfrak{r}+\frac12+l'-0,2)}\|f\|_{H^{0,l'}(\partial\Omega_1)} \tag{4.161}$$

$$\|\chi_1(\tilde{\mathcal{E}}_M-\tilde{\mathcal{E}}_0)\chi_1\psi(D)\chi_1f\|_{H^{N,l}(\partial\Omega_1)}\le CM^{\min(2\mathfrak{r}-0,\mathfrak{r}+\frac12-l-0,2)}\|f\|_{H^{-N,-N}(\partial\Omega_1)} \tag{4.162}$$

$$\|\chi_1(\tilde{\mathcal{E}}_M-\tilde{\mathcal{E}}_0)f\|_{H^{N,l}(\partial\Omega_1)}\le CM^{\min(2\mathfrak{r}-0,\mathfrak{r}+\frac12+\min(l',-l)-0,1+l'-l-0,2)}\|f\|_{H^{0,l'}(\partial\Omega_1)} \tag{4.163}$$

uniformly for all $f\in\mathcal{S}'(\partial\Omega_1)$ *and* $M\in[0,M_0]$.

Remark 4.20. If one sets $l = -\mathfrak{r} - \frac{1}{2} + 0$, $l' = \mathfrak{r} + \frac{1}{2} - 0$, all four M-terms are simplified to $M^{\min(2\mathfrak{r}-0,2)}$.

Proof. 0. In this proof, notations are inherited from Lemma 4.18 by making $l_1 = l_2 = l$, and all compatibility conditions there are satisfied. We have some $\epsilon > 0$ and some $m_1' > m_1 + \epsilon$, $m_2' > m_2 + \epsilon$ such that $m_2' \leq m_1'$ and $|m_1'| < \mathfrak{r}_{\Omega_1}^{r_0}$, $|m_1' - l'| < \frac{1}{2}$, $|m_2'| < \mathfrak{r}_{\Omega_2}^{r_0}$, $|m_2' - l'| < \frac{1}{2}$ as well. Note that given fixed l, l', one can always choose m_1, m_2, m_1', m_2' that $\epsilon = \min(2\mathfrak{r} - 0, \mathfrak{r} + \frac{1}{2} + \min(l', -l) - 0, 1 + l' - l - 0, 2)$: this is similar to Proposition 2.19. We confuse χ_1 with its extension $\chi_1 \in S^0(\Omega_1 \cap \Omega_2)$ be that $\partial_\nu \chi_1 \equiv 0$. Let $\tilde{\chi}_1 \in S^0(\Omega_1 \cap \Omega_2)$ that $\operatorname{supp}\chi_1 \cap \operatorname{supp}(1 - \tilde{\chi}_1) = \emptyset$, $\operatorname{supp}\tilde{\chi}_1 \cap \operatorname{supp}\varphi = \emptyset$. Decompose

$$\chi_1(\tilde{\mathcal{E}}_M - \tilde{\mathcal{E}}_0) = \chi_1 \partial_\nu \mathcal{R}_0^{\Omega_2}[-\Delta, \chi_2](\mathcal{H}_M^{\Omega_1} - \mathcal{H}_0^{\Omega_1}) + \chi_1 \partial_\nu (\mathcal{R}_M^{\Omega_2} - \mathcal{R}_0^{\Omega_2})[-\Delta, \chi_2]\mathcal{H}_M^{\Omega_1}. \tag{4.164}$$

We estimate those two terms separately to prove (4.163).

1. For the first term in (4.164), we claim

$$\|\chi_1 \partial_\nu \mathcal{R}_0^{\Omega_2}[-\Delta, \chi_2](\mathcal{H}_M^{\Omega_1} - \mathcal{H}_0^{\Omega_1}) f\|_{H_\partial^{N,l}} \leq C M^\epsilon \|f\|_{H_\partial^{0,l'}}. \tag{4.165}$$

Considering $\mathcal{H}_M^{\Omega_1} - \mathcal{H}_0^{\Omega_1} = -M^2 \mathcal{R}_0^{\Omega_1} \mathcal{H}_M^{\Omega_1}$, we can estimate

$$\begin{aligned}\|\chi_1 \partial_\nu \mathcal{R}_0^{\Omega_2}[-\Delta, \chi_2]\mathcal{R}_0^{\Omega_1}\mathcal{H}_M^{\Omega_1} f\|_{H_\partial^{N,l}} \leq C\|[-\Delta, \chi_2]\mathcal{R}_0^{\Omega_1}\mathcal{H}_M^{\Omega_1} f\|_{Y_{m_2}^l} \leq C\|\varphi \mathcal{R}_0^{\Omega_1}\mathcal{H}_M^{\Omega_1} f\|_{X_{m_1,-1}^{l,0}} \\ \leq C\|\mathcal{H}_M^{\Omega_1} f\|_{Y_{m_1,-1}^{l,0}},\end{aligned} \tag{4.166}$$

using $\chi_1 \partial_\nu \mathcal{R}_M^{\Omega_2} \varphi : \dot{Y}_{m_2}^l(\Omega_2) \to H_\partial^{N-\frac{1}{2},l}$ as in Step 4 in Lemma 4.18. Now Corollary 4.11 implies

$$M^{2-\epsilon}\|\mathcal{H}_M^{\Omega_1} f\|_{Y_{m_1,-1}^{l,0}} \leq C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\mathcal{H}_M^{\Omega_1} f\|_{Y_{m_1'-2,-1}^{l',0}} \leq C\|f\|_{H_\partial^{0,l'}}, \tag{4.167}$$

giving (4.165) as desired.

2. We address the second term in (4.164). We claim

$$\|\chi_1 \partial_\nu (\mathcal{R}_M^{\Omega_2} - \mathcal{R}_0^{\Omega_2})[-\Delta, \chi_2]\mathcal{H}_M^{\Omega_1} f\|_{H_\partial^{N,l}} \leq C M^\epsilon \|f\|_{H_\partial^{0,l'}}. \tag{4.168}$$

Denote by $v = [-\Delta, \chi_2]\mathcal{H}_M^{\Omega_1} f$. Consider

$$\chi_1 \partial_\nu (\mathcal{R}_M^{\Omega_2} - \mathcal{R}_0^{\Omega_2})\varphi = -M^2 \chi_1 \partial_\nu \mathcal{R}_0^{\Omega_2}\mathcal{R}_M^{\Omega_2}\varphi = -M^2 \chi_1 \partial_\nu \mathcal{R}_0^{\Omega_2}(1 - \tilde{\chi}_1)\mathcal{R}_M^{\Omega_2}\varphi - M^2 \chi_1 \partial_\nu \mathcal{R}_0^{\Omega_2}\tilde{\chi}_1 \mathcal{R}_M^{\Omega_2}\varphi, \tag{4.169}$$

Corollary 4.9 implies $\chi_1 \partial_\nu \mathcal{R}_0^{\Omega_2}(1 - \tilde{\chi}_1) : Y_{m_2,-N}^{l,0}(\Omega_2) \to H^{N-\frac{1}{2},l}(\partial\Omega_1)$ since χ_1 and $1 - \tilde{\chi}_1$ have disjoint support. Then the first term of (4.169) can be estimated by

$$\begin{aligned}M^{2-\epsilon}\|\chi_1 \partial_\nu \mathcal{R}_0^{\Omega_2}(1 - \tilde{\chi}_1)\mathcal{R}_M^{\Omega_2}\varphi v\|_{H_\partial^{N-\frac{1}{2},l}} \leq C M^{2-\epsilon}\|\mathcal{R}_M^{\Omega_2}\varphi v\|_{Y_{m_2}^l} \leq C M^{2-\epsilon}\|\mathcal{R}_M^{\Omega_2}\varphi v\|_{X_{m_2+2}^l} \\ \leq C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\mathcal{R}_M^{\Omega_2}\varphi v\|_{X_{m_2'}^{l'}} \leq C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\varphi v\|_{Y_{m_2'}^{l'}} \leq C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\varphi v\|_{Y_{m_1'}^{l'}}.\end{aligned} \tag{4.170}$$

Noting $\chi_1\partial_\nu\mathcal{R}_0^{\Omega_2}: Y^{l,0}_{m_2,N}(\Omega_2)\to H^{N-\frac12,l}(\partial\Omega_1)$, we can estimate the other term by

$$\begin{aligned}M^{2-\epsilon}\|\chi_1\partial_\nu\mathcal{R}_0^{\Omega_2}\tilde\chi_1\mathcal{R}_M^{\Omega_2}\varphi v\|_{H^{N-\frac12,l}_\partial}&\le CM^{2-\epsilon}\|\tilde\chi_1\mathcal{R}_M^{\Omega_2}\varphi v\|_{Y^{l,0}_{m_2,N}}\le CM^{2-\epsilon}\|\tilde\chi_1\mathcal{R}_M^{\Omega_2}\varphi v\|_{X^{l,0}_{m_2+2,N}}\\&\le C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\tilde\chi_1\mathcal{R}_M^{\Omega_2}\varphi v\|_{X^{l',0}_{m_2',N}}\le C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\varphi v\|_{Y^{l'}_{m_2'}}\le C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\varphi v\|_{Y^{l'}_{m_1'}},\end{aligned}\tag{4.171}$$

since $\tilde\chi_1$ and φ are disjointly supported, again using Corollary 4.9. Let $\kappa\in C_c^\infty(\overline{\Omega_1})$ be that $\operatorname{supp}\kappa\cap\operatorname{supp}\varphi=\emptyset$, $\operatorname{supp}(1-\kappa)\cap\partial\Omega_{\text{sing}}=\emptyset$. Now Step 1 of Corollary 4.11 and Lemma 2.24 imply

$$\begin{aligned}\|\langle M\mathbf{d}\rangle^{2-\epsilon}\varphi v\|_{Y^{l'}_{m_1'}}&\le C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\varphi\mathcal{H}_M^{\Omega_1}f\|_{X^{l',0}_{m_1',-1}}\le C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\varphi\mathcal{H}_M^{\Omega_1}(1-\kappa)f\|_{X^{l',0}_{m_1',-1}}\\&+C\|\langle M\mathbf{d}\rangle^{2-\epsilon}\varphi\mathcal{H}_M^{\Omega_1}\kappa f\|_{X^{l',0}_{m_1',-1}}\le C\|(1-\kappa)f\|_{H^{0,l'}_\partial}+C\|\kappa f\|_{H^{0,l'}_\partial}\le C\|f\|_{H^{0,l'}_\partial},\end{aligned}\tag{4.172}$$

giving (4.168) as desired. This concludes the proof of (4.163).

3. For the other three claims, it suffices to pay attention to (4.165) and (4.172). Use (4.119) and (4.117) to observe the claims analogously. □

Lemma 4.21. *Let $\chi_1,\chi_2\in C_c^\infty(\partial\Omega)$ that $\operatorname{supp}\chi_1\cap\operatorname{supp}(1-\chi_2)=\emptyset$. Then we have*

$$\|\chi_2(\Lambda_M^\Omega-\Lambda_0^\Omega)\chi_1 f\|_{L^2(\partial\Omega)}\le CM^{\min(2\mathfrak{r}_\Omega^{r_0}-0,2)}\|\chi_1 f\|_{L^2(\partial\Omega)},\tag{4.173}$$

uniformly for all $f\in\mathcal{S}'(\partial\Omega)$ and $M\in[0,M_0]$.

Proof. We confuse χ_1,χ_2 with their extension in $C_c^\infty(\overline{\Omega})$ that $\operatorname{supp}\chi_1\cap\operatorname{supp}(1-\chi_2)=\emptyset$. Let $\tilde\chi_2\in C_c^\infty(\overline{\Omega})$ that $\operatorname{supp}\chi_2\cap\operatorname{supp}(1-\tilde\chi_2)=\emptyset$ and $\partial_\nu\tilde\chi_2=0$. Let $\tilde\Omega$ be a bounded uniformly Lipschitz domain such that $\tilde\Omega,\Omega$ share the common boundary near $\operatorname{supp}\tilde\chi_2$. Lemma 4.17 implies

$$\chi_2(\Lambda_M^\Omega-\Lambda_0^\Omega)\chi_1=\chi_2(\Lambda_M^{\tilde\Omega}-\Lambda_0^{\tilde\Omega})\chi_1+\chi_2(\tilde{\mathcal{E}}_M-\tilde{\mathcal{E}}_0)\chi_1,\tag{4.174}$$

$$\tilde{\mathcal{E}}_M-\tilde{\mathcal{E}}_0=\partial_\nu\mathcal{R}_M^{\tilde\Omega}[-\Delta,\tilde\chi_2]\mathcal{H}_M^\Omega-\partial_\nu\mathcal{R}_0^{\tilde\Omega}[-\Delta,\tilde\chi_2]\mathcal{H}_0^\Omega.\tag{4.175}$$

Note $\Lambda_M^{\tilde\Omega}-\Lambda_0^{\tilde\Omega}=-M^2\partial_\nu\mathcal{R}_0^{\tilde\Omega}\mathcal{H}_M^{\tilde\Omega}$ and we can estimate the first term of (4.174) by

$$\|\chi_2(\Lambda_M^{\tilde\Omega}-\Lambda_0^{\tilde\Omega})\chi_1 f\|_{L^2_\partial}\le CM^2\|\mathcal{H}_M^{\tilde\Omega}\chi_1 f\|_{H^{\frac12}}\le CM^2\|\chi_1 f\|_{L^2_\partial}\tag{4.176}$$

since $\partial_\nu\mathcal{R}_0^{\tilde\Omega}:L^2\to L^2_\partial$ and $\mathcal{H}_M^{\tilde\Omega}:L^2_\partial\to H^{\frac12}$ are bounded per Lemma 2.5. On the other hand,

$$\chi_2(\tilde{\mathcal{E}}_M-\tilde{\mathcal{E}}_0)\chi_1=\chi_2\partial_\nu(\mathcal{R}_M^{\tilde\Omega}-\mathcal{R}_0^{\tilde\Omega})[-\Delta,\tilde\chi_2]\mathcal{H}_M^\Omega\chi_1+\chi_2\partial_\nu\mathcal{R}_0^{\tilde\Omega}[-\Delta,\tilde\chi_2](\mathcal{H}_M^\Omega-\mathcal{H}_0^\Omega)\chi_1,\tag{4.177}$$

whose first term can be estimated by $\mathcal{R}_M^{\tilde\Omega}-\mathcal{R}_0^{\tilde\Omega}=-M^2\mathcal{R}_0^{\tilde\Omega}\mathcal{R}_M^{\tilde\Omega}$ and

$$\begin{aligned}\|\chi_2\partial_\nu(\mathcal{R}_M^{\tilde\Omega}-\mathcal{R}_0^{\tilde\Omega})[-\Delta,\tilde\chi_2]\mathcal{H}_M^\Omega\chi_1 f\|_{L^2_\partial}&\le CM^2\|\mathcal{R}_M^{\tilde\Omega}[-\Delta,\tilde\chi_2]\mathcal{H}_M^\Omega\chi_1 f\|_{L^2}\\&\le CM^2\|[-\Delta,\tilde\chi_2]\mathcal{H}_M^\Omega\chi_1 f\|_{H^{-\frac32}}\le CM^2\|\chi_1 f\|_{L^2_\partial}\end{aligned}\tag{4.178}$$

from Lemma 2.24. Let $m = \min(\mathfrak{r}_\Omega^{r_0} - 0, 1)$. The second term of (4.177) can be estimated by

$$\|\chi_2 \partial_\nu \mathcal{R}_0^{\tilde{\Omega}}[-\Delta, \tilde{\chi}_2](\mathcal{H}_M^\Omega - \mathcal{H}_0^\Omega)\chi_1 f\|_{L^2_\partial} \leq CM^2 \|[-\Delta, \tilde{\chi}_2]\mathcal{R}_0^\Omega \mathcal{H}_M^\Omega \chi_1 f\|_{L^2} \leq CM^2 \|\mathcal{H}_M^\Omega \chi_1 f\|_{Y^{0,0}_{-m,-1}}$$
$$\leq CM^{2m} \|\langle M\mathbf{d}\rangle^{2-2m} \mathcal{H}_M^\Omega \chi_1 f\|_{Y^{0,0}_{m-2,-1}} \leq CM^{2m} \|\chi_1 f\|_{L^2_\partial} \quad (4.179)$$

via Corollary 4.11. This concludes the proof. □

We now record our main description of the DtN map on AC Lipschitz domains in $\mathbb{R}^d$:

Proposition 4.22. *Let $\chi, \tilde{\chi} \in S^0(\partial\Omega)$ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$, $\operatorname{supp}\tilde{\chi} \subset \partial\Omega_{\mathrm{reg}}$. Then we have*

$$\chi\Lambda_M^\Omega = \chi P\tilde{\chi} + \chi\mathcal{E}_M, \ \Lambda_M^\Omega\chi = \tilde{\chi}P\chi + \mathcal{E}'_M\chi, \quad (4.180)$$

where $\chi P\tilde{\chi}, \tilde{\chi}P\chi \in \Psi^{0,-\delta}_{1,\delta}(\partial\Omega)$ are the left/right microlocal description of Λ_M^Ω near $\operatorname{supp}\chi$ with

$$\sigma(P) = h^{-1}\langle x\rangle^{-\delta}\big\langle h\langle x\rangle^\delta \sqrt{|\xi|_g^2 + M^2}\big\rangle \quad (4.181)$$

given in Lemma 3.2. We confuse $\chi, \tilde{\chi}$ with extensions specified in the proof, and the errors $\mathcal{E}_M, \mathcal{E}'_M$ are given by

$$\chi\mathcal{E}_M = \chi\mathcal{E}_\flat\tilde{\chi} + \chi\tilde{\mathcal{E}}_M = \chi(\Lambda_M^\Omega - P)\tilde{\chi} + \chi\partial_\nu\mathcal{R}_M^\Omega[-\Delta, \tilde{\chi}]\mathcal{H}_M^\Omega, \quad (4.182)$$
$$\mathcal{E}'_M\chi = \tilde{\chi}\mathcal{E}_\flat\chi + \tilde{\mathcal{E}}^*_M\chi = \tilde{\chi}(\Lambda_M^\Omega - P)\chi - \partial_\nu\mathcal{R}_M^\Omega[-\Delta, \tilde{\chi}]\mathcal{H}_M^\Omega\chi. \quad (4.183)$$

Moreover, let $\psi(\xi) \in S^0(\mathbb{R}^{d-1})$ with $0 \notin \operatorname{supp}\psi$, $N \geq 0$, $|l| < \mathfrak{r}_\Omega^{r_0} + \frac{1}{2}$. We can estimate the error $\mathcal{E}_M$ by

$$\|\chi\psi(D)\chi\mathcal{E}_M\chi\psi(D)\chi f\|_{H^{N,N}(\partial\Omega)} \leq C\|f\|_{H^{-N,-N}(\partial\Omega)}, \quad (4.184)$$
$$\|\chi\psi(D)\chi\mathcal{E}_M f\|_{H^{N,N}(\partial\Omega)} \leq C\|f\|_{H^{0,l}(\partial\Omega)}, \quad (4.185)$$
$$\|\chi\mathcal{E}_M\chi\psi(D)\chi f\|_{H^{N,l}(\partial\Omega)} \leq C\|f\|_{H^{-N,-N}(\partial\Omega)}, \quad (4.186)$$
$$\|\chi\mathcal{E}_M f\|_{H^{N,l}(\partial\Omega)} \leq C\|f\|_{H^{0,\max(l-1,-\mathfrak{r}_\Omega^{r_0}-\frac{1}{2})+0}(\partial\Omega)}, \quad (4.187)$$

uniformly for all $f \in \mathcal{S}'(\partial\Omega)$ and $M \in [0, M_0]$. Same estimates hold for $\mathcal{E}'_M$ instead of $\mathcal{E}_M$. All of $P, \mathcal{E}_M, \mathcal{E}'_M$ depend on M continuously.

Proof. 1. Let U be a regular collar of Ω as in Remark 3.11 that $\operatorname{supp}\tilde{\chi} \subset U$. We confuse $\tilde{\chi}, \chi$ with their extensions in $S^0(U)$ that $\partial_\nu\tilde{\chi} = \partial_\nu\chi = 0$ and $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$. We address the case $\chi\mathcal{E}_M$ by proving estimates (4.184) to (4.187) with $\chi\mathcal{E}_M$ replaced by $\chi\tilde{\mathcal{E}}_M$ and $\chi\mathcal{E}_\flat\tilde{\chi}$ respectively. The case $\mathcal{E}'_M\chi$ then follows similarly.

2. We will use Lemma 4.18 for $\chi\tilde{\mathcal{E}}_M$. The compatibility condition reduces to $l_2 < l_1 + 1$. Estimates (4.184), (4.185) and (4.186) follow immediately. For (4.187), we take

$$l_2 = l, \ l_1 = \max(l-1, -\mathfrak{r}_\Omega^{r_0} - \frac{1}{2}) + 0, \quad (4.188)$$

which satisfies the compatibility condition and we have the claim.

3. Proposition 3.12 implies for any $u \in \mathcal{S}'(\partial\Omega)$,

$$\|\tilde{\chi}\Gamma^{N,l}_{N,\delta}\chi\mathcal{E}_\flat\tilde{\chi}u\|_{L^2(\partial\Omega)} \leq C\|\Gamma^{-N,\max(l+\frac{1}{2},1-\mathfrak{r}_\Omega^{r_0})-\frac{3}{2}\delta+0}_{-N,\delta}u\|_{L^2(\partial\Omega)}, \quad (4.189)$$

where choosing $\delta < 1$ close to 1 makes

$$\max(l + \frac{1}{2}, 1 - \mathfrak{r}_\Omega^{r_0}) - \frac{3}{2}\delta + 0 \leq \max(l - 1, -\mathfrak{r}_\Omega^{r_0} - \frac{1}{2}) + 0. \tag{4.190}$$

The estimate above then implies (4.184) to (4.187) with $\chi\mathcal{E}_M$ replaced with $\chi\mathcal{E}_\flat\tilde{\chi}$. This concludes the proof. □

Remark 4.23. Here the microlocal error $\mathcal{E}_\flat$ measures the nonlocality at the zero frequency of Λ_M^Ω due to its symbolic singularity in $\sqrt{|\xi|^2 + M^2}$, and the nonlocal error $\tilde{\mathcal{E}}_M$ measures the nonlocality at the zero frequency of Λ_M^Ω due to the Lipschitz singularity in $x \in \partial\Omega$.

Remark 4.24. All constants in this section can depend on Ω in the following way: the upper bound on uniform Lipschitz characters of $\partial\Omega$, the upper bound on the choice of $r_0 \geq 0$, and the lower bound on the radial Hardy constant $\mathfrak{r}_\Omega^{r_0}$, the upper bound on the diameter of $\partial\Omega \setminus \partial\tilde{\Omega}$ in Remark 3.11, for each multi-index α, the upper bounds on C_α of derivatives of the metric g_+ in (3.11), and lower bounds on $|\det g_+|$.

5. Microlocal analysis of the DtN map on AC Lipschitz domains

The purpose of this section is to prove a limiting absorption principle for the DtN map on AC Lipschitz domains. In fact, following the now standard ideas of Vasy and Melrose [Mel94, VZ00, Vas21] we show that the limiting absorption resolvent as an actual inverse on certain variable order Sobolev spaces.

Outline of the strategy

This section relies heavily on the microlocal description of the DtN map from Section 3. With that model available the proof of the limiting absorption principle follows steps similar to those used when proving the limiting absorption principle for the Laplacian on an asymptotically conic manifold.

(1) Prove elliptic estimates near $|x| = \infty$ and away from $|\xi|_g^2 + M^2 = \omega^2$.
In this step we use the standard elliptic parametrix construction away from $\xi = 0$. However, $\xi = 0$ requires special treatment since Λ_M^Ω is *not* pseudodifferential there. Instead, we show in Lemma 5.2 that the difference between Λ_M^Ω and a pseudodifferential operator is sufficiently well behaved to be absorbed into the error.

(2) Elliptic estimates on $|x| < \infty$, $|\xi| = \infty$.
Once again, away from the Lipschitz part of $\partial\Omega$, we are able to use standard elliptic parametrices. Near the Lipschitz part, we rely on the elliptic theory for the Laplacian as in Lemma 2.24.

(3) Propagation estimates on $\{|\xi|_g^2 + M^2 = \omega^2\} \cap \{|x| = \infty\}$.
With the microlocal description of the DtN map from Section 3, we can apply the standard real principal type propagation estimates and radial point estimates (see e.g. [DZ19b, Appendix E]). It is in this step where, as usual, we are required to use variable order Sobolev spaces.

(4) Absence of non-L^2 outgoing states.
Finally, we need to show that if u is purely outgoing an an appropriate microlocal sense and satisfies $(\Lambda_M^\Omega - \omega)u = 0$ for some $\omega > M$, then u is in L^2. Together with the

propagation estimates above, this will allow us to show that if ω is not an eigenvalue for the DtN map, then i) $(\Lambda_M^\Omega - \omega)$ is invertible on some appropriately defined spaces and ii) this inverse is also the limiting absorption resolvent.

5.1. **Geometric and microlocal preliminaries.** In this section, we work on an AC Lipschitz domain Ω, with its boundary $\partial\Omega$ being a scattering manifold.

We compactify $\partial\Omega$ and $T^*\partial\Omega$ in the picture of radial compactification in both x and ξ. We abuse the notation and may freely use the projective coordinates (ρ, y, τ, η) on $T^*\partial\Omega$, with trivialization $\tau\rho^{-2}d\rho + \eta\rho^{-1}dy$, or the local normal coordinates (x, ξ) on $T^*\partial\Omega$ induced from Fermi normal coordinates (x_0, x) locally near $\partial\Omega$ equipped with induced metric g, by restricting to $T^*\partial\Omega_{\mathrm{reg}}$ and $\partial\Omega_{\mathrm{reg}}$ respectively if necessary. One has the canonical transformation

$$\xi \cdot dx = \tau\rho^{-2}d\rho + \rho^{-1}\eta \cdot dy. \tag{5.1}$$

Note $\partial T^*\partial\Omega_{\mathrm{reg}} = \partial T^*\partial\Omega$. We say $a \in S^{s,l}_{k,\delta}(T^*\partial\Omega)$, if $a \in C^\infty(T^*\partial\Omega_{\mathrm{reg}})$ and for each $K \subset \partial\Omega_{\mathrm{reg}}$ closed, and for each multi-indices α, β, there is $C_{\alpha\beta K} > 0$ that

$$\left|(\langle x\rangle\partial_x)^\alpha(\langle x\rangle^{-\delta}\langle\langle x\rangle^\delta\sqrt{|\xi|_g^2 + M^2}\rangle\partial_\xi)^\beta a(x,\xi)\right| \le C_{\alpha\beta K}\langle\xi\rangle^s\langle x\rangle^l\langle\langle x\rangle^\delta\sqrt{|\xi|_g^2 + M^2}\rangle^k, \tag{5.2}$$

uniformly for all $(x, \xi) \in T^*K$.

Consider a finite family of charts covering (Φ_j, U_j) over $\partial\Omega$, associated with a partition of unity $\phi_j \in S^0(\partial\Omega)$ and $\operatorname{supp}\phi_j \subset U_j$ that $\sum_j \phi_j = 1$. For those $U_j \cap \partial\Omega_{\mathrm{sing}} = \emptyset$, let $\varphi_{j0} = \phi_j$ and $\varphi_{jk} = 0$ for all $k \ge 1$. For those $U_j \cap \partial\Omega_{\mathrm{sing}} \neq \emptyset$ consider a dyadic partition of unity $\varphi_{jk} \in S^0(\partial\Omega)$ that is locally finite on $\partial\Omega_{\mathrm{reg}}$, $\operatorname{supp}\varphi_{j0} \subset \{x : d(x, \partial\Omega_{\mathrm{sing}}) > \frac{1}{2}\}$,

$$\operatorname{supp}\varphi_{jk} \subset \{x : 2^{-k-1} < d(x, \partial\Omega_{\mathrm{sing}}) < 2^{-k+1}\} \text{ for } k \ge 1,\ \sum_{k=0}^\infty \varphi_{jk} = \phi_j \mathbb{1}_{U_j\cap\partial\Omega_{\mathrm{reg}}}. \tag{5.3}$$

Let $\tilde\varphi_{jk} \in S^0(\partial\Omega)$ with $\operatorname{supp}(1 - \tilde\varphi_{jk}) \cap \operatorname{supp}\varphi_{jk} = \emptyset$ and $d(\operatorname{supp}\tilde\varphi_{j0}, \partial\Omega_{\mathrm{sing}}) > \frac{1}{2}$ and $\operatorname{supp}\tilde\varphi_{jk} \subset \{x : 2^{-k-1} < d(x, \partial\Omega_{\mathrm{sing}}) < 2^{-k+1}\}$ for $k \ge 1$. Fix a localizer in distance $\psi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \subset [-\frac{1}{4}, \frac{1}{4}]$ and $\psi \equiv 1$ on $[-\frac{1}{8}, \frac{1}{8}]$. We define the individual quantization map on each dyadic piece by

$$\mathrm{Op}_{jk}(a) = \frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^{d-1}_\xi} e^{i\langle x-y,\xi\rangle}a(x,\xi)\psi(2^k|x-y|)\,d\xi \tag{5.4}$$

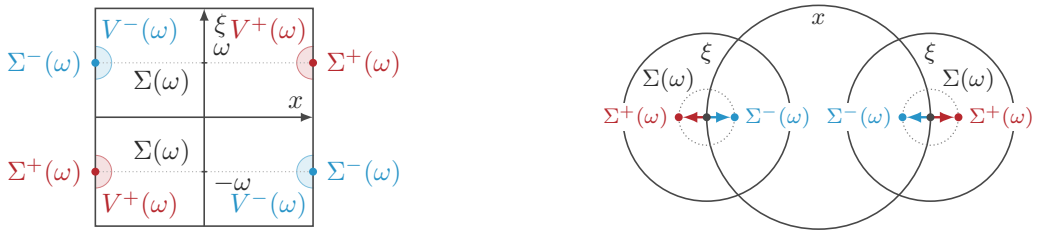
and thus the global quantization map

$$\mathrm{Op}(a) = \sum_j\sum_{k=0}^\infty \tilde\varphi_{jk}\Phi_j^*\,\mathrm{Op}_{jk}(\Phi_j^{-*}(\varphi_{jk}a))\Phi_j^{-*}\tilde\varphi_{jk}. \tag{5.5}$$

We say $A \in \Psi^{s,l}_{k,\delta}(\partial\Omega)$, if

(1) $A : C_c^\infty(\partial\Omega) \to L^2(\partial\Omega)$.
(2) There is $\chi \in S^0(\partial\Omega)$ with $\operatorname{supp}\chi \subset \partial\Omega_{\mathrm{reg}}$ and $a \in S^{s,l}_{k,\delta}(T^*\partial\Omega)$ that

$$A = \chi A\chi = \mathrm{Op}(a) + \chi O(\Psi^{-\infty,-\infty}(\partial\Omega))\chi. \tag{5.6}$$



(A) $d = 2$ (B) $d = 3$

FIGURE 8. Sink/source structures.

We say $A \in \Psi^{s,l}(\partial\Omega)$ if the above condition is met with $a \in S^{s,l}(T^*\partial\Omega)$. We define the wavefront set and elliptic set by

$$\mathrm{WF}\, A = \bigcup \{(x,\xi) \in \mathrm{WF}(\chi A \chi) : \chi \in S^0(\partial\Omega), \operatorname{supp}\chi \subset \partial\Omega_{\mathrm{reg}}\}, \tag{5.7}$$

$$\mathrm{Ell}\, A = \bigcup \{(x,\xi) \in \mathrm{Ell}(\chi A \chi) : \chi \in S^0(\partial\Omega), \operatorname{supp}\chi \subset \partial\Omega_{\mathrm{reg}}\}. \tag{5.8}$$

Let g be the induced metric of $\partial\Omega_{\mathrm{reg}} \subset \mathbb{R}^d$ and denote the principal symbol of the approximation P to Λ_M^Ω by

$$p(x,\xi;M) = \langle x\rangle^{-\delta}\langle\langle x\rangle^{\delta}\sqrt{|\xi|_g^2 + M^2}\rangle \in S^{0,-\delta}_{1,\delta}(\partial\Omega), \tag{5.9}$$

where the small parameter $h > 0$ was fixed small in Remark 3.5, and $\delta \in [0,1)$. In this section $\omega > 0$. Denote the characteristics of the approximation P to $\Lambda_M^\Omega - \sqrt{\omega^2 + M^2}$ by the set of dimension $2d-4$,

$$\Sigma(\omega) = \{(x,\xi) \in \partial T^*\partial\Omega : p(x,\xi;M) = \sqrt{\omega^2 + M^2}\}. \tag{5.10}$$

Near $\Sigma(\omega)$, we have $g(\rho, y) = g_\infty(y) + O(\rho S^0(\partial W_Y))$ and

$$p(\rho, y, \tau, \eta) = \sqrt{\tau^2 + |\eta|^2_{g_\infty} + M^2} + O(\rho S^{1,0}) + O(\rho^{2\delta} S^{-1,0}) \tag{5.11}$$

Without loss of generality assuming $\delta \geq \frac{1}{2}$ and noting the frequency is bounded near $\Sigma(\omega)$, we can ignore the two error terms above as subprincipal. Let $\hat{H}_p = \rho^{-1} H_p$ be the rescaled Hamiltonian vector field associated to p on $T^*\partial\Omega$, given by

$$\hat{H}_p = (\tau^2 + |\eta|^2_{g_\infty} + M^2)^{-\frac{1}{2}}\Big(\tau\rho\partial_\rho + \eta^t \cdot g_\infty \cdot \partial_y - |\eta|^2_{g_\infty}\partial_\tau + \sum_{j=1}^{d-2}(\tau\eta_j - \frac{1}{2}(\eta^t \cdot (\partial_{y_j} g_\infty) \cdot \eta)\partial_{\eta_j}\Big), \tag{5.12}$$

near $\Sigma(\omega)$. Let $e^{t\hat{H}_p}$ be the flow associated to $\hat{H}_p$ on $T^*\partial\Omega$. We let the set of dimension $d-2$,

$$\Sigma^\pm(\omega) = \{(\rho, y, \tau, \eta) \in \Sigma(\omega) : \rho = 0, \tau = \mp\omega, \eta = 0\} \tag{5.13}$$

as radial sink/source set of the approximation P to $\Lambda_M^\Omega - \sqrt{\omega^2 + M^2}$. See Figure 8.

The sink/source nature follows from the lemma below.

Lemma 5.1. *The radial set $\Sigma^-(\omega) \subset T^*\partial\Omega$ consists of sources, and $\Sigma^+(\omega)$ consists of sinks.*

Proof. It suffices to show the claim on an AC smooth Ω_e that $\Omega_e \cap \{\rho \leq \epsilon\} = \Omega \cap \{\rho \leq \epsilon\}$ for some $\epsilon > 0$ small. Note $p|_{\rho=0} = \sqrt{|\xi|_g^2 + M^2}$ and $2pH_p|_{\rho=0} = H_{-\Delta}|_{\rho=0}$. This implies the sink and source of P are the same of $-\Delta$ on $\partial\Omega_e$ and the claim follows from [MZ96, Lemma 2]. □

5.2. **Microlocal estimates through zeros, sinks and sources.** We start by showing an elliptic estimate for the operator $\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu$, on $\{|\xi|_g \neq \omega\}$. Let $\mathcal{I}_\omega \subseteq (0, \infty)$ be a compact interval. We define the decay threshold at the zero frequency by

$$L_0 = \mathfrak{r}_\Omega^{r_0} + \frac{1}{2}. \tag{5.14}$$

Fix $\kappa \in C_c^\infty(\overline{\Omega})$ that $\operatorname{supp}(1-\kappa) \cap \partial\Omega_{\text{sing}} = \emptyset$. Fix $\kappa_0, \tilde{\kappa}$ that satisfy the same requirement for κ with $\kappa_0 \prec \kappa \prec \tilde{\kappa}$.

Lemma 5.2 (Elliptic estimate at zero frequency)**.** *Let $\Omega \subset \mathbb{R}^d$ be an AC Lipschitz domain. Let $|l_0| < L_0$. Let $A_0, B_0 \in \Psi^{0,0}(\partial\Omega)$, $\operatorname{WF} A_0 \cap \Sigma(\omega) = \emptyset$, $\operatorname{WF} A_0 \subset \operatorname{Ell} B_0$, and*

$$\{(x,\xi) : x \in \operatorname{supp}(1-\kappa), \xi = 0\} \subset \operatorname{Ell} B_0, \ \operatorname{WF}(B_0) \subset \{(x,\xi) : |\xi| \leq \epsilon_0\} \tag{5.15}$$

for some $\epsilon_0 > 0$. Then for any $N > 0$ we have

$$\|A_0 u\|_{H^{N,l_0}} \leq C\|B_0(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu)u\|_{H^{-N,l_0}} + C\|\tilde{\kappa}u\|_{H^0} + C\|u\|_{H^{-N,-N}} \tag{5.16}$$

uniformly for all $u \in \mathcal{S}'$ with $\operatorname{WF}^{-\infty,-L_0+0}(u) \cap \{\xi = 0\} = \emptyset$, $\omega \in \mathcal{I}_\omega$ and $\mu \geq 0$ small, $M \in [0, M_0]$.

Proof. 1. We claim the lemma holds for some A_0' in place of A_0, that is,

$$\|A_0' u\|_{H^{N,l_0}} \leq C\|B_0(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu)u\|_{H^{-N,l_0}} + C\|\tilde{\kappa}u\|_{H^0} + C\|u\|_{H^{-N,-N}}, \tag{5.17}$$

where $A_0' \in \Psi^{0,0}(\partial\Omega)$ with $\operatorname{WF} A_0 \subset \operatorname{Ell} A_0'$, $\Pi_x(\operatorname{WF}(\mathrm{I} - A_0') \cap \{\xi = 0\}) \subset \operatorname{supp}\tilde{\kappa}$, $\operatorname{WF} A_0' \subset \operatorname{Ell} B_0$. Then the standard elliptic estimate implies the lemma. To show the claim, note since $B_0 = \chi B_0 \chi$ for some $\chi \in S^0$ whose support touches finitely many connected components of $\partial\Omega$, we can approximate Λ_M^Ω by $P \in \Psi_{1,\delta}^{0,-\delta}$ in Proposition 4.22. Note $A_0', B_0 \in \Psi_{0,\delta}^{-\infty,0}$ per the compact support in frequencies. We also have $B_0\chi(P - \sqrt{\omega^2 + M^2} - i\mu)\tilde{\chi} \in \Psi_{0,\delta}^{-\infty,0}$ is elliptic on $\operatorname{WF} A_0'$ for some $\tilde{\chi}$ satisfying all conditions for χ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$. By Lemma A.13 there is $E \in \Psi_{0,\delta}^{-\infty,0}$ such that

$$\Gamma_{k,\delta}^{s,l_0} A_0' = E\Gamma_{k,\delta}^{s,l_0} B_0 \chi(P - \sqrt{\omega^2 + M^2} - i\mu)\tilde{\chi} + O(\Psi^{-\infty,-\infty}). \tag{5.18}$$

Hence we have for any $s \in \mathbb{R}$,

$$\begin{aligned}\|\Gamma_{k,\delta}^{s,l_0} A_0' u\| &\leq C\|\Gamma_{k,\delta}^{s,l_0} B_0 \chi(P - \sqrt{\omega^2 + M^2} - i\mu)\tilde{\chi}u\| + C\|\Gamma_{-N,\delta}^{-N,-N} u\| \\ &\leq C\|\Gamma_{k,\delta}^{s,l_0} B_0 \chi(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu)u\| + C\|\Gamma_{k,\delta}^{s,l_0} B_0 \chi \mathcal{E}_M u\| + C\|\Gamma_{-N,\delta}^{-N,-N} u\|, \end{aligned} \tag{5.19}$$

where $\mathcal{E}_M$ is from Proposition 4.22. Taking $k = 0$ gives on $\partial\Omega$,

$$\|A_0' u\|_{H^{s,l_0}} \leq C\|B_0(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu)u\|_{H^{s,l_0}} + C\|\chi\mathcal{E}_M u\|_{H^{s,l_0}} + C\|u\|_{H^{-N,-N}}. \tag{5.20}$$

Let $\psi_0(\xi) \in C_c^\infty(\mathbb{R}^{d-1})$ be that $\operatorname{supp}(1-\psi_0) \cap \{\xi = 0\} = \emptyset$ and $\operatorname{supp}((1-\tilde\kappa(x))\psi_0(\xi)) \subset \operatorname{Ell} A_0'$. With the decomposition $\mathrm{I} = (1-\tilde\kappa)\psi_0(D)(1-\kappa) + (1-\tilde\kappa)(\mathrm{I}-\psi_0(D))(1-\kappa) + \tilde\kappa$, we have from Proposition 4.22 and the standard elliptic estimate that

$$\begin{aligned}\|\chi\mathcal{E}_M u\|_{H^{s,l_0}} &\le C\|(1-\tilde\kappa)\psi_0(D)(1-\kappa)u\|_{H^{-N,\max(l_0-1+0,-L_0+0)}} + C\|u\|_{H^{-N,-N}} + C\|\tilde\kappa u\|_{H^0}\\ &\le C\|A_0' u\|_{H^{-N,\max(l_0-1+0,-L_0+0)}} + C\|u\|_{H^{-N,-N}} + C\|\tilde\kappa u\|_{H^0}. \quad (5.21)\end{aligned}$$

2. It remains to reduce the $\|A_0'u\|_{H^{-N,\max(l_0-1+0,-L_0+0)}}$ term in (5.21). When $l_0 < -L_0+1$, one can interpolate

$$\|A_0'u\|_{H^{-N,\max(l_0-1+0,-L_0+0)}} \le C\epsilon^{-1}\|u\|_{H^{-N,-N}} + \epsilon\|A_0'u\|_{H^{s,-L_0+0}} \quad (5.22)$$

for small $\epsilon > 0$ to obtain (5.17) via absorption. For $-L_0+1 \le l_0 < -L_0+2$ with $l_0 < L_0$, interpolation implies

$$\begin{aligned}\|A_0'u\|_{H^{-N,\max(l_0-1+0,-L_0+0)}} &\le C\epsilon^{-1}\|u\|_{H^{-N,-N}} + \epsilon\|A_0'u\|_{H^{s,l_0-1+0}}\\ &\le C\|B_0(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{H^{-N,l_0-1+0}} + C\|\tilde\kappa u\|_{H^0} + C\|u\|_{H^{-N,-N}} \quad (5.23)\end{aligned}$$

where we used (5.17) for $l_0 - 1 < -L_0+1$. Iterate this procedure to obtain (5.17) for all $|l_0| < L_0$. □

Lemma 5.3 (Conormal regularity at zero frequency)**.** *Let $|l_0| < L_0$. Let $A_0, B_0 \in \Psi^{0,0}(\partial\Omega)$, $\operatorname{WF} A_0 \cap \Sigma(\omega) = \emptyset$, $\operatorname{WF} A_0 \subset \operatorname{Ell} B_0$, and*

$$\{(x,\xi) : x \in \operatorname{supp}(1-\kappa), \xi = 0\} \subset \operatorname{Ell} B_0,\ \operatorname{WF}(B_0) \subset \{(x,\xi) : |\xi| \le \epsilon_0\} \quad (5.24)$$

for some $\epsilon_0 > 0$. Let $k \in \mathbb{N}_0$. Let $\mathcal{V}_1, \dots, \mathcal{V}_k \in {}^\bullet\mathrm{Diff}_b^{1,1}(\partial\Omega)$ with support disjoint from $\operatorname{supp}\kappa$. Then for any $N > 0$ we have

$$\|A_0\mathcal{V}_1\cdots\mathcal{V}_k u\|_{H^{N,l_0}} \le C\|B_0(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{H^{-N,l_0+k+0}} + C\|\tilde\kappa u\|_{H^0} + C\|u\|_{H^{-N,-N}} \quad (5.25)$$

uniformly for all $u \in \mathcal{S}'$ with $\operatorname{WF}^{-\infty,-L_0+0}(u) \cap \{\xi = 0\} = \emptyset$, $\omega \in \mathcal{I}_\omega$ and $\mu \ge 0$ small, $M \in [0, M_0]$.

Proof. 0. Let $\tilde B_0, \tilde{\tilde B}_0$ satisfy the same requirement for B_0 with $\operatorname{WF} B_0 \cap \operatorname{WF}(\mathrm{I} - \tilde B_0) = \operatorname{WF}\tilde B_0 \cap \operatorname{WF}(\mathrm{I} - \tilde{\tilde B}_0) = \emptyset$. Without loss of generality assume $\chi, \tilde\chi \in S^0(\Omega)$ that $\nabla\chi, \nabla\tilde\chi$ is compactly supported, $\mathcal{V}_1 = \mathcal{V}_1\chi$, $\tilde{\tilde B}_0 = \chi\tilde{\tilde B}_0\chi$, $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde\chi) = \emptyset$, $\operatorname{supp}\tilde\chi \cap \operatorname{supp}\kappa = \emptyset$. Let $\varphi \in C_c^\infty(\Omega)$ that $\operatorname{supp}\nabla\tilde\chi \cap \operatorname{supp}(1-\varphi) = \emptyset$, $\operatorname{supp}\chi \cap \operatorname{supp}\varphi = \emptyset$.

1. We iteratively claim

$$\|A_0\mathcal{V}_1\cdots\mathcal{V}_k u\|_{H^{N,l_0}} \le C\|B_0(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{H^{-N,l_0+k+0}} + C\|\tilde\kappa u\|_{H^0} + C\|u\|_{H^{-N,-N}}. \quad (5.26)$$

The case $k = 0$ follows from Lemma 5.2. Assume the case holds for up to $k-1$. Applying (5.19) to $\mathcal{V}_1 \cdots \mathcal{V}_k u$ implies

$$\begin{aligned}\|A_0\mathcal{V}_1\cdots\mathcal{V}_k u\|_{H^{N,l_0}} &\leq C\|B_0\chi(P-\sqrt{\omega^2+M^2}-i\mu)\tilde{\chi}\mathcal{V}_1\mathcal{V}_2\cdots\mathcal{V}_k u\|_{H^{-N,l_0}} + C\|u\|_{H^{-N,-N}}\\ &\leq C\|B_0\mathcal{V}_1\cdots\mathcal{V}_k\chi(\Lambda_M^\Omega-\sqrt{\omega^2+M^2}-i\mu)u\|_{H^{-N,l_0}} + C\|B_0[\chi P\tilde{\chi},\mathcal{V}_1\cdots\mathcal{V}_k]u\|_{H^{-N,l_0}}\\ &+ C\|\tilde{B}_0\mathcal{V}_1\cdots\mathcal{V}_k\mathcal{E}_M u\|_{H^{-N,l_0}} + C\|u\|_{H^{-N,-N}} \leq C\|\tilde{B}_0(\Lambda_M^\Omega-\sqrt{\omega^2+M^2}-i\mu)u\|_{H^{-N,l_0+k+0}}\\ &+ C\|B_0[\chi P\tilde{\chi},\mathcal{V}_1\cdots\mathcal{V}_k]u\|_{H^{-N,l_0}} + C\|\tilde{B}_0\mathcal{V}_1\cdots\mathcal{V}_k\mathcal{E}_M u\|_{H^{-N,l_0}} + C\|\tilde{\kappa}u\|_{H^0} + C\|u\|_{H^{-N,-N}} \quad (5.27)\end{aligned}$$

by the iterative assumption.

2. It remains to bound the commutator term and the $\mathcal{E}$-term. Decompose

$$[\chi P\tilde{\chi},\mathcal{V}_1\cdots\mathcal{V}_k] = -\sum_{|\alpha|<k}\mathrm{ad}_{\mathcal{V}}^{\alpha^c}(\chi P\tilde{\chi})\mathcal{V}_\alpha, \quad (5.28)$$

as a summation over all proper subwords α of $(1,\ldots,k)$, defined in (4.38). Noting $\mathrm{ad}_{\mathcal{V}}^{\alpha^c}(\chi P\tilde{\chi}) \in \Psi_{0,\delta}^{1,0}$ we have

$$\begin{aligned}\|B_0[\chi P\tilde{\chi},\mathcal{V}_1\cdots\mathcal{V}_k]u\|_{H^{-N,l_0}} &\leq \sum_{|\alpha|<k}\|\tilde{B}_0\mathcal{V}_\alpha u\|_{H^{-N,l_0}} + C\|u\|_{H^{-N,-N}}\\ &\leq C\|\tilde{\tilde{B}}_0(\Lambda_M^\Omega-\sqrt{\omega^2+M^2}-i\mu)u\|_{H^{-N,l_0+k-1+0}} + C\|\tilde{\kappa}u\|_{H^0} + C\|u\|_{H^{-N,-N}} \quad (5.29)\end{aligned}$$

by the iterative assumption. For the other term, one can decompose

$$B_0\mathcal{V}_1\cdots\mathcal{V}_k\chi\mathcal{E}_M u = B_0\mathcal{V}_1\cdots\mathcal{V}_k\chi(\Lambda_M^\Omega-P)\tilde{\chi}u + B_0\mathcal{V}_1\cdots\mathcal{V}_k\chi\partial_\nu\mathcal{R}_M^\Omega[-\Delta,\tilde{\chi}]\mathcal{H}_M^\Omega u, \quad (5.30)$$

as in Proposition 4.22. Here Proposition 3.12 and the iterative assumption imply

$$\begin{aligned}\|B_0\mathcal{V}_1\cdots\mathcal{V}_k\chi(\Lambda_M^\Omega-P)\tilde{\chi}u\|_{H^{-N,l_0}} &\leq C\|\Gamma_{k,\delta}^{-N,l_0+(1-\delta)k}(\Lambda_M^\Omega-P)\tilde{\chi}u\| \leq C\|B_0\tilde{\chi}u\|_{H^{-N,l_0+0}}\\ &+ C\|u\|_{H^{-N,-N}} \leq C\|\tilde{\tilde{B}}_0(\Lambda_M^\Omega-\sqrt{\omega^2+M^2}-i\mu)u\|_{H^{-N,l_0+0}} + C\|\tilde{\kappa}u\|_{H^0} + C\|u\|_{H^{-N,-N}} \quad (5.31)\end{aligned}$$

by choosing $\delta \in [0,1)$ very close to 1. Lemma 4.8 gives

$$\begin{aligned}\|B_0\mathcal{V}_1\cdots\mathcal{V}_k\chi\partial_\nu\mathcal{R}_M^\Omega[-\Delta,\tilde{\chi}]\mathcal{H}_M^\Omega u\|_{H^{-N,l_0}} &\leq C\|\varphi\mathcal{H}_M^\Omega u\|_{X_{m,-N}^{l_0,-N}}\\ &\leq C\|\varphi\mathcal{H}_M^\Omega\chi(1-\tilde{\kappa})u\|_{X_{m,-N}^{l_0,-N}} + C\|\varphi\mathcal{H}_M^\Omega(1-\chi)(1-\tilde{\kappa})u\|_{X_{m,-N}^{l_0,-N}} + C\|\varphi\mathcal{H}_M^\Omega\tilde{\kappa}u\|_{X_{m,-N}^{l_0,-N}}\\ &\leq C\|B_0\chi u\|_{H^{-N,l_0}} + C\|u\|_{H^{-N,-N}} + C\|\tilde{\kappa}u\|_{H^{0,-N}}, \quad (5.32)\end{aligned}$$

where we used Lemma 4.10 and Lemma 4.12 for the second and third terms on the last line. Invoke the iterative assumption and rename $\tilde{\tilde{B}}_0$ to B_0 to conclude (5.26). □

Note that $\Psi_{1,\delta}^{0,-\delta} \subset \Psi_{0,\delta}^{1,0}$ and $\Psi^{s,l} \subset \Psi_{0,\delta}^{s,l}$. When away from $\xi = 0$, note $\Psi_{k,\delta}^{s,l} = \Psi^{s+k,l+k\delta}$. Here we record the following elliptic, propagation and source/sink estimates within the class $\Psi^{s,l}$ (see also [DZ19a, (25),(26) and (27)]). We denote the following error term, at the 0-frequency or in the Lipschitz part, by

$$O_0 = C(\|(1-\kappa)\psi_0(D)(1-\kappa_0)u\|_{H^{-N,-L_0+0}} + \|\kappa u\|_{H^0} + \|u\|_{H^{-N,-N}}), \quad (5.33)$$

if for each $N > 0$ and $\psi_0(\xi) \in C_c^\infty(\mathbb{R}^{d-1})$ with $\psi_0 \equiv 1$ near $\xi = 0$ there is such $C > 0$ that does not depend on u, ω or μ in the following microlocal statements.

Lemma 5.4 (Elliptic estimate away from zero frequency)**.** *Let $\Omega \subset \mathbb{R}^d$ be an AC Lipschitz domain. Let $|l_0| < L_0$. Let $s, l \in \mathbb{R}$. For each $A, B \in \Psi^{0,0}(\partial\Omega)$, $\operatorname{WF} A \subset \operatorname{Ell} B$,*

$$\operatorname{WF} B \cap \Sigma(\omega) = \emptyset,\ \operatorname{WF} B \subset \operatorname{Ell}(1-\kappa_0),\ \operatorname{WF} B \cap \{(x,\xi) : |\xi|_g \le \epsilon_0\} = \emptyset, \tag{5.34}$$

for some $\epsilon_0 > 0$, for any $N > 0$ we have

$$\|Au\|_{H^{s,l}} \le C\|B(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu)u\|_{H^{s-1,l}} + O_0 \tag{5.35}$$

uniformly for all $u \in \mathcal{S}'$, $\omega \in \mathcal{I}_\omega$ and $\mu \ge 0$ small, $M \in [0, M_0]$.

Proof. Let $B = \chi B \chi$ for some $\chi, \tilde\chi \in S^0(\Omega)$ away from $\partial\Omega_{\text{sing}}$ with $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde\chi) = \emptyset$. The standard elliptic estimate and Proposition 4.22 imply

$$\begin{aligned}\|Au\|_{H^{s,l}} &\le C\|B\chi(P - \sqrt{\omega^2+M^2} - i\mu)\tilde\chi u\|_{H^{s-1,l}} + C\|u\|_{H^{-N,-N}} \\ &\le C\|B\chi(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{H^{s-1,l}} + C\|B\chi\mathcal{E}_M u\|_{H^{s-1,l}} + C\|u\|_{H^{-N,-N}},\end{aligned} \tag{5.36}$$

and $B\chi\mathcal{E}_M(1-\kappa)(\mathrm{I} - \psi_0(D))(1-\kappa_0) : H^{-N,-N} \to H^{N,N}$ is bounded since $\operatorname{WF} B \cap \{\xi = 0\} = \emptyset$. Then

$$\|B\chi\mathcal{E}_M u\|_{H^{s-1,l}} \le C(\|(1-\kappa)\psi_0(D)(1-\kappa_0)u\|_{H^{-N,-L_0+0}} + \|u\|_{H^{-N,-N}} + \|\kappa u\|_{H^0}) = O_0 \tag{5.37}$$

where we decomposed $\mathrm{I} = (1-\kappa)\psi_0(D)(1-\kappa_0) + (1-\kappa)(\mathrm{I} - \psi_0(D))(1-\kappa_0) + \kappa$. □

With exactly the same treatment of $\Lambda_M^\Omega - P$, we have

Lemma 5.5 (Propagation Estimate)**.** *Let $\Omega \subset \mathbb{R}^d$ be an AC Lipschitz domain. Let $s, l \in \mathbb{R}$. Let $A, B, Q \in \Psi^{0,0}$ with $\operatorname{WF} B \subset \operatorname{Ell}(1-\kappa_0)$, $\operatorname{WF} B \cap \{\xi = 0\} = \emptyset$, $\operatorname{WF} A \cup \operatorname{WF} Q \subset \operatorname{Ell} B$. Moreover, assume there is $T > 0$ such that*

$$\operatorname{WF} A \subset \bigcup_{t\in[0,T]} e^{t\hat H_p}(\operatorname{Ell} Q), \quad \bigcup_{t\in[0,T]} e^{t\hat H_p}(\operatorname{WF} Q) \subset \operatorname{Ell}(B). \tag{5.38}$$

Then for each $N > 0$, we have

$$\|Au\|_{H^{s,l}} \le C\|B(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{H^{s-1,l+1}} + C\|Qu\|_{H^{s,l}} + O_0, \tag{5.39}$$

uniformly for all $u \in \mathcal{S}'$, $\omega \in \mathcal{I}_\omega$, $M \in [0, M_0]$ and $\mu > 0$ small.

Lemma 5.6 (Source estimate)**.** *Let $\Omega \subset \mathbb{R}^d$ be an AC Lipschitz domain. Let $s \in \mathbb{R}$, $l^- > -\frac{1}{2}$. Let $B_- \in \Psi^{0,0}$ such that $\operatorname{WF} B_- \subset \operatorname{Ell}(1-\kappa_0)$, $\operatorname{WF} B_- \cap \{\xi = 0\} = \emptyset$, $\Sigma^-(\omega) \subset \operatorname{Ell} B_-$. There exists $A_- \in \Psi^{0,0}$ with $\Sigma^-(\omega) \subset \operatorname{Ell} A_-$ such that for any $N > 0$ we have*

$$\|A_- u\|_{H^{s,l^-}} \le C\|B_-(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{H^{s-1,l^-+1}} + O_0, \tag{5.40}$$

uniformly for all $u \in \mathcal{S}'$ with $\operatorname{WF}^{s,-\frac{1}{2}+0}(u) \cap \Sigma^-(\omega) = \emptyset$, $\omega \in \mathcal{I}_\omega$, $M \in [0, M_0]$ and $\mu > 0$ small.



(A) $d = 2$ (B) $d = 3$

FIGURE 9. Variable order Sobolev spaces and regularity/decay indices.

Lemma 5.7 (Sink estimate)**.** *Let $\Omega \subset \mathbb{R}^d$ be an AC Lipschitz domain. Let $s \in \mathbb{R}$, $l^+ < -\frac{1}{2}$. Let $B_+ \in \Psi^{0,0}$ such that $\operatorname{WF} B_+ \subset \operatorname{Ell}(1-\kappa_0)$ $\operatorname{WF} B_+ \cap \{\xi = 0\} = \emptyset$, $\Sigma^+(\omega) \subset \operatorname{Ell} B_+$. There exist $A_+, Q \in \Psi^{0,0}$ with $\Sigma^+(\omega) \subset \operatorname{Ell} A_+$, $\operatorname{WF} Q \subset \operatorname{Ell} B_+ \setminus \Sigma^+(\omega)$ such that for any $N > 0$ we have*

$$\|A_+ u\|_{H^{s,l^+}} \leq C\|B_+(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu)u\|_{H^{s-1,l^+ +1}} + C\|Qu\|_{H^{s,l^+}} + O_0, \tag{5.41}$$

uniformly for all $u \in \mathcal{S}'$, $\omega \in \mathcal{I}_\omega$, $M \in [0, M_0]$ and $\mu > 0$ small.

5.3. **Variable order Sobolev spaces and Fredholm estimates.** In this subsection, we define the spaces on which the DtN map is Fredholm. Let $\omega > 0$. Let $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in C_c^\infty(\overline{\Omega})$ with $\tilde{\kappa} \prec \kappa_1 \prec \kappa_2 \prec \kappa_3 \prec \kappa_4$. Let $\psi_1, \tilde{\psi}_1$ in $C_c^\infty(\mathbb{R})$ such that $\psi_1(\xi) \equiv 1$ near $\xi = 1$ and $\tilde{\psi}_1(\xi) \equiv 0$ near $\xi = 0$, $\psi_1 \prec \tilde{\psi}_1$. We rescale by $\psi_\omega, \tilde{\psi}_\omega \in C_c^\infty(\mathbb{R})$ that $\psi_\omega(\xi) = \psi_1(\omega^{-1}\xi)$, $\tilde{\psi}_\omega(\xi) = \tilde{\psi}_1(\omega^{-1}\xi)$. Let $\psi_{00}, \psi_0 \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\psi_0 \cap \operatorname{supp}\psi_\omega = \emptyset$, $\psi_{00} \prec \psi_0$, $0 \notin \operatorname{supp}(1-\psi_0)$. The choice of ψ_{00}, ψ_0 can be made independent of ω if we ask $\omega \geq \omega_0$ for some $\omega_0 \geq 0$. Let $V^\pm(\omega)$ be neighborhoods of $\Sigma^\pm(\omega)$ in $T^*\partial\Omega$ and $\varphi \in C^\infty(\overline{T}^*\partial\Omega)$ such that $\operatorname{supp}(\pm 1 - \varphi) \cap V^\mp(\omega) = \emptyset$, and $\hat{H}_p\varphi \leq 0$ on $\Sigma(\omega)$.

Definition 5.8 (Variable-order Sobolev spaces on AC Lipschitz domains)**.** Let $s, l, l_0, l^-, l^+ \in \mathbb{R}$, and $|s_0| \leq 1$. We define the logarithmic weight $W = \frac{1}{2}(\operatorname{Op}(w) + \operatorname{Op}(w)^*)$ where

$$\begin{aligned} w = (1-\kappa_4(x))\Big(\psi_\omega(|\xi|_g)\big(\tfrac{l^+ + l^-}{2} + \tfrac{l^- - l^+}{2}\varphi\big) + l_0\psi_0(|\xi|_g) + l(1 - \psi_0(|\xi|_g) - \psi_\omega(|\xi|_g))\Big)\log\langle x\rangle \\ + \big(s_0(1-\kappa_0(x))\kappa_4(x) + s(1-\kappa_4(x))\big)\log\langle\omega^{-1}\xi\rangle. \end{aligned} \tag{5.42}$$

Note W is essentially self-adjoint on L^2, and $(e^W)^{-1} = e^{-W}$. Fix $N \geq \max(|s|, |s_0|, |l|, |l_0|, |l^-|, |l^+|)$ large. We define the variable-order Sobolev norm by

$$\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} = \|\kappa_3 u\|_{H^{s_0}} + \|(1-\kappa_2)e^W(1-\kappa_2)u\|_{L^2} + \|(1-\kappa_1)u\|_{H^{-N,-N}}. \tag{5.43}$$

For an AC Lipschitz domain Ω with Lipschitz $\partial\Omega$, we define the variable-order Sobolev spaces by

$$Z^{s,l,s_0}_{l_0,l^-,l^+}(\partial\Omega;\omega) = \Big\{u \in \mathcal{S}'(\partial\Omega) : \|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} < \infty\Big\}. \tag{5.44}$$

We may drop $\partial\Omega$ and ω whenever there is no confusion. The definition of $Z^{s,l,s_0}_{l_0,l^-,l^+}$ only depends on the choice of κ_4, not on κ_0 nor κ_1. When Ω is AC smooth and $\partial\Omega$ is smooth, one can take $\kappa_0 = \kappa_4 \equiv 0$ and w no longer depends on s_0. In this case, we abbreviate by

$$Z^{s,l}_{l_0,l^-,l^+}(\partial\Omega) = Z^{s,l,0}_{l_0,l^-,l^+}(\partial\Omega). \tag{5.45}$$

See Figure 9.

Lemma 5.9. $(Z^{s,l,s_0}_{l_0,l^-,l^+}(\partial\Omega))^* = Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}(\partial\Omega)$.

Proof. Throughout this proof, $\langle\cdot,\cdot\rangle$ indicates the inner product in L^2 or $(L^2)^3$. For $t\in\mathbb{R}$, the cutoffs $\kappa_t\in C_c^\infty(\partial\Omega)$ are chosen compatible with $\kappa_0,\kappa_1,\kappa_2,\kappa_3,\kappa_4$ in the manner that $\operatorname{supp}\kappa_t\cap\operatorname{supp}(1-\kappa_{t'})=\emptyset$ if $t<t'$.

1. We show $Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}\subset(Z^{s,l,s_0}_{l_0,l^-,l^+})^*$. For each $u\in Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}$ and $\phi\in Z^{s,l,s_0}_{l_0,l^-,l^+}$, we claim

$$|\langle u,\phi\rangle| = |\langle\kappa_{2.2}u,\kappa_3\phi\rangle + \langle(1-\kappa_{2.2})u,\phi\rangle| \le C\|u\|_{Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}}\|\phi\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}}. \tag{5.46}$$

Indeed, the first term can be estimated by

$$|\langle\kappa_{2.2}u,\kappa_3\phi\rangle| \le C\|\kappa_3 u\|_{H^{-s_0}}\|\kappa_3\phi\|_{H^{s_0}} \le C\|u\|_{Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}}\|\phi\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}}. \tag{5.47}$$

The second term can be estimated by

$$|\langle(1-\kappa_{2.2})u,\phi\rangle| = |\langle e^{-W}(1-\kappa_{2.2})u, e^{W}(1-\kappa_{2.1})\phi\rangle| \le C\|u\|_{Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}}\|\phi\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}}, \tag{5.48}$$

since $\mathrm{WF}(e^{-W}(1-\kappa_{2.2}))\subset\mathrm{Ell}((1-\kappa_2)e^{-W}(1-\kappa_2))$ implies

$$\|e^{-W}(1-\kappa_{2.2})u\| \le C\|(1-\kappa_2)e^{-W}(1-\kappa_2)u\| + C\|(1-\kappa_1)u\|_{H^{-N,-N}} \le C\|u\|_{Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}}, \tag{5.49}$$

and similarly we have

$$\|e^{W}(1-\kappa_{2.1})\phi\| \le C\|(1-\kappa_2)e^{W}(1-\kappa_2)\phi\| + C\|(1-\kappa_1)\phi\|_{H^{-N,-N}} \le C\|\phi\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}}. \tag{5.50}$$

This concludes the claim.

2. We show $(Z^{s,l,s_0}_{l_0,l^-,l^+})^*\subset Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}$. Let $u\in(Z^{s,l,s_0}_{l_0,l^-,l^+})^*$ and $\phi\in Z^{s,l,s_0}_{l_0,l^-,l^+}$. Consider the embedding $\Phi: Z^{s,l,s_0}_{l_0,l^-,l^+}\leftrightarrow\mathcal{V}\subset\mathcal{H}=H^{s_0}\times L^2\times H^{-N,-N}$, where

$$\Phi(\phi)=(\kappa_3\phi,(1-\kappa_2)e^{W}(1-\kappa_2)\phi,(1-\kappa_1)\phi),\ \mathcal{V}=\mathrm{Ran}(\Phi). \tag{5.51}$$

One can check Φ is injective and $\mathcal{V}=\mathrm{Ran}(\Phi)$ is closed: the invertibility of Φ then follows from the open mapping theorem. The functional $u\in(Z^{s,l,s_0}_{l_0,l^-,l^+})^*$ then admits a unique lift $\tilde u\in\mathcal{V}^*$ given by

$$\tilde u(\tilde\phi)=u(\Phi^{-1}(\tilde\phi)),\ \|\tilde u\|_{\mathcal{V}^*}\le C\|u\|_{(Z^{s,l,s_0}_{l_0,l^-,l^+})^*}, \tag{5.52}$$

for each $\tilde\phi\in\mathcal{V}$. Hahn-Banach theorem implies $\tilde u\in\mathcal{V}^*$ extends to an element in $\mathcal{H}^*$ represented by $\tilde{\tilde u}=(\tilde{\tilde u}_1,\tilde{\tilde u}_2,\tilde{\tilde u}_3)\in H^{-s_0}\times L^2\times H^{N,N}$, such that for each $\tilde\phi\in\mathcal{V}$,

$$\langle\tilde{\tilde u},\tilde\phi\rangle=\tilde u(\tilde\phi)=u(\Phi^{-1}(\tilde\phi)),\ \|\tilde{\tilde u}\|_{H^{-s_0}\times L^2\times H^{N,N}}=\|\tilde u\|_{\mathcal{V}^*}\le C\|u\|_{(Z^{s,l,s_0}_{l_0,l^-,l^+})^*}. \tag{5.53}$$

Now we set the representation

$$w=\kappa_3\tilde{\tilde u}_1+(1-\kappa_2)e^{W}(1-\kappa_2)\tilde{\tilde u}_2+(1-\kappa_1)\tilde{\tilde u}_3. \tag{5.54}$$

Since $\langle D\rangle^{-s_0}\kappa_3(1-\kappa_2)e^{W}(1-\kappa_2)\in\Psi^{0,-\infty}$, we have

$$\|\kappa_3 w\|_{H^{-s_0}}\le C\|\tilde{\tilde u}_1\|_{H^{-s_0}}+C\|\tilde{\tilde u}_2\|_{L^2}+\|\tilde{\tilde u}_3\|_{H^{N,N}}. \tag{5.55}$$

We can similarly estimate $\|(1-\kappa_2)e^{-W}(1-\kappa_2)w\|$ and $\|(1-\kappa_1)w\|_{H^{-N,-N}}$ to observe

$$\|w\|_{Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}} \leq C\|\tilde{\tilde{u}}_1\|_{H^{-s_0}} + C\|\tilde{\tilde{u}}_2\|_{L^2} + \|\tilde{\tilde{u}}_3\|_{H^{N,N}}. \tag{5.56}$$

Setting $\tilde{\phi} = \Phi(\phi)$ in (5.53), we end up with

$$u(\phi) = \langle \tilde{\tilde{u}}, \Phi(\phi)\rangle = \langle \tilde{\tilde{u}}_1, \kappa_3\phi\rangle + \langle \tilde{\tilde{u}}_2, (1-\kappa_2)e^{W}(1-\kappa_2)\phi\rangle + \langle \tilde{\tilde{u}}_3, (1-\kappa_1)\phi\rangle = \langle w, \phi\rangle, \tag{5.57}$$

that is we have represented u by $w \in Z^{-s,-l,-s_0}_{-l_0,-l^-,-l^+}$. This concludes the proof. □

Lemma 5.10. *Let Ω be an AC Lipschitz domain. Let $M_0 > 0$. Let $s, l, l^-, l^+ \in \mathbb{R}$. Let $s_0 \in [0,1]$, $|l_0| < L_0$. Then*

$$\Lambda^\Omega_M : Z^{s,l,s_0}_{l_0,l^-,l^+}(\partial\Omega;\omega) \to Z^{s-1,l,s_0-1}_{l_0,l^-,l^+}(\partial\Omega;\omega) \tag{5.58}$$

is continuous and uniformly bounded in $M \in [0, M_0]$ as bounded maps.

Proof. 0. In this proof, for $t \in \mathbb{R}$, the cutoffs $\kappa_t \in C^\infty_c(\partial\Omega)$ are chosen compatible with $\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4$ in the manner that $\operatorname{supp}\kappa_t \cap \operatorname{supp}(1-\kappa_{t'}) = \emptyset$ if $t < t'$.

1. Through Step 1–4, we show

$$\|\Lambda^\Omega_M u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-,l^+}} \leq C\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \tag{5.59}$$

uniformly for all $M \in [0, M_0]$. All estimates in this proof are uniform in this way.

2a. We estimate $\kappa_3\Lambda^\Omega_M u$ in this step, by claiming

$$\kappa_3\Lambda^\Omega_M\kappa_{3.5} = \kappa_3\Lambda^{\tilde{\Omega}}_M\kappa_{3.5} - \kappa_3\partial_\nu\mathcal{R}^\Omega_M\varphi[-\Delta,\kappa_{3.6}]\mathcal{H}^{\tilde{\Omega}}_M\kappa_{3.5} : H^{s_0} \to H^{s_0-1}, \tag{5.60}$$

where we take a bounded uniformly Lipschitz domain $\tilde{\Omega}$ that $\tilde{\Omega}, \Omega$ share the common boundary near $\operatorname{supp}\kappa_4 \cap \partial\Omega$, and $\varphi \in C^\infty_c(\overline{\tilde{\Omega}})$ that $\operatorname{supp}\nabla\kappa_{3.6} \cap \operatorname{supp}(1-\varphi) = \operatorname{supp}\varphi \cap \operatorname{supp}\kappa_3 = \emptyset$. Only the second term needs to be justified here. Lemma 2.5 implies

$$[-\Delta,\kappa_{3.6}]\mathcal{H}^{\tilde{\Omega}}_M : H^{s_0}(\partial\tilde{\Omega}) \to Y^N_N(\Omega). \tag{5.61}$$

Noting $\kappa_3\varphi = 0$, we have

$$\partial_\nu\kappa_3\mathcal{R}^\Omega_M\varphi = \partial_\nu\mathcal{R}^{\tilde{\Omega}}_M[-\Delta,\kappa_3]\mathcal{R}^\Omega_M\varphi : Y^N_N(\Omega) \to H^0(\partial\Omega), \tag{5.62}$$

where we used Proposition 2.16 and $\mathcal{R}^{\tilde{\Omega}}_M : L^2(\tilde{\Omega}) \to \dot{H}^{\frac{3}{2}}(\tilde{\Omega})$ from Lemma 2.5. This concludes the step.

2b. Let $\varphi' \in C^\infty_c(\overline{\Omega})$ that $\operatorname{supp}\nabla\kappa_{3.1} \cap \operatorname{supp}(1-\varphi') = \operatorname{supp}\varphi' \cap \operatorname{supp}\kappa_3 = \emptyset$. Proposition 4.22 implies

$$\kappa_3\Lambda^\Omega_M(1-\kappa_{3.5}) = \kappa_3\tilde{\mathcal{E}}^*_M(1-\kappa_{3.5}) = \kappa_3\partial_\nu\mathcal{R}^\Omega_M\varphi'[\Delta, 1-\kappa_{3.1}]\mathcal{H}^\Omega_M(1-\kappa_{3.5}). \tag{5.63}$$

Lemma 2.24 implies $\varphi'\mathcal{H}^\Omega_M\kappa_3 : H^{0,-N}(\partial\Omega) \to \dot{H}^{\frac{3}{2},N}(\Omega)$ bounded and so is its adjoint $-\kappa_3\partial_\nu\mathcal{R}^\Omega_M\varphi' : H^{-\frac{3}{2},-N}(\Omega) \to H^{0,N}(\partial\Omega)$. Away from the zero-frequency, note

$$\kappa_3\partial_\nu\mathcal{R}^\Omega_M\varphi'[\Delta, 1-\kappa_{3.1}]\mathcal{H}^\Omega_M(1-\kappa_{3.2})(1-\psi_0(D))(1-\kappa_{3.5}) : H^{-N,-N} \to H^{0,N} \tag{5.64}$$

is bounded, since $[\Delta, 1-\kappa_{3.1}]\mathcal{H}^\Omega_M(1-\kappa_{3.2})(1-\psi_0(D))(1-\kappa_{3.5}) : H^{-N,-N}(\partial\Omega) \to Y^{L_0-0,N}_{\mathfrak{r}^{r_0}_\Omega-0,N}(\Omega)$ bounded per (4.117). At the zero-frequency, we have

$$\kappa_3\partial_\nu\mathcal{R}^\Omega_M\varphi'[\Delta, 1-\kappa_{3.1}]\mathcal{H}^\Omega_M(1-\kappa_{3.2})\psi_0(D)(1-\kappa_{3.5}) : H^{-N,-L_0+0} \to H^{0,N}, \tag{5.65}$$

bounded since $\mathcal{H}^{\Omega}_{M} : H^{\frac{1}{2},-L_0+0} \to X^{-L_0+0}_{-\mathrm{r}^{r_0}_{\Omega}+0}(\Omega)$ per Proposition 2.19, and $\varphi'\mathcal{H}^{\Omega}_{M}\kappa_3$. Note this implies

$$\|\kappa_3\partial_\nu\mathcal{R}^{\Omega}_{M}\varphi'[\Delta,1-\kappa_{3.1}]\mathcal{H}^{\Omega}_{M}(1-\kappa_{3.2})\psi_0(D)(1-\kappa_{3.5})u\|_{H^0} \le C\|(1-\kappa_{4.1})u\|_{H^{-N,-L_0+0}} + C\|(1-\kappa_1)u\|_{H^{-N,-N}}. \quad (5.66)$$

Bringing the estimates together we have

$$\|\kappa_3\Lambda^{\Omega}_{M}u\|_{H^{s_0-1}} \le C\|\kappa_{3.9}u\|_{H^{s_0}}+C\|(1-\kappa_{4.1})u\|_{H^{-N,-L_0+0}}+C\|(1-\kappa_1)u\|_{H^{-N,-N}} \le C\|u\|_{Z^{-N,-N,s_0}_{-L_0+0,-N,-N}} \quad (5.67)$$

as desired.

3a. Let the weights W be with parameter $(s-1,l,s_0-1,l_0,l^-,l^+)$ and W' be with parameter (s,l,s_0,l_0,l^-,l^+) in (5.42). Denote $A=(1-\kappa_2)e^{W}(1-\kappa_2)$ and $A'=(1-\kappa_2)e^{W'}(1-\kappa_2)$. We begin with the first term of

$$A\Lambda^{\Omega}_{M} = APe^{-W'}e^{W'}(1-\kappa_{1.9})+A\mathcal{E}, \quad (5.68)$$

given by Proposition 4.22. Note $P\in\Psi^{0,-\delta}_{1,\delta}\subset\Psi^{1,0}_{0,\delta}$, $APe^{-W'}\in\Psi^{0,0}_{0,\delta}$ to see

$$\|AP(1-\kappa_{1.9})u\| \le C\|e^{W'}(1-\kappa_{1.9})u\| \le C\|A'u\|+C\|\kappa_3u\|_{H^{s_0}}+C\|(1-\kappa_1)u\|_{H^{-N,-N}}. \quad (5.69)$$

3b. Denote $B_{00}=(1-\kappa_{2.4})\psi_{00}(D)(1-\kappa_{2.5})$. Then

$$A\mathcal{E} = A\mathcal{E}\kappa_{2.5}+A\mathcal{E}(1-\kappa_{2.5}-B_{00})+A\mathcal{E}B_{00}. \quad (5.70)$$

Proposition 4.22 allows us to bound the error terms by

$$\|A\mathcal{E}\kappa_{2.5}u\| \le C\|\kappa_{2.5}u\|_{H^{0,-N}}, \quad (5.71)$$

$$\|A\mathcal{E}(1-\kappa_{2.5}-B_{00})u\| \le C\|A'u\|_{H^{-N,-N}}+C\|(1-\kappa_1)u\|_{H^{-N,-N}}, \quad (5.72)$$

$$\|A\mathcal{E}B_{00}u\| \le C\|A'u\|_{H^{-N,0}}+C\|(1-\kappa_1)u\|_{H^{-N,-N}}. \quad (5.73)$$

Together with Step 3a we have

$$\|A\Lambda^{\Omega}_{M}u\| \le C\|A'u\|+C\|\kappa_3u\|_{H^{s_0}}+C\|(1-\kappa_1)u\|_{H^{-N,-N}} \le C\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}}. \quad (5.74)$$

4. Step 2 and 3 imply

$$\|(1-\kappa_1)\Lambda^{\Omega}_{M}u\|_{H^{-N,-N}} \le \|(1-\kappa_1)\kappa_3\Lambda^{\Omega}_{M}u\|_{H^{-N,-N}}+\|(1-\kappa_3)\Lambda^{\Omega}_{M}u\|_{H^{-N,-N}} \le C\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}}. \quad (5.75)$$

This concludes the claim in Step 1.

5. That Λ^{Ω}_{M} is continuous in M follows from the continuity of all objects we used in the proof, established in earlier sections. □

We summarize the microlocal estimates in terms of the variable-order Sobolev spaces.

Lemma 5.11. *Let $s,l,l_0,l^{\pm}\in\mathbb{R}$, $s_0\in[0,1]$. Let $A,A_0,B_0,B,B^{\pm},Q\in\Psi^{0,0}(\partial\Omega)$. There exist some $C,\epsilon_0,T>0$ that the following statements are true for any $N>0$ and uniformly for all $u\in\mathcal{S}'$, $\omega\in\mathcal{I}_{\omega}$ and $\mu\ge 0$ small, $M\in[0,M_0]$:*

(1) Let $|l_0| < L_0$. If $\{(x,\xi) : x \in \operatorname{supp}(1-\kappa), \xi = 0\} \subset \operatorname{Ell} B_0$, $\operatorname{WF}(B_0) \subset \{(x,\xi) : |\xi|_g \leq \epsilon_0\}$, $\operatorname{WF} A_0 \cap \Sigma(\omega) = \emptyset$, $\operatorname{WF} A_0 \subset \operatorname{Ell} B_0$, then

$$\|A_0 u\|_{Z^{N,l,s_0}_{l_0,l^-,l^+}} \leq C\|B_0(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{Z^{-N,l,s_0-1}_{l_0,l^-,l^+}} + C\|\tilde{\kappa}u\|_{H^0} + C\|u\|_{H^{-N,-N}}. \quad (5.76)$$

uniformly for all u described above with $\operatorname{WF}^{-\infty,-L_0+0}(u) \cap \{\xi = 0\} = \emptyset$.

(2) If $\operatorname{WF} B \subset \operatorname{Ell}(1-\kappa_0)$, $\operatorname{WF} B \cap (\Sigma(\omega) \cup \{(x,\xi) : |\xi|_g \leq \epsilon_0\}) = \emptyset$, $\operatorname{WF} A \subset \operatorname{Ell} B$, then

$$\|Au\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|B(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-,l^+}} + O_0. \quad (5.77)$$

(3) If $\operatorname{WF} B \subset \operatorname{Ell}(1-\kappa_0)$, $\operatorname{WF} B \cap \{\xi = 0\} = \emptyset$, $\operatorname{WF} A \cup \operatorname{WF} Q \subset \operatorname{Ell} B$, and

$$\operatorname{WF} A \subset \bigcup_{t\in[0,T]} e^{t\hat{H}_p}(\operatorname{Ell} Q), \quad \bigcup_{t\in[0,T]} e^{t\hat{H}_p}(\operatorname{WF} Q) \subset \operatorname{Ell}(B) \quad (5.78)$$

for some $T > 0$, then

$$\|Au\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|B(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + C\|Qu\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} + O_0. \quad (5.79)$$

(4) Let $l^- > -\frac{1}{2}$. If $\operatorname{WF} B^- \subset \operatorname{Ell}(1-\kappa_0)$, $\operatorname{WF} B^- \cap \{\xi = 0\} = \emptyset$, $\Sigma^-(\omega) \subset \operatorname{Ell} B^-$, then there exists $A^- \in \Psi^{0,0}$ with $\Sigma^-(\omega) \subset \operatorname{Ell} A^-$ that

$$\|A^- u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|B^-(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + O_0, \quad (5.80)$$

uniformly for all u described above with $\operatorname{WF}^{s,-\frac{1}{2}+0}(u) \cap \Sigma^-(\omega) = \emptyset$.

(5) Let $l^+ < -\frac{1}{2}$. If $\operatorname{WF} B^+ \subset \operatorname{Ell}(1-\kappa_0)$, $\operatorname{WF} B^+ \cap \{\xi = 0\} = \emptyset$, $\Sigma^+(\omega) \subset \operatorname{Ell} B^+$ then there exist $A^+, Q^+ \in \Psi^{0,0}$ with $\Sigma^+(\omega) \subset \operatorname{Ell} A^+$, $\operatorname{WF} Q^+ \subset \operatorname{Ell} B^+ \setminus \Sigma^+(\omega)$ that

$$\|A^+ u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|B^+(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + C\|Q^+ u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} + O_0. \quad (5.81)$$

In the above estimates, one can replace $-i\mu$ by $+i\mu$ by switching $Z^{s,l,s_0}_{l_0,l^-,l^+}$ and $Z^{s,l,s_0}_{l_0,l^+,l^-}$, $\Sigma^-(\omega)$ and $\Sigma^+(\omega)$, that is, switching the source and sinks.

Combining the estimates above, we can write down the Fredholm estimates in the variable order Sobolev spaces.

Proposition 5.12 (Fredholm estimates)**.** *Let Ω be an AC Lipschitz domain. Let $M_0 > 0$. Let $s, l \in \mathbb{R}$, $s_0 \in [0,1]$. Let $|l_0| < L_0$. Then for any $N > 0$ we have the following estimates:*

(1) For $l^- > -\frac{1}{2}$, $l^+ < -\frac{1}{2}$, we have

$$\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} - i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + C\|\tilde{\kappa}u\|_{L^2} + C\|u\|_{H^{-N,-N}}, \quad (5.82)$$

uniformly for all $u \in Z^{-\infty,-\infty,0}_{-L_0+0,-\frac{1}{2}+0,-\infty}(\partial\Omega)$, $\omega \in \mathcal{I}_\omega$, $\mu \geq 0$ small, $M \in [0, M_0]$.

(2) For $l^- < -\frac{1}{2}$, $l^+ > -\frac{1}{2}$, we have

$$\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|(\Lambda_M^\Omega - \sqrt{\omega^2+M^2} + i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + C\|\tilde{\kappa}u\|_{L^2} + C\|u\|_{H^{-N,-N}}, \quad (5.83)$$

uniformly for all $u \in Z^{-\infty,-\infty,0}_{-L_0+0,-\infty,-\frac{1}{2}+0}(\partial\Omega)$, $\omega \in \mathcal{I}_\omega$, $\mu \geq 0$ *small,* $M \in [0, M_0]$.

(3) When $s_0 \in (0, 1]$, *the* $\|\tilde{\kappa}u\|_{L^2}$ *terms can be dropped from the above estimates.*

Proof. 1. Let $f = (\Lambda^\Omega_M - \sqrt{\omega^2 + M^2} - i\mu)u$. Lemma 5.11 implies

$$\|(1-\kappa)u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|(1-\kappa_0)f\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + O_0 \tag{5.84}$$

by applying (5.80), (5.79), (5.81), (5.77) and (5.76) sequentially. Invoke Lemma 5.2, the elliptic estimate at the zero frequency, to see for O_0,

$$\|(1-\kappa)\psi_0(D)(1-\kappa_0)u\|_{H^{N,l_0}} \leq C\|(1-\kappa_0)f\|_{H^{-N,l_0}} + C\|\tilde{\kappa}u\|_{H^0} + C\|u\|_{H^{-N,-N}}. \tag{5.85}$$

Thus we end up with

$$\|(1-\kappa)u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|(1-\kappa_0)f\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + C\|\tilde{\kappa}u\|_{H^0} + C\|u\|_{H^{-N,-N}}. \tag{5.86}$$

2. We now estimate $\|\tilde{\kappa}u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}}$. Note $\|\tilde{\kappa}u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \sim \|\tilde{\kappa}u\|_{H^{s_0}}$. By Lemma 4.16 and (5.86), we have

$$\begin{aligned}\|\tilde{\kappa}u\|_{H^{s_0}} &\leq C\|\tilde{\kappa}f\|_{H^{s_0-1}} + C\|(1-\kappa_1)u\|_{H^{-N,-N}} + C\|(1-\kappa_2)\psi_0(D)(1-\kappa_1)u\|_{H^{-N,-L_0+0}} \\ &+ C\|\kappa_2 u\|_{H^0} \leq C\|f\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + C\|(\kappa_2-\tilde{\kappa})u\|_{H^0} + C\epsilon^{-1}\|\tilde{\kappa}u\|_{H^{-N}} + \epsilon\|\tilde{\kappa}u\|_{H^{s_0}}\end{aligned} \tag{5.87}$$

where we used interpolation

$$\|\tilde{\kappa}u\|_{H^0} \leq C\epsilon^{-1}\|\tilde{\kappa}u\|_{H^{-N}} + \epsilon\|\tilde{\kappa}u\|_{H^{s_0}}. \tag{5.88}$$

Noting $\kappa_2 - \tilde{\kappa}$ is supported away from $\operatorname{supp}\kappa$, we have from (5.86) and interpolation that

$$\|(\kappa_2-\tilde{\kappa})u\|_{H^0} \leq C\|(1-\kappa_0)f\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + C\epsilon^{-1}\|\tilde{\kappa}u\|_{H^{-N}} + \epsilon\|\tilde{\kappa}u\|_{H^{s_0}}. \tag{5.89}$$

Choose small $\epsilon > 0$ to allow the absorption to see

$$\|\tilde{\kappa}u\|_{H^{s_0}} \leq C\|f\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} + C\|u\|_{H^{-N,-N}}, \tag{5.90}$$

and O_0 can be bounded by the same right hand side. Revisit (5.86) to conclude the proof.

3. Equation (5.83) follows similarly by switching the source and the sink. □

Lemma 5.13 (Absorption resolvent)**.** *Let* Ω *be an AC Lipschitz domain. Let* $M_0 > 0$. *Let* $s, l, l^-, l^+ \in \mathbb{R}$, $s_0 \in (0, 1)$, $|l_0| < L_0$. *Fix* $\mu > 0$. *Then for any* $N > 0$ *we have*

$$\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C_\mu\|(\Lambda^\Omega_M - \sqrt{\omega^2+M^2} \mp i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-,l^+}} + C_\mu\|u\|_{H^{-N,-N}}, \tag{5.91}$$

uniformly for all $u \in Z^{-\infty,-\infty,0}_{-L_0+0,-\frac{1}{2}+0,-\infty}$, $\omega \in \mathcal{I}_\omega$, $M \in [0, M_0]$. *Moreover, for each* $\mu > 0$,

$$(\Lambda^\Omega_M - \sqrt{\omega^2+M^2} \mp i\mu)^{-1} : Z^{s-1,l,s_0-1}_{l_0,l^-,l^+} \to Z^{s,l,s_0}_{l_0,l^-,l^+} \tag{5.92}$$

is bounded.

Proof. 1. We first briefly discuss (5.91). For fixed $\mu > 0$, $P - \sqrt{\omega^2 + M^2} \mp i\mu$ is elliptic. One can prove a similar version of Lemma 5.4 away from the zero frequency. Together with Lemma 5.2, Lemma 4.16, one can show (5.91) by repeating the proof of Proposition 5.12 without invoking any propagation or radial point estimates.

2. We show $\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu$ is invertible. Equation (5.91) implies it has closed range. It suffices to check it has trivial kernel and cokernel. Let $(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu)u = 0$ for $u \in Z^{s,l,s_0}_{l_0,l^-,l^+}$. Then (5.91) implies $u \in L^2$. Noting Λ_M^Ω is self-adjoint on L^2, we have

$$\mu\|u\|^2 = -\Im\langle(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu)u, u\rangle = 0, \tag{5.93}$$

yielding $u = 0$ and $\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} - i\mu$ has trivial kernel. The trivial cokernel and the case of $\Lambda_M^\Omega - \sqrt{\omega^2 + M^2} + i\mu$ follow similarly. □

5.4. **Limiting absorption principle.** We show that any outgoing resonant state is an eigenfunction for Λ_M^Ω.

Lemma 5.14 (No purely outgoing states)**.** *Let $\omega \in \mathcal{I}_\omega$, $s_0 \in [0, 1]$. Assume on $\partial\Omega$ we have*

$$u \in Z^{-\infty,-\infty,0}_{-L_0+0,-\frac{1}{2}+0,-\infty},\ (\Lambda_M^\Omega - \sqrt{\omega^2 + M^2})u \in Z^{\infty,\infty,s_0-1}_{L_0-0,\infty,\infty},\ \Im\langle(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2})u, u\rangle \geq 0. \tag{5.94}$$

Then we have

$$u \in Z^{\infty,\infty,s_0}_{L_0-0,\infty,\infty}(\partial\Omega) \subset L^2(\partial\Omega). \tag{5.95}$$

Proof. 1. The Fredholm estimate, Proposition 5.12, together with the assumptions on u and $(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2})u$, implies $u \in Z^{\infty,\infty,s_0}_{L_0-0,\infty,-\frac{1}{2}-0}$. It remains to show $\operatorname{WF} u \cap \Sigma^+(\omega) = \emptyset$ under the extra sign condition. To do this, we follow the idea in [DZ17, Lemma 2.3]. Throughout the proof, we will use the coordinate (ρ, y, ξ) where ρ is the boundary defining function of the scattering manifold and y the cross-section variable, and ξ is the dual variable to x.

Let $\varphi_0, \varphi, \tilde\varphi \in C_c^\infty((0,\infty)_\rho)$ that $[\frac{1}{9}, 9] \cap \operatorname{supp}(1-\varphi) = \operatorname{supp}\varphi_0 \cap \operatorname{supp}(1-\varphi) = \operatorname{supp}\varphi \cap \operatorname{supp}(1-\tilde\varphi) = \emptyset$. Let $\chi, \tilde\chi, \tilde{\tilde\chi} \in C_c^\infty([0,\infty)_\rho)$ with $\operatorname{supp}\tilde\varphi \cap \operatorname{supp}(1-\chi) = \operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde\chi) = \operatorname{supp}\tilde\chi \cap \operatorname{supp}(1-\tilde{\tilde\chi}) = \emptyset$. Fix $\epsilon_0 > 0$ and denote by $S^{s,\mathrm{comp}}$ those symbols in $S^{s,-\infty}$ with compact support in x, and $S^{\mathrm{comp},l}$ analogously. We will prove that for all $a(x,\xi) \in S^{0,0}$, with $\operatorname{supp} a \subset \{(x,\xi) : x \notin \operatorname{supp}\kappa, |\xi| > \epsilon_0\}$, there exists $b \in S^{0,0}$ with $\operatorname{supp} b \subset \{(x,\xi) : x \notin \operatorname{supp}\kappa, |\xi| > \epsilon_0\}$ such that for any $N > 0$, uniformly for $h > 0$ small,

$$\|\varphi_0(h^{-1}\rho)\operatorname{Op}(a)u\| \leq Ch^{\frac{1}{2}}\|\tilde\varphi(h^{-1}\rho)\operatorname{Op}(b)u\| + O(h^N\|(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2})u\|_{Z^{\infty,\infty,s_0}_{L_0-0,\infty,\infty}}). \tag{5.96}$$

Once we showed it, the iterative use of (5.96) gives for any $N > 0$,

$$\|\varphi_0(h^{-1}\rho)\operatorname{Op}(a)u\| \leq Ch^N(\|u\|_{H^{-N,-N}} + \|(\Lambda_M^\Omega - \sqrt{\omega^2 + M^2})u\|_{Z^{\infty,\infty,s_0}_{L_0-0,\infty,\infty}}) \tag{5.97}$$

uniformly in $h > 0$ small. This implies $\|\operatorname{Op}(a)u\|_{H^{0,N}}$ is bounded by the same bounds on the right and $\operatorname{WF} u \cap \Sigma^+(\omega) = \emptyset$ as desired.

2. We remind the reader that the neighborhood $V^+(\omega)$ of $\Sigma^+(\omega)$ was defined at the beginning of Section 5.3. Consider a function $f_1 \in S^{\mathrm{comp},1}$ such that

$$C_0^{-1}\rho^{-1} \leq f_1 \leq C_0\rho^{-1},\ H_p f_1 \geq C_0^{-1}\rho\tilde\varphi \text{ on } V^+(\omega). \tag{5.98}$$

Let $f_2 \in C_c^\infty(\mathbb{R})$ with

$$0 \notin \operatorname{supp}(1-f_2),\ f_2' \le 0 \text{ on } [0,\infty),\ f_2' < 0 \text{ on } [\frac{1}{2}C_0^{-1}, 2C_0],\ \operatorname{supp} f_2' \cap [0,\infty) \subset [\frac{1}{4}C_0^{-1}, 4C_0]. \tag{5.99}$$

Now construct the escape function

$$f = (1-f_2)(hf_1)\psi_\omega(\xi) \in S^{0,0},\ \operatorname{supp} f \subset \{0 \le \rho \le 4C_0^2 h, |\xi| > \frac{1}{2}\epsilon_0\}. \tag{5.100}$$

We have uniformly in $h > 0$ small that

$$H_p f = -h\psi_\omega(\xi) f_2'(hf_1) H_p f_1 + (1-f_2)(hf_1) H_p \psi_\omega(\xi), \tag{5.101}$$

the first term of which is supported inside $\{\frac{1}{4}C_0^{-2}h \le \rho \le 4C_0^2 h, |\xi| > \frac{1}{2}\epsilon_0\}$, disjoint from $\operatorname{supp}(1-\varphi_0(h^{-1}\rho))$, and the second term of which is supported away from $V^+(\omega)$. We have

$$H_p f > 0 \text{ on } V^+(\omega) \cap \{\rho \in [\frac{1}{2}h, 2h]\},\ H_p f \ge 0 \text{ on } V^+(\omega). \tag{5.102}$$

Fix $C_0 = \sqrt{2}$. Let

$$F = \frac{1}{2}\big(\operatorname{Op} f + (\operatorname{Op} f)^*\big),\ F^* = F,\ \operatorname{WF} F \subset \{0 \le \rho \le 8h, |\xi| > \frac{1}{2}\epsilon_0\},\ F = \chi(h^{-1}\rho) F \chi(h^{-1}\rho), \tag{5.103}$$

for $h > 0$ small since the quantization procedure does not allow off-diagonal communication further than distance $\frac{1}{4}$. Note $\rho \sim h$ on $\operatorname{supp}\varphi(h^{-1}\rho)$. There then exists $q \in S^{0,0}$ with $Q = \operatorname{Op}(q)$,

$$\operatorname{supp} q \subseteq \{x \notin \operatorname{supp}\kappa, |\xi| > \epsilon_0\},\ \operatorname{supp} q \cap V^+(\omega) = \emptyset, \tag{5.104}$$

such that for some $C > 0$,

$$\frac{1}{2}H_p f + \rho\varphi(h^{-1}\rho)^2 |q(\rho, y, \xi)|^2 \ge C^{-1}\rho\varphi_0(h^{-1}\rho)^2 |a(\rho, y, \xi)|^2. \tag{5.105}$$

Note that $\operatorname{WF} Q$ is away from $\operatorname{WF} u$ implies $Qu \in \mathcal{S}$ and $\varphi(h^{-1}\rho)Qu = O_{H^{N,N}}(h^\infty \|u\|_{H^{-N,-N}})$. Now we choose $b \in S^{0,0}$ with $B = \operatorname{Op}(b)$ such that

$$\operatorname{supp} H_p f \cup \operatorname{supp} q \cup \operatorname{supp} a \subset \{(x,\xi) : b(x,\xi) = 1\},\ \operatorname{supp} b \cap \operatorname{supp}\kappa = \emptyset. \tag{5.106}$$

Note $[P, \mathrm{I} - F] \in \Psi^{0,-1}$ even when $\mathrm{I} - F$ is in general not in our calculus. Apply the sharp Gårding inequality with $\frac{1}{2i}[P, \mathrm{I}-F] + Q^*\varphi^2(h^{-1}\rho)\rho Q - C^{-1}A^*\varphi_0^2(h^{-1}\rho)\rho A$ on $\tilde\varphi(h^{-1}\rho)Bu$ to obtain

$$\|\rho^{\frac{1}{2}}\varphi_0(h^{-1}\rho)A\tilde\varphi(h^{-1}\rho)Bu\|^2 \le C\|\rho^{\frac{1}{2}}\varphi(h^{-1}\rho)Q\tilde\varphi(h^{-1}\rho)Bu\|^2 + \frac{C}{2i}\langle [P, \mathrm{I}-F]\tilde\varphi(h^{-1}\rho)Bu, \tilde\varphi(h^{-1}\rho)Bu\rangle$$

$$+ C\|\tilde\varphi(h^{-1}\rho)Bu\|^2_{H^{-\frac{1}{2},-1}} \le -\frac{C}{2i}\langle [P,F]u, u\rangle + Ch^2\|\tilde\varphi(h^{-1}\rho)Bu\|^2 + O(h^\infty\|u\|^2_{H^{-N,-N}}) \tag{5.107}$$

since $[P,F](\mathrm{I} - \tilde\varphi(h^{-1}\rho)B) \in h^\infty\Psi^{-\infty,-\infty}$ uniformly . This implies

$$\|\varphi_0(h^{-1}\rho)Au\|^2 \le -\frac{C}{2i}\langle [P,F]u, u\rangle + Ch\|\tilde\varphi(h^{-1}\rho)Bu\|^2 + O(h^\infty\|u\|^2_{H^{-N,-N}}) \tag{5.108}$$

uniformly in $h > 0$ small.

3. It remains to estimate $\langle [P,F]u,u\rangle$. Note $F=\chi(h^{-1}\rho)F\chi(h^{-1}\rho)$ and $(\Lambda_M^\Omega-\sqrt{\omega^2+M^2})u\in Z^{\infty,\infty,s_0-1}_{L_0-0,\infty,\infty}$. This implies $\|F(\Lambda_M^\Omega-\sqrt{\omega^2+M^2})u\|=O(h^\infty)$. On another hand, noting $\Im\langle(\Lambda_M^\Omega-\sqrt{\omega^2+M^2})u,u\rangle\geq 0$, we have

$$\begin{aligned}O(h^\infty)=\Im\langle F(\Lambda_M^\Omega-\sqrt{\omega^2+M^2})u,u\rangle&=\Im\langle(\Lambda_M^\Omega-\sqrt{\omega^2+M^2})u,u\rangle-\Im\langle(\mathrm{I}-F)(\Lambda_M^\Omega-\sqrt{\omega^2+M^2})u,u\rangle\\&\geq-\frac{1}{2i}\big(\langle[P,F]u,u\rangle+\langle Fu,(\mathcal{E}')^*u\rangle-\langle\mathcal{E}u,Fu\rangle\big)=-\frac{1}{2i}\langle[P,F]u,u\rangle+O(h^\infty),\end{aligned}\tag{5.109}$$

where we used $(\mathrm{I}-F)u\in H^{s_0,L_0-0}\subset L^2$ implies

$$-\Im\langle(\mathrm{I}-F)(\Lambda_M^\Omega-\sqrt{\omega^2+M^2})u,u\rangle=-\frac{1}{2i}\langle[\Lambda_M^\Omega,F]u,u\rangle\tag{5.110}$$

and we used Proposition 4.22 for

$$\begin{aligned}\Lambda_M^\Omega F=\Lambda_M^\Omega\chi(h^{-1}\rho)F\chi(h^{-1}\rho)&=\tilde\chi(h^{-1}\rho)P\chi(h^{-1}\rho)F\chi(h^{-1}\rho)+\mathcal{E}'\chi(h^{-1}\rho)F\chi(h^{-1}\rho)\\&=PF+\mathcal{E}'F+O(h^\infty\Psi^{-\infty,-\infty}),\end{aligned}\tag{5.111}$$

since $(1-\tilde\chi)(h^{-1}\rho)P\chi(h^{-1}\rho)F\chi(h^{-1}\rho)\in h^\infty\Psi^{-\infty,-\infty}$. The $F\Lambda_M^\Omega$ term is addressed similarly. Here the error $O(h^\infty)$ comes from

$$\begin{aligned}|\langle\mathcal{E}u,Fu\rangle|&\leq C\|Fu\|_{H^{-N,-N}}\|\chi(h^{-1}\rho)\tilde\psi_\omega(D)\chi(h^{-1}\rho)\mathcal{E}u\|_{H^{N,N}}+Ch^N\|u\|_{H^{-N,-N}}\|\chi(h^{-1}\rho)\mathcal{E}u\|_{H^{-N,-N}}\\&\leq Ch^N\|u\|^2_{Z^{\infty,\infty,s_0}_{-L_0+0,\infty,-\infty}},\end{aligned}\tag{5.112}$$

where $F=\chi(h^{-1}\rho)\tilde\psi_\omega(D)\chi(h^{-1}\rho)F+O(h^\infty\Psi^{-\infty,-\infty})$. For $\langle Fu,(\mathcal{E}')^*u\rangle$, note in Proposition 4.22, $(\mathcal{E}')^*=\mathcal{E}_0^*\tilde\chi(h^{-1}\rho)+\mathcal{E}_0'$ where $\langle Fu,\mathcal{E}_0'u\rangle$ can be estimated in a way similar to (5.112). It remains to estimate $\langle Fu,\mathcal{E}_0^*\tilde\chi(h^{-1}\rho)u\rangle$. Proposition 3.12 implies $\Gamma^{N,0}_{N,\delta}\tilde\chi(h^{-1}\rho)\mathcal{E}_0\chi(h^{-1}\rho)\tilde\psi_\omega(D)\chi(h^{-1}\rho)\Gamma^{N,0}_{N,\delta}$ is bounded on $L^2(\partial\Omega)$ and so is $\Gamma^{N,0}_{N,\delta}\chi(h^{-1}\rho)\tilde\psi_\omega(D)\chi(h^{-1}\rho)\mathcal{E}_0^*\tilde\chi(h^{-1}\rho)\Gamma^{N,0}_{N,\delta}$. Thus we have

$$\begin{aligned}\|\Gamma^{N,0}_{N,\delta}\chi(h^{-1}\rho)\tilde\psi_\omega(D)\chi(h^{-1}\rho)\mathcal{E}_0^*\tilde\chi(h^{-1}\rho)u\|&\leq C\|\psi_\omega(D)\tilde{\tilde\chi}(h^{-1}\rho)u\|_{H^{-N,-N}}\\&\quad+C\|(1-\psi_\omega(D))\tilde{\tilde\chi}(h^{-1}\rho)u\|_{L^2}\leq C\|u\|_{Z^{0,0,0}_{0,-\infty,-\infty}}\end{aligned}\tag{5.113}$$

and thus

$$|\langle Fu,\mathcal{E}_0^*\tilde\chi(h^{-1}\rho)u\rangle|\leq C\|\Gamma^{-N,0}_{-N,\delta}Fu\|\|\Gamma^{N,0}_{N,\delta}\chi(h^{-1}\rho)\tilde\psi_\omega(D)\chi(h^{-1}\rho)\mathcal{E}_0^*\tilde\chi(h^{-1}\rho)u\|\leq Ch^{N\delta}\|u\|^2_{Z^{0,0,0}_{0,-\infty,-\infty}}.\tag{5.114}$$

This justifies $|\langle Fu,(\mathcal{E}')^*u\rangle-\langle\mathcal{E}u,Fu\rangle|=O(h^\infty)$ in (5.109), and thus $\langle[P,F]u,u\rangle=O(h^\infty)$. Bring it back in (5.108) to see (5.96) as desired. □

Now we show that there are finitely many eigenvalues for Λ_M^Ω in any given $\mathcal{I}_\omega$.

Lemma 5.15 (Eigenvalues are isolated)**.** *Let $M_0>0$. Denote the point spectrum of Λ_M^Ω by $\mathrm{Spec}_{\mathrm{pp}}(\Lambda_M^\Omega)$. Then*

$$|\{\sqrt{\omega^2+M^2}:\omega\in\mathcal{I}_\omega\}\cap\mathrm{Spec}_{\mathrm{pp}}(\Lambda_M^\Omega)|<\infty\tag{5.115}$$

for each $M\in[0,M_0]$. Without loss of generality, assume

$$\{\sqrt{\omega^2+M^2}:\omega\in\partial\mathcal{I}_\omega\}\cap\mathrm{Spec}_{\mathrm{pp}}(\Lambda_M^\Omega)=\emptyset\tag{5.116}$$

for all $M \in [0, M_0]$. Then

$$\left\{(\sqrt{\omega^2+M^2}, M) : \sqrt{\omega^2+M^2} \in \mathrm{Spec}_{\mathrm{pp}}(\Lambda_M^\Omega), M \in [0,M_0], \omega \in \mathcal{I}_\omega\right\} \subset \mathbb{R} \times [0, M_0] \tag{5.117}$$

is closed. We can define Π_M as the projection operator

$$\Pi_M : Z^{-\infty,-\infty,0}_{-L_0+0,-\infty,-\infty} \to \bigoplus_\omega \{\ker_{L^2}(\Lambda_M^\Omega - \sqrt{\omega^2+M^2}) : \omega \in \mathcal{I}_\omega\} \subset Z^{\infty,\infty,1}_{L_0-0,\infty,\infty}, \tag{5.118}$$

and $\Pi_M^\perp = \mathrm{I} - \Pi_M : Z^{s,l,s_0}_{l_0,l^-,l^+} \to Z^{s,l,s_0}_{l_0,l^-,l^+}$.

Remark 5.16. For $M_j \to M$, one can prove $\Pi_M^\perp \Pi_{M_j} \to 0$, but to show $\Pi_{M_j}^\perp \Pi_M \to 0$ and thus $\Pi_{M_j} \to \Pi_M$ would require more. We will not pursue this direction in the paper.

Proof. 1. Let $\omega \in \mathcal{I}_\omega$, $M \geq 0$. Assume against contradiction that there is a sequence of distinct $\omega_j \in \mathcal{I}_\omega$ with $\omega_j \to \omega$, $\|u_j\|_{L^2} \equiv 1$, $(\Lambda_M^\Omega - \sqrt{\omega_j^2+M^2})u_j = 0$. Then $\{u_j\}$ is an orthonormal sequence, and $u_j \rightharpoonup 0$ weakly in L^2, hence $u_j \to 0$ in the norm topology in $H^{0-0,0-0}$ via some subsequences. On another hand, Proposition 5.12 implies

$$\|u_j\|_{Z^{N,N,1}_{L_0-0,N,-\frac{1}{2}-0}} \leq C\|u_j\|_{H^{-N,-N}}, \ \|u_j\|_{Z^{N,N,1}_{L_0-0,-\frac{1}{2}-0,N}} \leq C\|u_j\|_{H^{-N,-N}}. \tag{5.119}$$

Note here we have both estimates since Lemma 5.14 implies $u_j \in Z^{\infty,\infty,1}_{L_0-0,\infty,\infty}$. Then we have $\|u_j\|_{L^2} = o(1)$, yielding the desired contradiction for the finiteness of point spectrum.

2. Let $\sqrt{\omega^2+M^2} \notin \mathrm{Spec}_{\mathrm{pp}}(\Lambda_M^\Omega)$. Assume against contradiction that there is a sequence of distinct $M_j \to M$, $\omega_j \to \omega$ inside $\mathcal{I}_\omega$, $\|u_j\|_{L^2} \equiv 1$, $(\Lambda_{M_j}^\Omega - \sqrt{\omega_j^2+M_j^2})u_j = 0$. There then exists $u \in L^2$ that $u_j \rightharpoonup u$ weakly in L^2 and $u_j \to u$ in $H^{0-0,0-0}$ via some subsequences. Since Λ_M^Ω is continuous in M, we have

$$(\Lambda_M^\Omega - \sqrt{\omega^2+M^2})u_j = (\Lambda_M^\Omega - \Lambda_{M_j}^\Omega)u_j - (\sqrt{\omega^2+M^2} - \sqrt{\omega_j^2+M_j^2})u_j = o_{Z^{\infty,\infty,0}_{L_0-0,\infty,\infty}}(1). \tag{5.120}$$

This implies $(\Lambda_M^\Omega - \sqrt{\omega^2+M^2})u = 0$ and thus $u = 0$. Similar to (5.119) we have $\|u_j\|_{L^2} = o(1)$. The contradiction gives the stated closed graph property of $(\sqrt{\omega^2+M^2}, M)$. □

Lemma 5.17 (Uniform absorption resolvent)**.** *Let Ω be an AC Lipschitz domain. Let $M_0 > 0$. Let $s, l \in \mathbb{R}$, $s_0 \in (0,1]$. Let $|l_0| < L_0$. For each fixed $M \in [0, M_0]$, we have the following estimates:*

(1) For $l^- > -\frac{1}{2}$, $l^+ < -\frac{1}{2}$, we have

$$\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \sqrt{\omega^2+M^2} - i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} \tag{5.121}$$

uniformly for all $u \in Z^{-\infty,-\infty,0}_{-L_0+0,-\frac{1}{2}+0,-\infty}(\partial\Omega)$, $\omega \in \mathcal{I}_\omega$, $\mu \geq 0$ small.

(2) For $l^- < -\frac{1}{2}$, $l^+ > -\frac{1}{2}$, we have

$$\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}} \leq C\|(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \sqrt{\omega^2+M^2} + i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}} \tag{5.122}$$

uniformly for all $u \in Z^{-\infty,-\infty,0}_{-L_0+0,-\infty,-\frac{1}{2}+0}(\partial\Omega)$, $\omega \in \mathcal{I}_\omega$, $\mu \geq 0$ small.

(3) If additionally

$$\{\sqrt{\omega^2+M^2}:\omega\in\mathcal{I}_\omega\}\cap \mathrm{Spec}_{\mathrm{pp}}(\Lambda_M^\Omega)=\emptyset,\ \textit{for all } M\in[0,M_0], \tag{5.123}$$

then the above estimates are also uniform in $M\in[0,M_0]$.

Proof. Assume (5.121) does not hold and there are $\|u_j\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}}\equiv 1$ with

$$(\Pi_M^\perp\Lambda_M^\Omega\Pi_M^\perp-\sqrt{\omega_j^2+M^2}-i\mu_j)u_j=o_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}}(1),\ \omega_j\to\omega \text{ within } \mathcal{I}_\omega,\ \mu_j\to\mu \tag{5.124}$$

This implies $\Pi_M u_j=o(1)$ in $Z^{\infty,\infty,1}_{L_0-0,\infty,\infty}$ and thus

$$(\Lambda_M^\Omega-\sqrt{\omega_j^2+M^2}-i\mu_j)\Pi_M^\perp u_j=o_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}}(1). \tag{5.125}$$

Via a subsequence there is $u\in Z^{s,l,s_0}_{l_0,l^-,l^+}$ that $\Pi_M^\perp u_j\rightharpoonup u$ weakly in $Z^{s,l,s_0}_{l_0,l^-,l^+}$ and $\Pi_M^\perp u_j\to u$ in the norm of $Z^{s-0,l-0,s_0-0}_{l_0-0,l^--0,l^+-0}$. The limit u solves $(\Lambda_M^\Omega-\sqrt{\omega^2+M^2}-i\mu)u=0$ and $\Pi_M u=0$, and thus $u=0$. Now the Fredholm estimates, Proposition 5.12 implies $\Pi_M^\perp u_j=o(1)$ in $Z^{s,l,s_0}_{l_0,l^-,l^+}$. This contradicts the normalization of u_j, and yields the desired estimates at fixed M. For uniformity in M, note the extra assumption yields $\Pi_M=0$ and $\Pi_M^\perp=\mathrm{I}$ and the Fredholm estimates are uniform in M and the proof is similar. □

We show the limiting absorption principle.

Proposition 5.18 (Limiting absorption principle)**.** *Let* Ω *be an AC Lipschitz domain. Let* $\omega\in\mathcal{I}_\omega$, $M_0>0$. *Let* $M\in[0,M_0]$. *Let* $s,l\in\mathbb{R}$, $s_0\in(0,1)$. *Let* $|l_0|<L_0$, $l^->-\frac12$, $l^+<-\frac12$. *Then the following are true for each* $f\in Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}(\partial\Omega)$*:*

(1) The limit

$$u=(\Pi_M^\perp\Lambda_M^\Omega\Pi_M^\perp-\sqrt{\omega^2+M^2}-i0)^{-1}f=\lim_{\mu\to0^+}(\Pi_M^\perp\Lambda_M^\Omega\Pi_M^\perp-\sqrt{\omega^2+M^2}-i\mu)^{-1}f \tag{5.126}$$

as a limit in $Z^{s-0,l-0,s_0-0}_{l_0-0,l^--0,l^+-0}$ *exists.*

(2) The limit $u\in Z^{s,l,s_0}_{l_0,l^-,l^+}$, *and is the unique solution in* $Z^{s,l,s_0}_{l_0,l^-,l^+}$ *to*

$$(\Pi_M^\perp\Lambda_M^\Omega\Pi_M^\perp-\sqrt{\omega^2+M^2})u=f. \tag{5.127}$$

(3) The limit u *depends continuously in* $\omega\in\mathcal{I}_\omega$ *for each fixed* $M\in[0,M_0]$.

(4) If additionally,

$$\{\sqrt{\omega^2+M^2}:\omega\in\mathcal{I}_\omega\}\cap \mathrm{Spec}_{\mathrm{pp}}(\Lambda_M^\Omega)=\emptyset,\ \textit{for all } M\in[0,M_0], \tag{5.128}$$

then the limit u *depends continuously in both* $\omega\in\mathcal{I}_\omega$ *and* $M\in[0,M_0]$.

(5) For any $k\in\mathbb{N}$,

$$\partial^k_{\omega_M}(\Pi_M^\perp\Lambda_M^\Omega\Pi_M^\perp-\omega_M-i0)^{-1}f=k!(\Pi_M^\perp\Lambda_M^\Omega\Pi_M^\perp-\omega_M-i0)^{-k-1}f\in Z^{s,l,s_0}_{l_0,l^-,l^+-k}, \tag{5.129}$$

for any $f\in Z^{s-1-k,l,s_0-1}_{l_0,l^-+1+k,l^++1}$ *and* $\omega_M=\sqrt{\omega^2+M^2}$.

Proof. 1. We show $u \in Z^{s,l,s_0}_{l_0,l^-,l^+}$, the solution to $(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \sqrt{\omega^2+M^2})u = f$, is unique in $Z^{s,l,s_0}_{l_0,l^-,l^+}$. Let $v \in Z^{-\infty,-\infty,0}_{-L_0+0,-\frac{1}{2}+0,-\infty}$, the difference between any two such solutions, solve $(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \sqrt{\omega^2+M^2})v = 0$. Lemma 5.17 implies $v = 0$ and u must be unique in $Z^{s,l,s_0}_{l_0,l^-,l^+}$.

2a. Let a sequence $\mu \to 0^+$ and $\omega_\mu \to \omega$. Lemma 5.13 implies the μ-sequence

$$u_\mu = (\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \sqrt{\omega_\mu^2+M^2} - i\mu)^{-1} f = (\Lambda_M^\Omega - \sqrt{\omega_\mu^2+M^2} - i\mu)^{-1}\Pi_M^\perp f - (\sqrt{\omega_\mu^2+M^2}+i\mu)^{-1}\Pi_M f \tag{5.130}$$

exists in $Z^{s,l,s_0}_{l_0,l^-+1,l^++1}$ and are uniformly bounded in $Z^{s,l,s_0}_{l_0,l^-,l^+}$ per Lemma 5.17. There then exists $u \in Z^{s,l,s_0}_{l_0,l^-,l^+}$ that $u_\mu \rightharpoonup u$ weakly in $Z^{s,l,s_0}_{l_0,l^-,l^+}$ via a subsequence.

2b. We claim $(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \sqrt{\omega^2+M^2})u = f$. Indeed, for any $\phi \in C_c^\infty$ we have $(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \sqrt{\omega^2+M^2})\phi \in Z^{\infty,\infty,0}_{L_0-0,\infty,\infty}$ and

$$\begin{aligned}\langle (\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \omega_M)u, \phi\rangle &= \langle u, (\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \omega_M)\phi\rangle = \lim_{\mu\to 0^+}\langle u_\mu, (\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \omega_M)\phi\rangle \\ &= \lim_{\mu\to 0^+}\langle (\Lambda_M^\Omega - \omega_M)u_\mu, \phi\rangle - \langle \Lambda_M^\Omega \Pi_M u_\mu, \phi\rangle = \langle f, \phi\rangle, \end{aligned} \tag{5.131}$$

as desired. Then u is the unique limit of u_μ with respect to any sequence of $\mu \to 0$ via Step 1. This justifies (5.126), (5.127) and the continuity in ω.

2c. The continuity in $M \in [0, M_0]$ is immediate since the additional assumption gives uniform estimates in Lemma 5.17.

3. We show (5.129). For $\mu > 0$, the resolvent identity gives

$$\begin{aligned}(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \omega_M' - i\mu)^{-1} &= (\omega_M' - \omega_M)(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \omega_M - i\mu)^{-1}(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \omega_M' - i\mu)^{-1} \\ &\quad + (\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \omega_M - i\mu)^{-1}. \end{aligned} \tag{5.132}$$

Taking $\omega_M' \to \omega_M$ gives $\partial_{\omega_M}(\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \omega_M - i\mu)^{-1} = (\Pi_M^\perp \Lambda_M^\Omega \Pi_M^\perp - \omega_M - i\mu)^{-2}$. Take $\mu \to 0$ and iterate for (5.129). □

Remark 5.19. All constants in this section can depend on Ω in the following way: the upper bound on uniform Lipschitz characters of $\partial\Omega$, the upper bound on the choice of $r_0 \geq 0$, and the lower bound on the radial Hardy constant $\mathfrak{r}_\Omega^{r_0}$, the upper bound on the diameter of $\partial\Omega \setminus \partial\tilde{\Omega}$ in Remark 3.11, for each multi-index α, the upper bounds on the derivatives of the metric g_+ in (3.11), and lower bounds on $|\det g_+|$.

6. The scattering matrix

This section concerns the scattering matrix for the Dirichlet-to-Neumann map $\Lambda_M^{W_Y}$ on the boundary ∂W_Y, boundary of the AC Lipschitz domain W_Y with smooth section Y. We remind the readers that $L_0 = \mathfrak{r}_{W_Y}^{r_0} + \frac{1}{2}$.

The construction of the scattering matrix relies heavily on the microlocal parametrix for the DtN map from Section 3. Indeed, once we have this structure, the construction of the scattering matrix follows familiar lines (see e.g. [DZ19a, Mel94]). In lemma 6.1, we show that outgoing

functions solving $(\Lambda_M^{W_Y} - \sqrt{\omega^2 + M^2})u = 0$ near the radial sink have the form $a(y,\rho)e^{i\sqrt{\omega^2+M^2}/\rho}$ near $|x| = \infty$. To do this, we commute pseudodifferential operators defining Σ_+ through the DtN map, eventually showing that u has Lagrangian regularity and Σ^+. We are then able to extract the asymptotic expansion by applying $\Lambda_M^{W_Y} - \sqrt{\omega^2 + M^2}$ to u and computing its symbol as a Lagrangian distribution.

To construct the scattering matrix, given incoming data $a_-(y)$, we then construct an incoming function $u^- = \tilde{a}_-(y,\rho)e^{-i\sqrt{\omega^2+M^2}/\rho}$ supported near $\rho = 0$ that solves $)(\Lambda_M^{W_Y} - \sqrt{\omega^2 + M^2})u^- = 0$ microlocally near the source. An application of the outgoing resolvent together with the asymptotic expansion from Lemma 6.1 then yields the scattering matrix.

6.1. **Asymptotic expansions of outgoing solutions.** We show that we can decompose resonant states of $\Lambda_M^{W_Y}$ into plane waves.

Lemma 6.1 (Plane wave expansion)**.** *Let $W_Y \subset \mathbb{R}^d$ be an AC Lipschitz domain with smooth cross-section Y. Let $M \geq 0$ and $\omega > 0$. Let microlocalizer $B^+ \in \Psi^{0,0}(\partial W_Y)$ be that $\mathrm{WF}(B^+) \subset \{|\xi| > c\omega\}$ for some $c > 0$, $\Sigma^+(\omega) \subset \mathrm{Ell}(B^+)$. Let ω-family $u \in Z^{-\infty,-\infty,0}_{-L_0+0,-\frac{1}{2}+0,-\infty}(\partial W_Y;\omega)$ with*

$$(\Lambda_M^{W_Y} - \sqrt{\omega^2 + M^2})u \in Z^{\infty,\infty,-1}_{-L_0+0,\infty,\infty}(\partial W_Y;\omega),\ B^+(\Lambda_M^{W_Y} - \sqrt{\omega^2 + M^2})u = O(\omega^{-\infty}\mathcal{S}), \quad (6.1)$$

the second condition is uniform in $\omega > 0$ large. Then there exist $\rho_0 > 0$, constant $\alpha(\omega) \geq 1$, and $\tilde{a}^+(\rho, y;\omega) \in S^0(\partial W_Y)$ that, for each $A \in \Psi^{0,0}(\partial W_Y)$ with $\mathrm{WF}\, A \subset \{\rho < \rho_0, |\xi| > c\omega\}$ for some $c > 0$, we have

$$(Au)(\rho, y) = \alpha(\omega)\big(A\rho^{\frac{d}{2}-1}\tilde{a}^+(\rho, y;\omega)e^{i\omega/\rho} + O(\omega^{-\infty}\mathcal{S})\big), \quad (6.2)$$

in local coordinates y on ∂Y, uniformly in $\omega > 0$ large. Given $\alpha(\omega)$, such $\tilde{a}^+ \in S^0(\partial W_Y)$ is unique in $\{\rho < \rho_0\}$ modulo $\omega^{-\infty}\mathcal{S}(\partial W_Y)$. Moreover, there exist $a^+(y;\omega) \in S^0(\partial Y)$, $a_k^+(\rho, y;\omega) \in S^0(\partial W_Y)$ for $k \geq 1$ such that

$$\tilde{a}^+(\rho, y;\omega) = a^+(y;\omega) + \sum_{k=1}^{\infty} \omega^{-k}\rho^k a_k^+(\rho, y;\omega) + O(\omega^{-\infty}\mathcal{S}). \quad (6.3)$$

Proof. We mostly follow the strategy of [DZ19a, Lemma 4.1] and [Mel94] with a small modification in Step 2, where we commute $\rho\Delta_y$ with the operator, rather than Δ_y in coordinates where Σ^+ is given by $\{(0, y, -\omega, 0)\}$.

1. We set up some notations. Since $A \in \Psi^{0,0}(\partial W_Y)$, there is $\chi \in S^0(\partial W_Y)$ where ∂W_Y is smooth over $\operatorname{supp}\chi$ and $A = A\chi$. Let $\tilde{\chi}, \tilde{\tilde{\chi}} \in S^0(\partial W_Y)$ such that $\chi \prec \tilde{\chi} \prec \tilde{\tilde{\chi}}$, and ∂W_Y is smooth over $\operatorname{supp}\tilde{\tilde{\chi}}$. Let $\sigma(B^+) = b^+$ and we assume without loss of generality $\mathrm{WF}(\mathrm{I} - B^+) \cap \Sigma^+(\omega) = \emptyset$. Throughout the proof, we ask any microsupport separation to be of distance at least $c\omega$ in ξ for some $c > 0$. Consider microlocalizer $B \in \Psi^{0,0}$ near $\Sigma^+(\omega)$,

$$\mathrm{WF}(\mathrm{I} - B) \cap \Sigma^+(\omega) = \emptyset,\ \mathrm{WF}(\mathrm{I} - B^+) \cap \mathrm{WF}\, B = \emptyset. \quad (6.4)$$

For convenience, let the microlocal representation of $\Lambda_M^{W_Y} - \sqrt{\omega^2 + M^2}$ near the sink $\Sigma^+(\omega)$ be

$$P_\omega^+ = B^+\chi(P - \sqrt{\omega^2 + M^2})\tilde{\chi} \in \Psi^{1,0},\ p_\omega^+ = \sigma(P_\omega^+) = b^+\chi(\sqrt{|\xi|_g^2 + M^2} - \sqrt{\omega^2 + M^2}). \quad (6.5)$$

We know by assumption $\mathrm{WF}^{-\infty,-\frac{1}{2}+0}u\cap\Sigma^-(\omega)=\emptyset$. The Fredholm estimate, Proposition 5.12, then implies $u\in Z^{\infty,\infty,0}_{-L_0+0,\infty,-\frac{1}{2}-0}$, and thus $B\tilde{\tilde{\chi}}u\in H^{\infty,-\frac{1}{2}-0}$, for each ω not uniformly. We normalize them by $\alpha(\omega)=\max(\|B\tilde{\tilde{\chi}}u\|_{H^{0,-\frac{1}{2}-0}},1)$, and denote $u^+=\alpha(\omega)^{-1}B\tilde{\tilde{\chi}}u\in H^{\infty,-\frac{1}{2}-0}$. Note Proposition 4.22 implies

$$P^+_\omega u^+=\alpha(\omega)^{-1}\big(B^+\chi(\Lambda^{W_Y}_M-\sqrt{\omega^2+M^2})u-B^+\chi\mathcal{E}_Mu-B^+\chi(P-\sqrt{\omega^2+M^2})\tilde{\chi}(\mathrm{I}-B)\tilde{\tilde{\chi}}u\big)=O(\omega^{-\infty}\mathcal{S}),\tag{6.6}$$

where we used that $\mathrm{WF}(B^+\chi(P-\sqrt{\omega^2+M^2})\tilde{\chi}(\mathrm{I}-B)\tilde{\tilde{\chi}})$ is disjoint from $\Sigma^+(\omega)$ and $\{\xi=0\}$. From this point onwards, all estimates in this proof are uniform in $\omega>0$ large.

2. We show that u near $\Sigma^+(\omega)$ is a Lagrangian distribution associated with the sink $\Sigma^+(\omega)=\{(0,y,-\omega,0)\}$, that is to claim for all $k\geq 0$ and all $L_1,\dots,L_k\in\Psi^{1,1}$ such that $\omega^{-1}L_j$ is uniformly bounded in ω, $\sigma(L_j)|_{\Sigma^+(\omega)}=0$ for $1\leq j\leq k$, we have

$$L_1\cdots L_ku^+\in H^{\infty,-\frac{1}{2}-0}.\tag{6.7}$$

Consider in local coordinates near $(0,y)$, the testing operators in $\Psi^{1,1}$ for $\Sigma^+(\omega)$,

$$V=(V_j)_{j=0}^{d-2},\ V_0=\rho^{-1}P^+_\omega,\ V_j=D_{y_j},\ 1\leq j\leq d-2.\tag{6.8}$$

They correspond to $S^{1,1}$-defining functions of $\Sigma^+(\omega)$,

$$v_0=\sigma(V_0)=\rho^{-1}(\sqrt{\tau^2+|\eta|^2_{g_\infty}+M^2}-\sqrt{\omega^2+M^2}),\ v_j=\sigma(V_j)=\rho^{-1}\eta_j.\tag{6.9}$$

In Step 2, we iterate in $(k,n)\in\mathbb{N}_0\times\mathbb{N}_0$ to claim for any $k'\leq k,n'\leq n$, any word β of length k',

$$\beta=(\beta_1,\dots,\beta_{k'}),\ 0\leq\beta_l\leq d-2\text{ for each }1\leq l\leq k',\tag{6.10}$$

we have, with the notation $V_\beta=V_{\beta_1}\cdots V_{\beta_{k'}}$,

$$(\rho\Delta_y)^{n'}V_\beta u^+\in H^{\infty,-\frac{1}{2}-0}.\tag{6.11}$$

Once we have shown (6.11), (6.7) follows immediately. Indeed, noting L_j has principal symbols vanishing on $\Sigma^+(\omega)$, we have

$$L_j=\sum_{l=0}^{d-2}O(\Psi^{0,0})V_l+O(\Psi^{0,0}),\tag{6.12}$$

and thus

$$L_1\cdots L_ku^+=\sum_{|\beta|\leq k}O(\Psi^{0,0})V_\beta u^+\in H^{\infty,-\frac{1}{2}-0},\tag{6.13}$$

by the case $(k,0)$ of (6.11). The rest of Step 2 is devoted to showing (6.11).

2a. We provide a priori estimates on commutators. We check near $\Sigma^+(\omega)$,

$$H_pv_0=if_0v_0,\ f_0=\frac{i\rho\tau}{\sqrt{\tau^2+|\eta|^2_{g_\infty}+M^2}},\tag{6.14}$$

$$H_pv_j=-\sum_{l,m=1}^{d-2}\frac{\eta_l(\partial_{y_j}(g_\infty)^{lm})\eta_m}{2\sqrt{\tau^2+|\eta|^2_{g_\infty}+M^2}},\ 1\leq j\leq d-2,\tag{6.15}$$

and thus we have for $1 \leq j \leq d-2$, modulo $O(\Psi^{1,0})$ supported away from $\Sigma^+(\omega)$,

$$[P_\omega^+, V_0] = F_0 V_0 + O(\Psi^{0,-1}),\ \sigma(F_0) = f_0 \in S^{0,-1}, \tag{6.16}$$

$$[P_\omega^+, V_j] = \sum_l O(\Psi^{0,-1}) V_l + O(\Psi^{0,-1}) = \sum_{l,m=1}^{d-2} O(\Psi^{-1,0})\rho^2 V_l V_m + O(\Psi^{0,-1}). \tag{6.17}$$

Noting $V_0 = \rho^{-1} P_\omega^+$, we have for each $1 \leq j, m \leq d-2$,

$$[V_0, V_j] = O(\Psi^{0,0}) V_0 + \sum_{l=1}^{d-2} O(\Psi^{0,0}) V_l + O(\Psi^{0,0}),\ [V_j, V_m] = \sum_{l=1}^{d-2} O(\Psi^{0,0}) V_l \tag{6.18}$$

modulo $O(\Psi^{1,1})$ supported away from $\Sigma^+(\omega)$, and implies with the same modulo class,

$$\mathrm{ad}_V^\gamma V_0 = O(\Psi^{0,0}) V_0 + \sum_{l=1}^{d-2} O(\Psi^{0,0}) V_l + O(\Psi^{0,0}). \tag{6.19}$$

Moreover, if $\gamma_j \neq 0$ for all j, then modulo $O(\Psi^{1,0})$ supported away from $\Sigma^+(\omega)$,

$$\mathrm{ad}_V^\gamma P_\omega^+ = \sum_{l,m=1}^{d-2} O(\Psi^{-1,0})\rho^2 V_l V_m + O(\Psi^{0,-1}). \tag{6.20}$$

We have near $\Sigma^+(\omega)$,

$$H_p(\rho^{-1}|\eta|_{g_\infty}^2) = \frac{\tau|\eta|_{g_\infty}^2 + \tau\rho\eta^t \cdot \partial_\rho g_\infty \cdot \eta}{\sqrt{\tau^2 + |\eta|_{g_\infty}^2 + M^2}} = \frac{\tau|\eta|_{g_\infty}^2 + O(\rho\tau|\eta|_{g_\infty}^2)}{\sqrt{\tau^2 + |\eta|_{g_\infty}^2 + M^2}}. \tag{6.21}$$

This implies, modulo $O(\Psi^{2,0})$ supported away from $\Sigma^+(\omega)$,

$$[\rho\Delta_y, P_\omega^+] = F_1 \rho\Delta_y + O(\Psi^{1,-1}),\ F_1 \in \Psi^{0,-1},\ \sigma(F_1) = \frac{i\rho\tau}{\sqrt{\tau^2 + |\eta|_{g_\infty}^2 + M^2}}. \tag{6.22}$$

Then for all $n \in \mathbb{N}$, we have modulo $O(\Psi^{n+1,0})$ supported away from $\Sigma^+(\omega)$,

$$\mathrm{ad}_{\rho\Delta_y}^n P_\omega^+ = O(\Psi^{n-1,-1})\rho\Delta_y + O(\Psi^{n,-1}). \tag{6.23}$$

2b. We bootstrap by showing the case $(0, n)$ holds for every n. Step 1 implies the case $(0, 0)$ holds. Assume the case $(0, n-1)$ holds and consider

$$P_\omega^+ (\rho\Delta_y)^n u^+ = (\rho\Delta_y)^n P_\omega^+ u^+ - \sum_{n'<n} \binom{n}{n'} (\mathrm{ad}_{\rho\Delta_y}^{n-n'} P_\omega^+)(\rho\Delta_y)^{n'} u^+ \tag{6.24}$$

For $F_1 \in \Psi^{0,-1}$, we have

$$\sum_{n'<n} \binom{n}{n'} (\mathrm{ad}_{\rho\Delta_y}^{n-n'} P_\omega^+)(\rho\Delta_y)^{n'} u^+ = \sum_{n'=0}^{n-1} O(\Psi^{n-n',-1})(\rho\Delta_y)^{n'} u^+ + nF_1(\rho\Delta_y)^n u^+$$
$$= nF_1(\rho\Delta_y)^n u^+ + O(H^{\infty,\frac{1}{2}-0}), \tag{6.25}$$

by the case $(0, n-1)$. This gives

$$(P_\omega^+ + nF_1)(\rho\Delta_y)^n u^+ \in H^{\infty,\frac{1}{2}-0}. \tag{6.26}$$

Applying the standard radial sink estimate to $(P_\omega^+ + nF_1)$ gives for any $s, N > 0$,

$$\|(\rho\Delta_y)^n u^+\|_{H^{s,-\frac{1}{2}-0}} \le C\|(P_\omega^+ + nF_1)(\rho\Delta_y)^n u^+\|_{H^{s+1,\frac{1}{2}-0}} + C\|Q_+(\rho\Delta_y)^n u^+\|_{H^{s,-\frac{1}{2}-0}} + C\|u^+\|_{H^{-N,-N}} \tag{6.27}$$

for some $Q_+ \in \Psi^{0,0}$ supported near $\Sigma^+(\omega)$ with $\mathrm{WF}\, Q_+ \cap \Sigma^+(\omega) = \emptyset$. One can choose Q_+ that $\mathrm{WF}\, Q_+ \cap \mathrm{WF}\, B = \emptyset$ and this yields $(\rho\Delta_y)^n u^+ \in H^{\infty,-\frac{1}{2}-0}$ as desired, and validates the case $(0, n)$ iteratively.

2c. We claim if $V_{\beta_j} = V_0$ for some $1 \le j \le k$, then (6.11) holds for case (k, n) for any n. Indeed if so, then

$$(\rho\Delta_y)^n V_\beta u^+ = (\rho\Delta_y)^n V_{\beta_1} \cdots V_{\beta_{j-1}} \big(V_{\beta'} \rho^{-1} P_\omega^+ u^+ - \sum_{\gamma \subsetneq \beta'} \binom{\beta'}{\gamma} (\mathrm{ad}_V^{\beta'\setminus\gamma} V_0) V_\gamma u^+ \big) \tag{6.28}$$

is in $H^{\infty,-\frac{1}{2}-0}$ as desired, for $\beta' = (\beta_{j+1}, \dots, \beta_k)$, since

$$(\rho\Delta_y)^n V_{\beta_1} \cdots V_{\beta_{j-1}} (\mathrm{ad}_V^{\beta'\setminus\gamma} V_0) V_\gamma u^+ = \sum_{n' \le n} \sum_{|\beta''|<k} O(\Psi^{n-n',0})(\rho\Delta_y)^{n'} V_{\beta''} u^+ \in H^{\infty,-\frac{1}{2}-0}, \tag{6.29}$$

per iterative assumption (6.11) of the case $(k-1, n)$. Here we used $\mathrm{ad}_V^\gamma O(\Psi^{0,0}) = O(\Psi^{0,0})$ and $\mathrm{ad}_{\rho\Delta_y}^{n'} O(\Psi^{0,0}) = O(\Psi^{n',0})$.

2d. It then suffices to show (6.11) for case (k, n) under the extra condition that $\beta_j \ne 0$ for all $1 \le j \le k$. We proceed by induction on n. We first show the case $(k, 0)$ holds. Consider

$$P_\omega^+ V_\beta u^+ = V_\beta P_\omega^+ u^+ - \sum_{\gamma \subsetneq \beta} \big(\mathrm{ad}_V^{\beta\setminus\gamma} P_\omega^+ \big) V_\gamma u^+ = O(H^{\infty,\frac{1}{2}-0}), \tag{6.30}$$

where we used

$$\sum_{\gamma \subsetneq \beta} \big(\mathrm{ad}_V^{\beta\setminus\gamma} P_\omega^+ \big) V_\gamma u^+ = \sum_{\gamma \subsetneq \beta} \sum_{j,l=1}^{d-2} O(\Psi^{-1,0}) \rho^2 V_j V_l V_\gamma u^+ + O(\Psi^{0,-1} V_\gamma u^+) = O(H^{\infty,\frac{1}{2}-0}), \tag{6.31}$$

since Δ_y is elliptic over $V_j V_l$ and $\rho\Delta_y V_\gamma u^+ \in H^{\infty,-\frac{1}{2}-0}$ for $|\gamma| \le k-1$ per the case $(k-1, 1)$. This implies the case $(k, 0)$.

2e. Now assume the case $(k, n-1)$ holds. Consider

$$P_\omega^+ (\rho\Delta_y)^n V_\beta u^+ = (\rho\Delta_y)^n V_\beta P_\omega^+ u^+ - \sum_{n'<n} \binom{n}{n'} (\mathrm{ad}_{\rho\Delta_y}^{n-n'} P_\omega^+)(\rho\Delta_y)^{n'} V_\beta u^+ - \sum_{\gamma \subsetneq \beta} (\rho\Delta_y)^n \big(\mathrm{ad}_V^{\beta\setminus\gamma} P_\omega^+ \big) V_\gamma u^+. \tag{6.32}$$

For $F_1 \in \Psi^{0,-1}$, we have

$$\sum_{n'<n} \binom{n}{n'} (\mathrm{ad}_{\rho\Delta_y}^{n-n'} P_\omega^+)(\rho\Delta_y)^{n'} V_\beta u^+ = \sum_{n'=0}^{n-1} O(\Psi^{n-n',-1})(\rho\Delta_y)^{n'} V_\beta u^+ + nF_1(\rho\Delta_y)^n V_\beta u^+$$
$$= nF_1(\rho\Delta_y)^n V_\beta u^+ + O(H^{\infty,\frac{1}{2}-0}), \tag{6.33}$$

by the case $(k, n-1)$. On another hand,

$$\sum_{\gamma\subsetneq\beta}(\rho\Delta_y)^n\big(\operatorname{ad}_V^{\beta\setminus\gamma}P_\omega^+\big)V_\gamma u^+ = \sum_{\gamma\subsetneq\beta}\sum_{n'\le n}\sum_{j,l=1}^{d-2} O(\Psi^{n-n'-1,0})(\rho\Delta_y)^{n'}\rho^2V_jV_lV_\gamma u^+ + O(\Psi^{n-n',-1})(\rho\Delta_y)^{n'}V_\gamma u^+ = O(H^{\infty,\frac12-0}), \quad (6.34)$$

where we used $(\Delta_y)^{n'+1}$ being elliptic over $(\rho\Delta_y)^{n'}V_jV_l$ and $(\rho\Delta_y)^{n'+1}V_\gamma u^+\in H^{\infty,-\frac12-0}$ for $|\gamma|\le k-1$ and $n'\le n$ per the case $(k-1,n+1)$. We ended up with

$$(P_\omega^+ + nF_1)(\rho\Delta_y)^nV_\beta u^+\in H^{\infty,\frac12-0}. \quad (6.35)$$

Similar to Step 2b, applying the standard radial sink estimate to $(P_\omega^+ + nF_1)$ yields $(\rho\Delta_y)^nV_\beta u^+\in H^{\infty,-\frac12-0}$ as desired. By induction we have shown (6.11) for all (k,n).

3. Denote $h=\omega^{-1}$. Now we invoke [Ḧ85b, Theorem 25.2.4] to see the asymptotic structure at the sink: (6.7) implies there is $\tilde a_+(\rho,y)\in S^0$ such that

$$u^+ = \alpha(\omega)^{-1}B\tilde{\tilde\chi}u = \rho^{\frac d2-1}\tilde a^+(\rho,y)e^{i/h\rho}+O(h^\infty\mathcal{S}). \quad (6.36)$$

It remains to show that there exist $a^+\in S^0(\partial Y)$, $a_k^+\in S^{0,0}(\partial W_Y)$ that

$$\tilde a^+(\rho,y) = a^+(y)+\sum_{k=1}^\infty (h\rho)^k a_k^+(\rho,y)+O(h^\infty\mathcal{S}). \quad (6.37)$$

Note (6.36) implies

$$O(h^\infty\mathcal{S}) = P_\omega^+u^+ = -\frac{i\omega}{\sqrt{\omega^2+M^2}}\rho^{\frac d2}\Big(-\rho\partial_\rho\tilde a^+(\rho,y)+\sum_{k=2}^\infty(h\rho)^{k-1}L_k^+\tilde a^+(\rho,y)\Big)e^{i/h\rho}+O(h^\infty\mathcal{S}), \quad (6.38)$$

for L_k^+ being b-differential operators locally involving $\rho\partial_\rho$ and ∂_y up to order $2k-1$. Note $L_k^+\tilde a_+\in S^0$. We then have the governing transport equation:

$$\rho\partial_\rho\tilde a^+(\rho,y)-\sum_{k=0}^\infty(h\rho)^{k+1}L_{k+2}^+\tilde a^+(\rho,y)=O(h^\infty\mathcal{S}). \quad (6.39)$$

4a. We proceed by induction to show (6.37). It is equivalent to showing for any $K\ge1$, there exist $a^+=a_0^+,a_1^+,\dots,a_{K-1}^+\in S^0$ and $\tilde a_K^+$ that

$$\tilde a^+(\rho,y)=a^+(y)+\sum_{k=1}^{K-1}(h\rho)^ka_k^+(\rho,y)+(h\rho)^K\tilde a_K^+,\ \ \rho\partial_\rho((h\rho)^K\tilde a_K^+)=O((h\rho)^K). \quad (6.40)$$

We address the case $K=1$ first. From (6.39) we have

$$\rho\partial_\rho\tilde a^+(\rho,y)=h\rho L_2^+\tilde a^+(\rho,y)+O(h^2\rho^2). \quad (6.41)$$

We can solve the transport equation $\rho\partial_\rho a^+(y)=0$ for $a_0^+(y)\in S^0$ by $a^+(y)=\tilde a^+(0,y)$. Plugging

$$\tilde a_1^+(\rho,y)=(h\rho)^{-1}(\tilde a^+(\rho,y)-a^+(y)) \quad (6.42)$$

into (6.41) gives $\rho\partial_\rho(h\rho\tilde a_{+,1})=O(h\rho)$. This validates the case $K=1$.

4b. We show if (6.40) holds for some $K \geq 1$, then there exist $a_K^+ \in S^{0,0}$ and $\tilde{a}_{K+1}^+$ that

$$\tilde{a}^+(\rho, y) = a^+(y) + \sum_{k=1}^{K} (h\rho)^k a_k^+(\rho, y) + (h\rho)^{K+1} \tilde{a}_{K+1}^+, \ \rho\partial_\rho((h\rho)^{K+1}\tilde{a}_{K+1}^+) = O((h\rho)^{K+1}). \quad (6.43)$$

Indeed, substituting (6.40) into the governing equation (6.39) gives

$$\rho\partial_\rho((h\rho)^K \tilde{a}_K^+) = \sum_{k=0}^{K-1} (h\rho)^{K-k} L_{K+1-k}^+((h\rho)^k a_k^+) + O((h\rho)^{K+1}), \quad (6.44)$$

where we used $\rho\partial_\rho((h\rho)^K\tilde{a}_K^+)$ is of size $(h\rho)^K$. Solving the transport equation

$$\rho\partial_\rho((h\rho)^K a_K^+) = \sum_{k=0}^{K-1} (h\rho)^{K-k} L_{K+1-k}^+((h\rho)^k a_k^+) \quad (6.45)$$

gives the desired $a_K^+ \in S^0$. Now letting

$$\tilde{a}_{K+1}^+ = (h\rho)^{-1}(\tilde{a}_K^+ - a_K^+) \quad (6.46)$$

gives $\rho\partial_\rho((h\rho)^{K+1}\tilde{a}_{K+1}^+) = O((h\rho)^{K+1})$ in (6.44). One can then iterate and this shows (6.37) at the sink. To observe (6.2), it suffices to note for any $A = A\tilde{\tilde{\chi}}$ described in the statement of the lemma, one has that $\alpha(\omega)^{-1}Au = \alpha(\omega)^{-1}AB\tilde{\tilde{\chi}}u + O(h^\infty\mathcal{S})$ by noting $\alpha(\omega)^{-1}A(\mathrm{I}-B)\tilde{\tilde{\chi}}u = O(h^\infty\mathcal{S})$.

5. To show uniqueness, it suffices to show for $u^+ \in H^{-\frac{1}{2}-0}$ with $P_\omega^+ u^+ = O(h^\infty\mathcal{S})$,

$$u^+ = \Big(\rho^{\frac{d}{2}-1}\tilde{a}^+(\rho, y)e^{i/h\rho}\Big) + O(h^\infty\mathcal{S}), \ a_+(y) = 0 \quad (6.47)$$

implies $u^+ = O(h^\infty\mathcal{S})$. Indeed, given $a_0^+, \dots, a_{K-1}^+ = 0$, (6.45) implies $(h\rho)^K a_K^+$ does not depend on ρ while $a_K^+ \in S^0$, hence $a_K^+ = 0$. Induction concludes the uniqueness claim. □

6.2. **Unitarity and existence of the scattering matrix.** We prove a pairing formula.

Lemma 6.2 (Green's second identity for $\Lambda_M^{W_Y}$)**.** *Let $M \geq 0$, $\omega > 0$ fixed. Let $\chi \in S^0(\partial W_Y)$ with ∂W_Y smooth over* $\operatorname{supp}\chi$, $\operatorname{supp}(1-\chi)$ *compact. Let $u, v \in Z^{\frac{1}{2},0,\frac{1}{2}}_{0,-\frac{1}{2}-0,-\frac{1}{2}-0}(\partial W_Y)$ where*

$$u = \chi\rho^{\frac{d-2}{2}}\Big(u^+e^{i\omega/\rho} + u^-e^{-i\omega/\rho}\Big) + O(Z^{\frac{1}{2},0,\frac{1}{2}}_{0,-\frac{1}{2}+0,-\frac{1}{2}+0}), \quad (6.48)$$

$$v = \chi\rho^{\frac{d-2}{2}}\Big(v^+e^{i\omega/\rho} + v^-e^{-i\omega/\rho}\Big) + O(Z^{\frac{1}{2},0,\frac{1}{2}}_{0,-\frac{1}{2}+0,-\frac{1}{2}+0}), \quad (6.49)$$

for $u^\pm, v^\pm \in L^2(\partial Y)$. Assume

$$(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})u, (\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})v \in Z^{-\frac{1}{2},0,-\frac{1}{2}}_{0,\frac{1}{2}+0,\frac{1}{2}+0}(\partial W_Y) \quad (6.50)$$

Then we have

$$\langle(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})u, v\rangle_{\partial W_Y} - \langle u, (\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})v\rangle_{\partial W_Y} = \frac{\omega}{i\sqrt{\omega^2+M^2}}\Big(\langle u^+, v^+\rangle_{\partial Y} - \langle u^-, v^-\rangle_{\partial Y}\Big). \quad (6.51)$$

Proof. 1. We first turn the left of (6.51) into its microlocal representation. Let $\tilde{\chi} \in S^0(\partial W_Y)$ that $\operatorname{supp}\chi \cap \operatorname{supp}(1-\tilde{\chi}) = \emptyset$ with ∂W_Y smooth over $\operatorname{supp}\tilde{\chi}$. Denote the microlocal representation of $\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2}$ by

$$P_\omega = \chi(P - \sqrt{\omega^2+M^2})\tilde{\chi} \in \Psi^{1,0},\ p_\omega = \sigma(P_\omega) = \chi(\sqrt{|\xi|_g^2+M^2} - \sqrt{\omega^2+M^2}). \tag{6.52}$$

1a. Let $\varphi(\rho) \in S^0(\partial W_Y)$ that only depends on ρ, with $\operatorname{supp}\varphi \cap \operatorname{supp}(1-\chi) = \emptyset$, $\varphi(\rho) \equiv 1$ near $\rho = 0$ and $\operatorname{supp}\varphi \subset \{\rho < 1\}$. We remind the reader that $\psi_\omega \in C_c^\infty(\mathbb{R}_\xi)$ defined in Definition 5.8 satisfies $\omega \notin \operatorname{supp}(1-\psi_\omega)$ and $0 \notin \operatorname{supp}\psi_\omega$. Let $\varphi_0(\rho), \varphi_{00}, \tilde{\varphi} \in S^0(\partial W_Y)$ satisfy the same requirement for φ, with $\varphi_{00} \prec \varphi_0 \prec \varphi \prec \tilde{\varphi}$. Let $0 < h < \frac{1}{2}$. Let $Q \in \Psi^{0,0}$ be that

$$Q = \operatorname{Op}(\varphi_{00}(h^{-1}\rho)\psi_\omega(|\xi|_g)),\ \sigma(Q) = q(\rho, y; \tau, \eta) = \varphi_{00}(h^{-1}\rho)\psi_\omega(|\xi|_g). \tag{6.53}$$

Note $Q = \varphi_0(h^{-1}\rho)Q\varphi_0(h^{-1}\rho)$ since the quantization procedure does not move the support more than $\frac{1}{4}$. We have

$$Q = Q^* + O_{\Psi^{-1,0}}(h),\ \operatorname{WF}(\mathrm{I}-Q) \cap \Sigma(\omega) = \emptyset,\ \operatorname{WF} Q \cap \{(\tau,\eta) = 0\} = \emptyset. \tag{6.54}$$

Noting $\langle(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})u, Qv\rangle = o_h(1)$ and $(\mathrm{I}-Q^*)v \in L^2$, we have

$$\langle(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})u, v\rangle = \langle(\mathrm{I}-Q)(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})u, v\rangle + o_h(1), \tag{6.55}$$

since $\Lambda_M^{W_Y}$ is self-adjoint on L^2. Similar arguments around $\langle Qu, (\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})v\rangle$ give

$$\langle(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})u, v\rangle - \langle u, (\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})v\rangle = \langle[\Lambda_M^{W_Y}, Q]u, v\rangle + o_h(1). \tag{6.56}$$

1b. We further reduce the $[\Lambda_M^{W_Y}, Q]$ into $[P_\omega^\varphi, Q]$ for $P_\omega^\varphi = \varphi(h^{-1}\rho)P_\omega\tilde{\varphi}(h^{-1}\rho)$. By Proposition 4.22 we have

$$\varphi(h^{-1}\rho)\Lambda_M^{W_Y} = \varphi(h^{-1}\rho)P\tilde{\varphi}(h^{-1}\rho) + \varphi(h^{-1}\rho)\mathcal{E}_M,\ \Lambda_M^{W_Y}\varphi(h^{-1}\rho) = \tilde{\varphi}(h^{-1}\rho)P\varphi(h^{-1}\rho) + \mathcal{E}'_M\varphi(h^{-1}\rho). \tag{6.57}$$

With $Q\mathcal{E}_M : H^{0,-L_0+0} \to H^{N,N}$ by (4.185), this implies

$$Q\Lambda_M^{W_Y}u = Q\varphi(h^{-1}\rho)P\tilde{\varphi}(h^{-1}\rho)u + O_{H^{N,N}}(h^\infty). \tag{6.58}$$

On another hand, note $Q = \varphi_0(h^{-1}\rho)Q = \tilde{\varphi}(h^{-1}\rho)Q$ implies

$$\tilde{\varphi}(h^{-1}\rho)P\varphi(h^{-1}\rho)Q = \varphi(h^{-1}\rho)P\tilde{\varphi}(h^{-1}\rho)Q + (\tilde{\varphi}-\varphi)(h^{-1}\rho)P\varphi_0(h^{-1}\rho)Q, \tag{6.59}$$

the second term of which is $O_{\Psi^{-\infty,-\infty}}(h^\infty)$. With $\mathcal{E}'_M Q : H^{-N,-N} \to H^{0,\mathfrak{r}_{W_Y}^{r_0}+\frac{1}{2}-0}$, we have

$$\Lambda_M^{W_Y}Qu = \varphi(h^{-1}\rho)P\tilde{\varphi}(h^{-1}\rho)Qu + O_{H^{0,\mathfrak{r}_{W_Y}^{r_0}+\frac{1}{2}-0}}(h^\infty). \tag{6.60}$$

This implies

$$\langle(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})u, v\rangle - \langle u, (\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})v\rangle = \langle[P_\omega^\varphi, Q]u, v\rangle + o_h(1). \tag{6.61}$$

2. It remains to look at $[P_\omega^\varphi, Q] \in \Psi^{0,-1}$. We remind the reader that the neighborhood $V^+(\omega)$ of $\Sigma^+(\omega)$ inside $T^*\partial W_Y$ was defined near (5.42). Noting $\rho \sim h$ on $\operatorname{WF}[P_\omega^\varphi, Q]$, we have in

$V^+(\omega)\cap\{(\tau,\eta)=(-\omega,0)\}$,

$$H_pq=-\frac{\omega\rho^2\partial_\rho q}{\sqrt{\omega^2+M^2}}+O(S^{-2,-3+0})=-\frac{\omega h^{-1}\rho^2(\partial_\rho\varphi_{00})(h^{-1}\rho)\psi_\omega(|\xi|_g)}{\sqrt{\omega^2+M^2}}+O(S^{-2,-3+0})\in S^{0,-1} \tag{6.62}$$

with remainder terms up to size $h^{1-0}S^{-1,-1-0}$. Acting on a Lagrangian distribution, we have

$$[P_\omega^\varphi,Q]\varphi\rho^{\frac{d-2}{2}}u^+(y)e^{i\omega/\rho}=\frac{i\omega}{\sqrt{\omega^2+M^2}}\rho^{\frac{d-2}{2}}h^{-1}\rho^2(\partial_\rho\varphi_{00})(h^{-1}\rho)u^+(y)e^{i\omega/\rho}$$
$$+O_{\Psi^{-1,-1-0}}(h^{1-0})\big(\varphi\rho^{\frac{d-2}{2}}u^+(y)e^{i\omega/\rho}\big), \tag{6.63}$$

the last remainder term of which is of size $O_{H^{1,\frac12+0}}(h^{1-0})$. Now

$$\begin{aligned}\langle[P_\omega^\varphi,Q]\varphi\rho^{\frac{d-2}{2}}u^+(y)e^{i\omega/\rho},\varphi\rho^{\frac{d-2}{2}}v^+(y)e^{i\omega/\rho}\rangle&\\ =\frac{i\omega}{\sqrt{\omega^2+M^2}}\langle u^+,v^+\rangle_{\partial Y}\int_0^1\rho^{d-2}h^{-1}\rho^2(\partial_\rho\varphi_{00})(h^{-1}\rho)\,\rho^{-d}d\rho+O(h^{1-0})&\\ =\frac{\omega}{i\sqrt{\omega^2+M^2}}\langle u^+,v^+\rangle_{\partial Y}+O(h^{1-0}),&\end{aligned} \tag{6.64}$$

where that integral in ρ is just $\varphi_{00}(h^{-1})-\varphi_{00}(0)=-1$ for small h. Similar arguments with H_pq in $V^-(\omega)\cap\{(\tau,\eta)=(\omega,0)\}$ give

$$\begin{aligned}\langle(\Lambda_M^{W_Y}-\sqrt{\omega^2+M^2})u,v\rangle_{\partial W_Y}-\langle u,(\Lambda_M^{W_Y}-\sqrt{\omega^2+M^2})v\rangle_{\partial W_Y}&\\ =\frac{\omega}{i\sqrt{\omega^2+M^2}}\Big(\langle u^+,v^+\rangle_{\partial Y}-\langle u^-,v^-\rangle_{\partial Y}\Big)+o_h(1).&\end{aligned} \tag{6.65}$$

Note all terms above except $o_h(1)$ are independent of h to conclude the proof. □

Now we show the existence of the scattering matrix.

Proposition 6.3 (The scattering matrix)**.** *Let W_Y be an AC Lipschitz domain with smooth cross-section Y. Let $M\geq 0$ and $\omega>0$. Then for all $a^-(y)\in C^\infty(\partial Y)$, there exists*

$$u\in Z^{\infty,\infty,1-0}_{L_0-0,-\frac12-0,-\frac12-0}(\partial W_Y;\omega),\ (\Lambda_M^{W_Y}-\sqrt{\omega^2+M^2})u=0 \tag{6.66}$$

unique modulo $Z^{\infty,\infty,1}_{L_0-0,\infty,\infty}(\partial W_Y)$, such that there exist $\rho_0>0$, $\tilde a^\pm(\rho,y)\in S^0(\partial W_Y)$ that for each $A\in\Psi^{0,0}(\partial W_Y)$ with $\mathrm{WF}\,A\subset\{\rho<\rho_0,|\xi|>c\omega\}$ for some $c>0$, we have

$$Au(\rho,y)=A\rho^{\frac d2-1}\sum_\pm\tilde a^\pm(\rho,y;\omega)e^{\pm i\omega/\rho}+O(\omega^{-\infty}\mathcal S), \tag{6.67}$$

in local coordinates y on ∂Y, uniformly in $\omega>0$. Such $\tilde a^\pm(\rho,y;\omega)$ in $\{\rho<\rho_0\}$ modulo $\omega^{-\infty}\mathcal S(\partial W_Y)$ are uniquely determined by $a^-(y)$ modulo $\omega^{-\infty}\mathcal S(\partial W_Y)$. Moreover, there exist $a^+(y;\omega)\in S^0(\partial Y)$, $a_k^\pm(\rho,y;\omega)\in S^0(\partial W_Y)$ for $k\geq1$ such that

$$\tilde a^\pm(\rho,y;\omega)=a^\pm(y)+\sum_{k=1}^\infty\omega^{-k}\rho^k a_k^\pm(\rho,y;\omega)+O(\omega^{-\infty}\mathcal S). \tag{6.68}$$

Moreover, the map $\Sigma_{M,\omega}^{W_Y}:a^-(y)\mapsto a^+(y;\omega)$ is $L^2(\partial Y)$-unitary.

Proof. 1. Let $\chi, \tilde{\chi}, \tilde{\tilde{\chi}} \in S^0(\partial W_Y)$ that ∂W_Y is smooth over $\operatorname{supp}\tilde{\tilde{\chi}}$, $\operatorname{supp}(1-\chi)$ is compact and $\chi \prec \tilde{\chi} \prec \tilde{\tilde{\chi}}$. Pick $B^- \in \Psi^{0,0}$ with $\mathrm{WF}(\mathrm{I}-B^-)\cap\Sigma^-(\omega)=\emptyset$, $\mathrm{WF}\, B^- \subset \{\rho<\rho_0, |\xi|>c\omega\}$ for some $c>0$. Let

$$P_\omega^- = B^-\chi(P-\sqrt{\omega^2+M^2})\tilde{\chi} \in \Psi^{1,0}. \tag{6.69}$$

In this step, we build $\tilde{a}^-(\rho,y;\omega), a_k^-(\rho,y;\omega) \in S^0$ such that $P_\omega^- v^i = O(\omega^{-\infty}\mathcal{S})$ with

$$v^i = \tilde{\tilde{\chi}}\rho^{\frac{d-2}{2}}\tilde{a}^-(\rho,y;\omega)e^{-i\omega/\rho} \in Z^{\infty,\infty,1}_{L_0-0,-\frac{1}{2}-0,\infty}, \tag{6.70}$$

$$\tilde{a}^-(\rho,y;\omega) = a^-(y) + \sum_{k=1}^{\infty}(\omega^{-1}\rho)^k a_k^-(\rho,y;\omega) + O(\omega^{-\infty}\mathcal{S}). \tag{6.71}$$

Denote $h=\omega^{-1}$. Indeed, one notes the Lagrangian structure of v^i with respect to $\Sigma^-(\omega)$, and via computations similar to (6.38) we have

$$P_\omega^-\rho^{\frac{d-2}{2}}\tilde{a}^-(\rho,y;h)e^{-i/h\rho} = \frac{i\omega}{\sqrt{\omega^2+M^2}}\rho^{\frac{d}{2}}\Big(-\rho\partial_\rho\tilde{a}^-(\rho,y;h)+\sum_{k=2}^{\infty}(h\rho)^{k-1}L_k^-\tilde{a}^-(\rho,y;h)\Big)e^{-i/h\rho}+O(h^\infty\mathcal{S}), \tag{6.72}$$

where L_k^- are b-differential operators locally involving $\rho\partial_\rho$ and ∂_y up to order k. We solve the governing equation

$$\rho\partial_\rho\tilde{a}^-(\rho,y;h) - \sum_{k=0}^{\infty}(h\rho)^{k+1}L_{k+2}^-\tilde{a}^-(\rho,y;h) = O(h^\infty\mathcal{S}),\ \tilde{a}^-(\rho,y;h)\sim a^-(y)+\sum_{k=1}^{\infty}(h\rho)^k a_k^-(\rho,y;h). \tag{6.73}$$

At the heights of $h\rho$ and $(h\rho)^2$, the transport equations are

$$\rho\partial_\rho(h\rho a_1^-(\rho,y;h)) = h\rho L_2^- a^-(y),\ \rho\partial_\rho(h^2\rho^2 a_2^-(\rho,y;h)) = (h\rho)^2 L_3^- a^-(y) + h\rho L_2^-(h\rho a_1^-(\rho,y;h)). \tag{6.74}$$

One obtains a_1^-, a_2^-. The transport equations always give a_k^- based on $a^-, a_1^-, \dots, a_{k-1}^-$ for $k\geq 2$. One can iteratively solve for all $a_k^- \in S^0$ and this concludes the step. Note $a^-(y)$ uniquely determines each coefficient and hence $\tilde{a}^-(\rho,y;\omega)$, modulo $\omega^{-\infty}\mathcal{S}$, by arguments similar to Step 5 of Lemma 6.1.

2. For some B^+ stated in Lemma 6.1, we claim

$$f = (\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})v^i \in Z^{\infty,\infty,0}_{L_0-0,\infty,\infty},\ B^+f = O(\omega^{-\infty}\mathcal{S}). \tag{6.75}$$

Here the first bound on f is not uniform in ω. Note $v^i \in Z^{\infty,\infty,1}_{L_0-0,-\frac{1}{2}-0,\infty}$ immediately implies that $f \in Z^{\infty,\infty,0}_{L_0-0,-\frac{1}{2}-0,\infty}$ in a non-uniform way. It remains to show $B^-\chi f \in \mathcal{S}$. Indeed, Proposition 4.22 implies

$$B^-\chi f = P_\omega^- v^i + B^-\chi\mathcal{E}_M v^i = O(\omega^{-\infty}\mathcal{S}), \tag{6.76}$$

since $B^-\chi\mathcal{E}_M : H^{0,-L_0+0} \to \omega^{-N}H^{N,N}$ is bounded. On the other hand, we have

$$B^+\chi f = B^+\chi(P-\sqrt{\omega^2+M^2})\tilde{\chi}v^i + B^+\chi\mathcal{E}_M v^i = O(\omega^{-\infty}\mathcal{S}), \tag{6.77}$$

as claimed.

3. Without loss of generality assume $\mathrm{ran}(\Pi_M) = \ker_{L^2}(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2}) \subset Z^{\infty,\infty,1}_{L_0-0,\infty,\infty}$ by shrinking $\mathcal{I}_\omega$ if necessary. This implies $f = \Pi_M^\perp f$. The limiting absorption principle, Proposition 5.18, implies there is

$$v^o = -(\Pi_M^\perp \Lambda_M^{W_Y} \Pi_M^\perp - \sqrt{\omega^2+M^2} - i0)^{-1} f \in Z^{\infty,\infty,1-0}_{L_0-0,\infty,-\frac{1}{2}-0} \tag{6.78}$$

that solves $(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})v^o = -f$. A priori, we do not have uniform control of v^o. Apply Lemma 6.1 to see with WF A away from the zero frequency and $\Sigma^-(\omega)$,

$$Av^o(\rho,y) = \alpha(\omega)\big(A\rho^{\frac{d-2}{2}}\tilde{a}^+(\rho,y)e^{i\omega/\rho} + O(\omega^{-\infty}\mathcal{S})\big), \tag{6.79}$$

where

$$\tilde{a}^+(\rho,y) = a^+(y) + \sum_{k=1}^{\infty} \omega^{-k}\rho^k a_k^+(\rho,y) + O(\omega^{-\infty}\mathcal{S}), \tag{6.80}$$

for some $a^+(y;\omega), a_k^+(\rho,y;\omega) \in S^0$, uniquely determined modulo $\omega^{-\infty}\mathcal{S}$ by $a^-(y)$ once $\alpha(\omega)$ is fixed. Now set $u = v^o + v^i$. We have $(\Lambda_M^{W_Y} - \sqrt{\omega^2+M^2})u = 0$ with

$$Au(\rho,y) = A\rho^{\frac{d}{2}-1}\Big(\tilde{a}^-(\rho,y;\omega)e^{-i\omega/\rho} + \alpha(\omega)\tilde{a}^+(\rho,y;\omega)e^{i\omega/\rho}\Big) + O((1+\alpha(\omega))\omega^{-\infty}\mathcal{S}). \tag{6.81}$$

We remind the reader that we defined in Step 1 of Lemma 6.1 that uniformly for $\omega > 0$ large,

$$\alpha(\omega) = \max(\|B\tilde{\tilde{\chi}}v^o\|_{H^{0,-\frac{1}{2}-0}}, 1) \le \max(K\alpha(\omega)\|a^+(y;\omega)\|_{L^2(\partial Y)}, 1) \tag{6.82}$$

for some constant $K > 0$. Now a posteriori, the Green's identity, Lemma 6.2 implies

$$\alpha(\omega)^2\|a^+(y;\omega)\|^2_{L^2(\partial Y)} = \|a^-(y)\|^2_{L^2(\partial Y)}. \tag{6.83}$$

The above implies

$$\alpha(\omega) \le \max(K\|a^-(y)\|_{L^2(\partial Y)}, 1) \tag{6.84}$$

is uniformly bounded for $\omega > 0$ large. This allows us to renormalize $\tilde{a}^+(\rho,y;\omega)$ such that $\alpha(\omega) \equiv 1$. This gives (6.67) and $\Sigma^{W_Y}_{M,\omega} : a^- \mapsto a^+$ is L^2-unitary. □

As a corollary, we show that there is no meaningful reflection on an AC smooth domain of $d = 2$.

Corollary 6.4 (Scattering matrix on a smooth curve)**.** *Let Ω be a 2-dimensional AC smooth domain whose boundary $\partial\Omega$ is a connected smooth curve. Let $M \ge 0$ and $\omega > 0$. Then for all $\mathbf{a}^i = (a^i_+, a^i_-) \in \mathbb{C}^2$, there exists a unique $\mathbf{a}^o = (a^o_+, a^o_-) \in \mathbb{C}^2$ depending on M, ω that* (6.67), (6.68) *hold with*

$$a^+(\pm\infty) = a^o_\pm,\ a^-(\pm\infty) = a^i_\mp. \tag{6.85}$$

Moreover, there exist $\theta_\pm(M,\omega) \in [0, 2\pi)$ that

$$\begin{pmatrix} a^o_- \\ a^o_+ \end{pmatrix} = \Sigma^\Omega_{M,\omega}\begin{pmatrix} a^i_+ \\ a^i_- \end{pmatrix},\ \Sigma^\Omega_{M,\omega} = \begin{pmatrix} 0 & e^{i\theta_-} \\ e^{i\theta_+} & 0 \end{pmatrix} + O(\omega^{-\infty}), \tag{6.86}$$

uniformly for $\omega > 0$ large.

Proof. Proposition 6.3 gives the desired $u \in Z^{\infty,\infty}_{L_0-0,-\frac{1}{2}-0,-\frac{1}{2}-0}$, $(\Lambda^\Omega_M - \sqrt{\omega^2+M^2})u = 0$. It suffices to show (6.86). We claim

$$\psi_\omega(D)(\Lambda^\Omega_M - \sqrt{\omega^2+M^2})\psi_\omega(D)u = O(\omega^{-\infty}\mathcal{S}), \tag{6.87}$$

where $\psi_\omega \in C^\infty_c((0,\infty))$ defined in Definition d:Zspace satisfies $\omega \in \operatorname{supp}\psi_\omega$ and $0 \notin \operatorname{supp}\psi_\omega$. Indeed,

$$[\Lambda^\Omega_M, \psi_\omega(D)] = [P, \psi_\omega(D)] + \mathcal{E}_\flat\psi_\omega(D) - \psi_\omega(D)\mathcal{E}_\flat. \tag{6.88}$$

Note $\operatorname{WF} u, \operatorname{WF}[P,\psi_\omega(D)]$ are separated at scale ω in ξ, and $\mathcal{E}_\flat$ is in $\Psi^{-\infty,-\infty}$ away from the zero frequency in Proposition 3.12. They imply (6.87). Now Green's identity, Lemma 6.2 implies

$$O(\omega^{-\infty}) = 2i\Im\langle(\Lambda^\Omega_M - \sqrt{\omega^2+M^2})\psi_\omega(D)u, \psi_\omega(D)u\rangle_{\partial\Omega} = \frac{\omega}{i\sqrt{\omega^2+M^2}}\Big(|a^o_+|^2 - |a^i_+|^2\Big). \tag{6.89}$$

The proof giving $|a^o_-| \sim |a^i_-|$ is similar. □

Remark 6.5. All constants in this section can depend on Ω in the following way: the upper bound on uniform Lipschitz characters of $\partial\Omega$, the upper bound on the choice of $r_0 \geq 0$, and the lower bound on the radial Hardy constant $\mathfrak{r}^{r_0}_\Omega$, the upper bound on the diameter of $\partial\Omega \setminus \partial\tilde{\Omega}$ in Remark 3.11, for each multi-index α, the upper bounds on the derivatives of the metric g_+ in (3.11), and lower bounds on $|\det g_+|$.

Part 2. The Steklov spectrum on curvilinear polygons

7. Convergence of the scattering matrix in two dimension

This section considers scattering matrices on simply connected AC Lipschitz domains in $\mathbb{R}^2$ with a single point singularity in the high energy limit. We call these domains *curved wedges with angle* α if the two smooth components of the boundary meet at angle $\alpha \in (0, 2\pi)$. The main goal if this section is to determine the extent to which the scattering matrix is determined by α together with the derivatives of the smooth parts of the boundary at the singular point.

Specifically, letting $\mathfrak{r}_\alpha = \mathfrak{r}^0_{W_\alpha} = \frac{\pi}{\alpha}$ and $\mathcal{W}_\alpha, \tilde{\mathcal{W}}_\alpha \subset \mathbb{R}^2$ curved wedges of corner angle α with strong radial Hardy constants $\mathfrak{r}^0_{\mathcal{W}_\alpha}, \mathfrak{r}^0_{\tilde{\mathcal{W}}_\alpha}$. Without loss of generality, we assume that the singular point for $\mathcal{W}_\alpha$ and $\tilde{\mathcal{W}}_\alpha$ is at 0. Then, setting $\min(\mathfrak{r}^0_{\mathcal{W}_\alpha}, \mathfrak{r}^0_{\tilde{\mathcal{W}}_\alpha}) = \mathfrak{r}_\alpha - 0$, we obtain

Proposition 7.1. *Let $M_0 > 0$, $\omega \in [\frac{1}{2}, 2]$, $k_0 \geq 0$ such that the first (k_0+1) derivatives of the boundary of $\mathcal{W}_\alpha$ agreeing with those for $\tilde{\mathcal{W}}_\alpha$ at 0. Then there exists $h_0 > 0$ such that uniformly for all $M \in [0, M_0]$, all $h \in (0, h_0]$, we have*

$$\Sigma^{\tilde{\mathcal{W}}_\alpha}_{M,h^{-1}\omega} = \Sigma^{\mathcal{W}_\alpha}_{0,\omega} + O(h^{\min(2\mathfrak{r}_\alpha, 1+k_0)-0}) + O((hM)^{2-0}). \tag{7.1}$$

Remark 7.2. In Section 9, for any $\epsilon > 0$, we will apply Proposition 7.1 to domains that satisfy $\mathfrak{r}^0_{\mathcal{W}_\alpha} > \mathfrak{r}_\alpha - \epsilon$, $\mathfrak{r}^0_{\tilde{\mathcal{W}}_\alpha} > \mathfrak{r}_\alpha - \epsilon$.

Outline of the strategy

We prove Proposition 7.1 in two steps. We first compare $\Sigma^{\tilde{\mathcal{W}}_\alpha}_{M,h^{-1}\omega}$ with $\Sigma^{\mathcal{W}_\alpha}_{M,h^{-1}\omega}$ and then compare $\Sigma^{\mathcal{W}_\alpha}_{M,h^{-1}\omega}$ with $\Sigma^{\mathcal{W}_\alpha}_{0,h^{-1}\omega}$. The first comparison modifies the underlying geometry, while the

second only modifies the mass. These correspond to Lemmas 7.4, and 7.5, respectively, which together imply the Proposition.

Geometric comparison:

When comparing scattering matrices on two different wedges, we consider a solution, u, to

$$(\Lambda_M^{\tilde{\mathcal{W}}_\alpha} - h^{-1}\sqrt{\omega^2 + h^2M^2})u = 0.$$

Our goal is then to transplant u onto $\mathcal{W}_\alpha$, as the function v', in such a way that it satisfies

$$(\Lambda_M^{\mathcal{W}_\alpha} - h^{-1}\sqrt{\omega^2 + h^2M^2})v' = \text{small}.$$

More precisely, we find $F : B(0,\epsilon) \cap \mathcal{W}_\alpha \to \tilde{\mathcal{W}}_\alpha$ a diffeomorphism that is as close to the identity as possible. We then zoom in to scale h, considering $\mathcal{W}_\alpha^h =: h^{-1}\mathcal{W}_\alpha$ and $\tilde{\mathcal{W}}_\alpha^h := h^{-1}\tilde{\mathcal{W}}_\alpha$, the map $F_h := h^{-1}F(hx)$ and consider $u_h := u(hx)$ satisfying

$$(\Lambda_{hM}^{\tilde{\mathcal{W}}_\alpha^h} - \sqrt{\omega^2 + h^2M^2})u_h = 0. \tag{7.2}$$

Recall from Proposition 6.3 that such a u_h exists and is can be decomposed as

$$u = u_{h,\text{low}} + u_{h,\text{wave}}, \tag{7.3}$$

where $u_{h,\text{low}}$ is low frequency near infinity and $u_{h,\text{wave}}$ is an approximate plane wave. Moreover, $u_{h,\text{wave}}$ is a sum of approximate plane waves that is determined by only the scattering data.

We then build v_h' out of three pieces:

$$v'(h\cdot) := v_h' = \chi^\delta F_h^* u + (\chi - \chi^\delta)F_h^* u_{h,\text{low}} + (1-\chi^\delta)(w_{\text{right}} + w_{\text{left}}). \tag{7.4}$$

Here, χ^δ is a localizer to $|x| \lesssim h^{-\delta}$ for $0 < \delta \ll 1$, and χ to $|x| \lesssim h^{-1}$, $u_{h,\text{low}}$ denotes the low frequencies of u, and $w_{\text{right/left}}$ are approximate plane waves on the right and left of the singularity of $\mathcal{W}_\alpha$. The reason for the choice of scales is as follows:

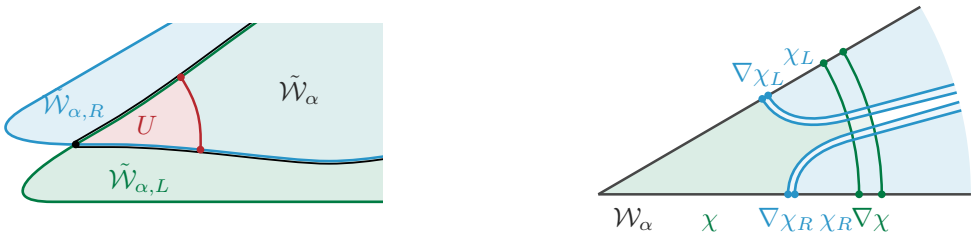
(1) The asymptotic properties of $u_{h,\text{wave}}$ set in (modulo h^∞) at scale $h^{-\delta}$ so, at this scale, we can construct $w_{\text{right/left}}$ on $\mathcal{W}_\alpha^h$ that are very close to $F^* u_{h,\text{wave}}$ at scale $h^{-\delta}$ and are very accurate quasimodes on $\mathcal{W}_\alpha^h$. This allows us to match $F^* u_{h,\text{wave}}$ at scale $h^{-\delta}$ and appropriately extend to infinity.

(2) We have very little control over u_{low} and, in addition, we expect its properties to be global. Therefore, although we must truncate these low frequencies somewhere, we want to wait as long as possible to do so. Since we can only accurately compare the geometries to scale h^{-1}, this is the scale we true.

Estimating

$$(\Lambda_{hM}^{\mathcal{W}_\alpha^h} - \sqrt{\omega^2 + h^2M^2})v_h' \tag{7.5}$$

is technically involved, so we do not discuss details here. We instead point out one important feature that is crucial in proving Proposition 7.1. From the Fredholm estimates in Section 5, one apriori knows that $u_{h,\text{low}}$ decays like $|x|^{-\frac{\pi}{\alpha}-\frac{1}{2}+0}$. At first, it seems that this is not sufficient to obtain control better than $h^{\frac{\pi}{\alpha}+\frac{1}{2}-0}$. However, by measuring the error in a space that allows growth at low frequency like $|x|^{\frac{\pi}{\alpha}+\frac{1}{2}-0}$, we are able to recover the difference.



(A) Domain extensions (B) Cutoffs

FIGURE 10. Illustration of domain extensions and cutoffs

Once we show that the right $(\Lambda_{hM}^{\mathcal{W}_\alpha^h} - \sqrt{\omega^2 + h^2M^2})v_h'$ is small enough, an application of the outgoing resolvent will finish the proof.

Mass comparison:

The mass comparison is significantly simpler than the geometric comparison, but the process is similar. We begin we a solution of

$$(\Lambda_M^{\mathcal{W}_\alpha} - h^{-1}\sqrt{\omega^2 + h^2M^2})u = 0 \tag{7.6}$$

and show that u satisfies

$$(\Lambda_0^{\mathcal{W}_\alpha} - h^{-1}\omega)u = \text{small}. \tag{7.7}$$

Applying the outgoing resolvent then finishes the proof.

The keys to estimating the error in (7.7) are, again, the apriori structure of u and an accurate microlocal description of $\Lambda_M^{\mathcal{W}_\alpha}$ near infinity.

7.1. **Rescaling AC Lipschitz domains with exactly one corner.** Consider an AC Lipschitz domain $\tilde{\mathcal{W}}_\alpha \subset \mathbb{R}^2$ with exactly one corner of central angle $\alpha \in (0, 2\pi)$ positioned at $z = 0$, and suppose that we have an h-family of u_h that $(h\Lambda_M^{\tilde{\mathcal{W}}_\alpha} - \omega)u_h = 0$ for $\omega \in [\frac{1}{2}, 2]$ fixed. Its boundary $\partial\tilde{\mathcal{W}}_\alpha$ can be parametrized by the arclength variable x, where $x = 0$ at the corner. Let $\partial\tilde{\mathcal{W}}_\alpha = \Gamma_L \cup \{x = 0\} \cup \Gamma_R$, where each of $\Gamma_L = \{x \in \partial\tilde{\mathcal{W}}_\alpha : x < 0\}$ and $\Gamma_R = \{x \in \partial\tilde{\mathcal{W}}_\alpha : x > 0\}$ is a smooth ray that originates from the corner and is asymptotically linear as $|x| \to \infty$. There then exist AC smooth extension $\tilde{\mathcal{W}}_{\alpha,L/R} \subset \mathbb{R}^2$ such that $\Gamma_{L/R} \subset \partial\tilde{\mathcal{W}}_{\alpha,L/R}$ with $\mathfrak{r}^\infty_{\tilde{\mathcal{W}}_{\alpha,L/R}} \geq \mathfrak{r}_\alpha$. See Figure 10(A).

Let $k_0 \geq 0$. Our goal in this subsection is to relate $\tilde{\mathcal{W}}_\alpha$ to $\mathcal{W}_\alpha$, another curved wedge in $\mathbb{R}^2$ cornered at $z = 0$ with corner angle α, with first $(k_0 + 1)$ derivatives of the boundary at $z = 0$ agreeing with those for $\tilde{\mathcal{W}}_\alpha$. In this section, we use $\mathrm{cl}(\cdot)$ to denote the closure of an open set to avoid confusion with bar and tilde notations. Let $U = \mathrm{cl}\left(\tilde{\mathcal{W}}_\alpha \cap B(0, R)\right)$ for some $R > 0$ fixed large, and $F : U \to F(U) \subset \mathrm{cl}(\mathcal{W}_\alpha)$ be a smooth map such that $F|_{\partial\tilde{\mathcal{W}}_\alpha \cap \partial U}$ is an isometry, $dF(0) = \mathrm{I}$, with $dF(\partial_{\nu_{\tilde{\mathcal{W}}_\alpha}}) = \partial_{\nu_{\mathcal{W}_\alpha}}$ for $\partial_{\nu_{\tilde{\mathcal{W}}_\alpha}}$ and $\partial_{\nu_{\mathcal{W}_\alpha}}$ being respectively the outward pointing

normal to $\partial\tilde{\mathcal{W}}_\alpha$ and $\partial\mathcal{W}_\alpha$. We ask

$$F(z) = z + \sum_{k=2}^{\infty} F_k(z) = z + O(|z|^{k_0+2}),\ F_k = \frac{1}{k!}\nabla_z^k F(0)(z,\dots,z), F_k \equiv 0 \text{ for } 2 \le k \le k_0+2, \quad (7.8)$$

where $F_k(\lambda z) = \lambda^k F_k(z)$ is homogeneous of degree k for $\lambda \in \mathbb{R}$. The existence of such F follows from Lemma B.4. Its pushforward is

$$DF(z) = \mathrm{I} + \sum_{k=2}^{\infty} DF_k(z) = \mathrm{I} + O(|z|^{k_0+1}),\ DF_k(z)z' = \frac{1}{(k-1)!}\nabla_z^k F(0)(z,\dots,z,z'), \quad (7.9)$$

with DF_k homogeneous of degree $k-1$ in z. We denote $k_1 = k_0 + 1$.

We now zoom in near $z = 0$ and rescale the domains by

$$\tilde{\mathcal{W}}_\alpha^h = \{z \in \mathbb{R}^2 : hz \in \tilde{\mathcal{W}}_\alpha\},\ \mathcal{W}_\alpha^h = \{z \in \mathbb{R}^2 : hz \in \mathcal{W}_\alpha\}. \quad (7.10)$$

Then, setting $u(z) = u_h(hz)$, we have

$$(\Lambda_{hM}^{\tilde{\mathcal{W}}_\alpha^h} - \omega)u = 0. \quad (7.11)$$

Let $U^h = \{z : hz \in U\} \subset \tilde{\mathcal{W}}_\alpha^h$, and $F_h(z) = h^{-1}F(hz)$ as a map from $U^h \subset \tilde{\mathcal{W}}_\alpha^h$ to $\mathcal{W}_\alpha^h$. This implies that $dF_h(z) = \mathrm{I} + O(h^{k_1}|z|^{k_1})$. For the convenience of our analysis, we will introduce another scale $0 < \delta \ll 1$, by

$$\tilde{\mathcal{W}}_\alpha^\delta = \{z \in \mathbb{R}^2 : h^{1-\delta}z \in \tilde{\mathcal{W}}_\alpha\},\ \mathcal{W}_\alpha^\delta = \{z \in \mathbb{R}^2 : h^{1-\delta}z \in \mathcal{W}_\alpha\}, \quad (7.12)$$

and $U^\delta = \{z : h^{1-\delta}z \in U\}$. We also have $F_\delta = h^{-1+\delta}F(h^{1-\delta}z)$ and

$$(h^\delta \Lambda_{h^{1-\delta}M}^{\tilde{\mathcal{W}}_\alpha^\delta} - \omega)u_\delta = 0,\ u_\delta(z) = u_h(h^{1-\delta}z). \quad (7.13)$$

We see $\tilde{\mathcal{W}}_\alpha, \tilde{\mathcal{W}}_\alpha^\delta, \tilde{\mathcal{W}}_\alpha^h$ are the domains that we see at the physical scale, matching scale, and semiclassical scale. We rescale $\tilde{\mathcal{W}}_{\alpha,L/R}$ to obtain $\tilde{\mathcal{W}}_{\alpha,L/R}^h, \tilde{\mathcal{W}}_{\alpha,L/R}^\delta$ accordingly.

We denote the unitary rescaling operator

$$T_h : L^2(\partial\mathcal{W}_\alpha) \to L^2(\partial\mathcal{W}_\alpha^h),\ (T_h u)(z) = h^{\frac{1}{2}}u(hz), \quad (7.14)$$

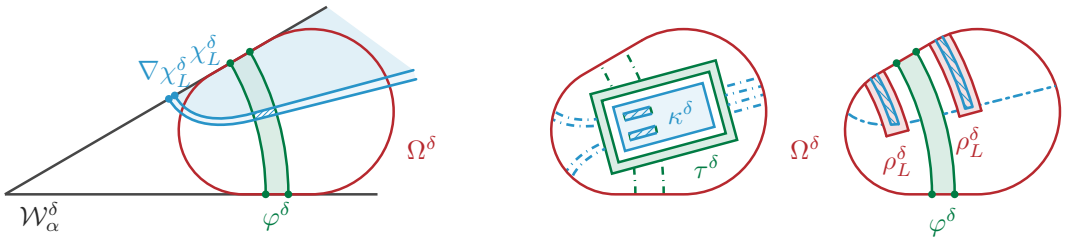
$$T_\delta : L^2(\partial\mathcal{W}_\alpha) \to L^2(\partial\mathcal{W}_\alpha^\delta),\ (T_\delta u)(z) = h^{\frac{1-\delta}{2}}u(h^{1-\delta}z). \quad (7.15)$$

These rescaling operators naturally extends to other rescaled domains.

We now set up some cutoff functions we will need for Lemma 7.4. Throughout this subsection and Lemma 7.4, we use superscripts h and δ to denote functions that lives on $\mathcal{W}_\alpha$ at a scale $|x| \sim h$ and $|x| \sim h^{1-\delta}$ respectively, and to denote domains in which a unit ball looks likes a ball on $\mathcal{W}_\alpha$ at a scale $|x| \le h$ and $|x| \le h^{1-\delta}$ respectively. On $\mathcal{W}_\alpha$, consider $\chi \in C_c^\infty(F(U))$ that $0 \notin \operatorname{supp}(1-\chi) \cap \partial\mathcal{W}_\alpha$ and $\partial_\nu\chi|_{\partial\mathcal{W}_\alpha} \equiv 0$. Let $\tilde{\chi}, \tilde{\tilde{\chi}}$ satisfy the same conditions as χ, and $\chi \prec \tilde{\chi} \prec \tilde{\tilde{\chi}}$. Consider the following rescaled functions

$$\chi^h(z) = \chi(h^{-1}z) \in C_c^\infty(F^h(U^h)),\ \chi^\delta(z) = \chi(h^{\delta-1}z) \in C_c^\infty(F^\delta(U^\delta)), \quad (7.16)$$

We understand $\chi \in C_c^\infty(U)$ via the pullback induced by F, and $\chi^h \in C_c^\infty(U^h)$, $\chi^\delta \in C_c^\infty(U^\delta)$ similarly. All functions with superscripts h or δ are scaled accordingly, such as $\tilde{\chi}^h, \tilde{\chi}^\delta, \tilde{\tilde{\chi}}^h, \tilde{\tilde{\chi}}^\delta$. On $\mathcal{W}_\alpha$, we let $\chi_{L/R} \in S^0(\mathcal{W}_\alpha)$ such that $0 \notin \operatorname{supp}\chi_{L/R}$, $\chi_{L/R} \equiv 0$ on $\Gamma_{R/L}$, $\chi_L + \chi_R \equiv 1$ on



(A) Domain Ω^δ (B) Cutoffs on Ω^δ

FIGURE 11. Illustration of Ω^δ and cutoffs on Ω^δ.

$\partial\mathcal{W}_\alpha \cap \operatorname{supp}(1-\chi)$, and $\partial_\nu \chi_{L/R}|_{\partial\mathcal{W}_\alpha} \equiv 0$. We ask $\nabla\chi_{L/R}$ is supported inside $\{C^{-1} < |x|, x_0 > C^{-1}\langle x\rangle - C\}$ in $\tilde{\mathcal{W}}_{\alpha,L/R}$. Rescale $\chi_{L/R}$ to obtain $\chi_{L/R}^h$, $\chi_{L/R}^\delta$. See Figure 10 (B).

On $\mathcal{W}_\alpha^\delta$ we fix h-independent φ^δ, $\tilde\varphi^\delta \in C^\infty(\mathrm{cl}(\mathcal{W}_\alpha^\delta))$ such that $(\nabla\chi^\delta)\varphi^\delta = \nabla\chi^\delta$, $\varphi^\delta\tilde\varphi^\delta = \varphi^\delta$. Note $\operatorname{supp}\varphi^\delta$ is compact on $\mathcal{W}_\alpha^\delta$ and away from 0, there exists an h-uniform compact smooth domain $\Omega^\delta \subset \mathcal{W}_\alpha^\delta$ such that $\operatorname{supp}\tilde\varphi^\delta \subset \Omega^\delta$, $\partial\Omega^\delta \cap \operatorname{supp}\tilde\varphi^\delta = \partial\mathcal{W}_\alpha \cap \operatorname{supp}\tilde\varphi^\delta$. Then φ^δ, $\tilde\varphi^\delta$ can be viewed as functions in $C^\infty(\Omega^\delta)$. We can pull back Ω^δ to obtain $\tilde\Omega^\delta = F_\delta^{-1}(\Omega^\delta) \subset \tilde{\mathcal{W}}_\alpha^\delta$, whose shapes then depends on h. Making Ω^δ larger if necessary, we let $\kappa^\delta, \tilde\kappa^\delta \in C_c^\infty(\Omega^\delta)$ be that $\tilde\varphi^\delta(\nabla\chi_\pm^\delta) \prec \kappa^\delta$ and $\kappa^\delta \prec \tilde\kappa^\delta$, and let $\tau^\delta \in C_c^\infty(\Omega^\delta)$ be that $\nabla\tilde\kappa^\delta \prec \tau^\delta$ and $\operatorname{supp}\tau^\delta \cap \operatorname{supp}\kappa^\delta = \emptyset$. Note $\tilde\kappa^\delta$ and τ^δ are supported away from $\partial\Omega^\delta$. Let $\rho_{L/R}^\delta \in C^\infty(\bar\Omega^\delta)$ be that $\nabla\tilde\varphi^\delta\chi_{L/R}^\delta \prec \rho_{L/R}^\delta$ and $\operatorname{supp}\rho_{L/R}^\delta \cap \operatorname{supp}\varphi^\delta = \emptyset$. See Figure 11.

7.2. **Uniform resolvent estimates along family of curves.** Denote $L_0 = \mathfrak{r}_\alpha + \frac12 - 0$. We also record the 2D resolvent estimates in terms of the variable-order Sobolev spaces:

Lemma 7.3 (Small-mass 2D uniform absorption resolvent)**.** *Let $M_0 > 0$. Let $s, l \in \mathbb{R}$, $s_0 \in (0,1]$, $l^- > -\frac12$, $l^+ < -\frac12$, $|l_0| < L_0$. Then there are $C > 0$ and $h_0 > 0$ that we have*

$$\|u\|_{Z^{s,l,s_0}_{l_0,l^-,l^+}(\partial\tilde{\mathcal{W}}_\alpha^h)} \le C\|(\Lambda_{hM}^{\tilde{\mathcal{W}}_\alpha^h} - \omega - i\mu)u\|_{Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}(\partial\tilde{\mathcal{W}}_\alpha^h)} \tag{7.17}$$

uniformly for $u \in Z^{-\infty,-\infty,0}_{-L_0+0,-\frac12+0,-\infty}(\partial\tilde{\mathcal{W}}_\alpha^h;\omega)$, $h \in (0,h_0]$, $\mu \ge 0$, $\omega \in [\frac14, 4]$, $M \in [0, M_0]$.

Proof. 1. Observe that $\partial\tilde{\mathcal{W}}_\alpha^h$ can be isometrically identified by $\partial\mathcal{W}_\alpha^h$. Denote

$$\mathcal{X} = Z^{s,l,s_0}_{l_0,l^-,l^+}(\partial\mathcal{W}_\alpha^h),\ \mathcal{Y} = Z^{s-1,l,s_0-1}_{l_0,l^-+1,l^++1}(\partial\mathcal{W}_\alpha^h),\ \mathcal{K} = H^{-N,-N}(\partial\mathcal{W}_\alpha^h), \tag{7.18}$$

which then identify their counterparts on $\partial\tilde{\mathcal{W}}_\alpha^h$. Note $\mathcal{X} \hookrightarrow \mathcal{K}$ compactly. Assume against contradiction that there are u_k, $\mu_k \to 0^+$ with $\|u_k\|_{\mathcal{X}} = 1$, $h_k \to 0$, such that $\|(\Lambda_{h_kM}^{\tilde{\mathcal{W}}_\alpha^{h_k}} - \omega - i\mu_k)u_k\|_{\mathcal{Y}} \to 0$. From Proposition 5.12, we have

$$\|u_k\|_{\mathcal{X}} \le C\|(\Lambda_{h_kM}^{\tilde{\mathcal{W}}_\alpha^{h_k}} - \omega - i\mu_k)u_k\|_{\mathcal{Y}} + C_N\|u_k\|_{\mathcal{K}}, \tag{7.19}$$

since $\{\tilde{\mathcal{W}}_\alpha^{h_k}\}$ have uniformly controlled constants needed for Proposition 5.12. Observe that all estimates we used are uniform in ω, M in the prescribed range, and only the limit $\mu_k \to 0^+$ is

interesting here. Since u_k are bounded in $\mathcal{X}$, $u_k \rightharpoonup v$ weakly in $\mathcal{X}$ and $u_k \to v$ for some $v \in \mathcal{X}$ in norm in $\mathcal{K}$ via a subsequence.

2. We claim $(\Lambda_0^{\mathcal{W}_\alpha^h} - \omega)v = 0$. Indeed, for a test function $\phi \in C_c^\infty(\partial\mathcal{W}_\alpha^h)$, we have

$$\begin{aligned}\langle(\Lambda_0^{\mathcal{W}_\alpha^h} - \omega)v, \phi\rangle = \lim_{k\to\infty}\langle u_k, (\Lambda_0^{\mathcal{W}_\alpha^h} - \omega)\phi\rangle = \lim_{k\to\infty}\langle u_k, (\Lambda_0^{\mathcal{W}_\alpha^h} - \Lambda_{h_kM}^{\tilde{\mathcal{W}}_\alpha^{h_k}})\phi\rangle + \langle(\Lambda_{h_kM}^{\tilde{\mathcal{W}}_\alpha^{h_k}} - \omega)u_k, \phi\rangle \\ = \lim_{k\to\infty}\langle u_k, (\Lambda_0^{\mathcal{W}_\alpha^{h_k}} - \Lambda_{h_kM}^{\tilde{\mathcal{W}}_\alpha^{h_k}})\phi\rangle. \quad (7.20)\end{aligned}$$

Therefore, since u_k is bounded in $\mathcal{X}$, it is enough to show that $\Lambda_{hM}^{\tilde{\mathcal{W}}_\alpha^h}\phi \to \Lambda_0^{\mathcal{W}_\alpha^h}\phi$ in $\mathcal{X}^*$ as $h \to 0$. Now noting $\phi = \chi^\delta\phi$ for small h and $\tilde{\chi}^\delta F_h^* \mathcal{H}_0^{\mathcal{W}_\alpha^h}\phi$ extends ϕ from $\partial\tilde{\mathcal{W}}_\alpha^h$ to $\tilde{\mathcal{W}}_\alpha^h$, we have

$$\mathcal{H}_{hM}^{\tilde{\mathcal{W}}_\alpha^h}\phi = \tilde{\chi}^\delta F_h^*\mathcal{H}_0^{\mathcal{W}_\alpha^h}\phi - \mathcal{R}_{hM}^{\tilde{\mathcal{W}}_\alpha^h}(-\Delta + h^2M^2)\tilde{\chi}^\delta F_h^*\mathcal{H}_0^{\mathcal{W}_\alpha^h}\phi, \quad (7.21)$$

and thus by $dF_h(\partial_\nu) = \partial_\nu$ on $\operatorname{supp}\tilde{\chi}^\delta$, we have

$$\Lambda_{hM}^{\tilde{\mathcal{W}}_\alpha^h}\phi = \tilde{\chi}^\delta\Lambda_0^{\mathcal{W}_\alpha^h}\phi + \partial_\nu\mathcal{R}_{hM}^{\tilde{\mathcal{W}}_\alpha^h}(-\Delta + h^2M^2)\tilde{\chi}^\delta F_h^*\mathcal{H}_0^{\mathcal{W}_\alpha^h}\phi. \quad (7.22)$$

One can check $[-\Delta, \tilde{\chi}^\delta] = O(h^\delta)\nabla + O(h^{2\delta})$ and

$$(F_h^{-1})^*(-\Delta)(F_h)^* = -\Delta + L,\ L = O(h^{k_1}|z|^{k_1})\nabla^2 + O(h^{k_1}|z|^{k_1-1})\nabla, \quad (7.23)$$

which implies

$$\partial_\nu\mathcal{R}_{hM}^{\tilde{\mathcal{W}}_\alpha^h}(-\Delta+h^2M^2)\tilde{\chi}^\delta F_h^*\mathcal{H}_0^{\mathcal{W}_\alpha^h}\phi = \partial_\nu\mathcal{R}_{hM}^{\tilde{\mathcal{W}}_\alpha^h}F_h^*([-\Delta,\tilde{\chi}^\delta]+(h^2M^2+L)\tilde{\chi}^\delta)\mathcal{H}_0^{\mathcal{W}_\alpha^h}\phi = O_{\mathcal{X}^*}(h^\delta) \quad (7.24)$$

decays, since $|z| \le Ch^{-\delta}$ on $\operatorname{supp}\tilde{\chi}^\delta$ in $\mathcal{W}_\alpha^h$. Now we have

$$(\Lambda_{hM}^{\tilde{\mathcal{W}}_\alpha^h} - \Lambda_0^{\mathcal{W}_\alpha^h})\phi = -(1-\tilde{\chi}^\delta)\Lambda_0^{\mathcal{W}_\alpha^h}\chi^\delta\phi + O_{\mathcal{X}^*}(h^\delta) = O_{\mathcal{X}^*}(h^\delta), \quad (7.25)$$

goes to zero as desired, where we used (4.142) to obtain

$$(1-\tilde{\chi}^\delta)\Lambda_0^{\mathcal{W}_\alpha^h}\chi^\delta\phi = \partial_\nu\mathcal{R}_0^{\mathcal{W}_\alpha^h}[-\Delta, 1-\tilde{\chi}^\delta]\mathcal{H}_0^{\mathcal{W}_\alpha^h}\chi^\delta\phi = O_{\mathcal{X}^*}(h^\delta), \quad (7.26)$$

since $[-\Delta, 1-\tilde{\chi}^\delta] = O(h^\delta)\nabla + O(h^{2\delta})$. This concludes the claim.

3. Now set $\mathcal{W}_\alpha = W_\alpha$, which is scaling-invariant. Then $(\Lambda_0^{W_\alpha^h} - \omega)v = 0$ and $v \in \mathcal{X}$. The Fredholm estimate, Proposition 5.12 then implies $v \in Z^{\infty,\infty,1}_{L_0-0,\infty,-\frac{1}{2}-0}$, and thus $\operatorname{WF} v \subset \Sigma^+(\omega) \cup \{\xi = 0\}$. Lemma 5.14 gives $\operatorname{WF} v \subset \{\xi = 0\}$ and thus $v \in L^2$, and ω is an eigenvalue to $\Lambda_0^{W_\alpha^h}$. Note W_α^h is a straight wedge and is invariant under rescaling. For each $\lambda \in \mathbb{R}_{>0}$, we have

$$(\Lambda_0^{W_\alpha^h} - \lambda\omega)v(\lambda x) = \lambda((\Lambda_0^{W_\alpha^h} - \omega)v)(\lambda x) = 0. \quad (7.27)$$

This implies $(0,\infty) \subset \operatorname{Spec}_{\mathrm{pp}}(\Lambda_0^{W_\alpha^h})$, contradicting with its discreteness per Lemma 5.15. □

7.3. **Comparison between curved wedges.** In this subsection, we will show the scattering matrices of $\tilde{\mathcal{W}}_j$ and $\mathcal{W}_{\alpha_j}$ are close to each other.

Lemma 7.4. *Let $M_0 > 0$. Then there exists $h_0 > 0$ that uniformly for all $M \in [0, M_0]$, $\omega \in [\frac{1}{2}, 2]$, $h \in (0, h_0]$, we have*

$$\Sigma^{\tilde{\mathcal{W}}_\alpha}_{M,h^{-1}\omega} = \Sigma^{\mathcal{W}_\alpha}_{M,h^{-1}\omega} + O(h^{\min(2\mathfrak{r}_\alpha, 1+k_0)-0}) + O(M^2 h^{2-0}), \tag{7.28}$$

given $k_0 \geq 0$ that the first (k_0+1) derivatives of the boundaries of $\mathcal{W}_\alpha, \tilde{\mathcal{W}}_\alpha$ agree at the corner.

Proof. 1. We begin with some generalized eigenfunctions u on $\partial\mathcal{W}^h_\alpha$. Let $a^i = (a^i_+, a^i_-) \in \mathbb{C}^2$ with $|a^i| = 1$. Proposition 6.3 implies the existence of $u \in Z^{-\infty,-\infty,1-0}_{L_0-0,-\frac{1}{2}-0,-\frac{1}{2}-0}(\partial\mathcal{W}^h_\alpha;\omega)$ such that $(\Lambda^{\mathcal{W}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2})u = 0$, $u \sim \tilde{a}^i_\mp(x)e^{-i\omega|x|} + \tilde{a}^o_\pm(x)e^{i\omega|x|}$ away from the zero frequency as $x \to \pm\infty$, where

$$\tilde{a}^i_\pm(x;h) = a^i_\pm + O_{S^0}(\langle x\rangle^{-1}),\ \tilde{a}^o_\pm(x;h) = a^o_\pm(h) + O(\langle x\rangle^{-1}),\ a^o = (a^o_+, a^o_-) = \Sigma^{\mathcal{W}_\alpha}_{M,h^{-1}\omega} a^i = \Sigma^{\mathcal{W}^h_\alpha}_{hM,\omega} a^i. \tag{7.29}$$

Denote its harmonic extension

$$\bar{u} = \mathcal{H}^{\mathcal{W}^h_\alpha}_{hM} u \in H^1_{\mathrm{loc}}(\mathcal{W}^h_\alpha). \tag{7.30}$$

In Step 1, we do some a priori analysis of $\bar{u}$.

1a. Lemma C.3 gives the existence of quasimodes $\tilde{e}_{\pm,L/R,h}(x_0, x; h, \omega, M)$ on $\tilde{\mathcal{W}}_{L/R}$ that

$$(-\Delta + M^2)\tilde{e}_{\pm,L/R,h} = O_{\mathcal{S}(\tilde{\mathcal{W}}_{L/R})}(h^\infty),\ (h\partial_\nu - \sqrt{\omega^2+M^2})\tilde{e}_{\pm,L/R,h} = O_{\mathcal{S}(\partial\tilde{\mathcal{W}}_{L/R})}(h^\infty), \tag{7.31}$$

whose boundary restriction $e_{\pm,L/R,h}(x; h, \omega, M) = \tilde{e}_{\pm,L/R,h}(0, x; h, \omega, M)$ satisfies

$$e_{\pm,L/R,h}(x; h, \omega, M) = e^{\pm i\omega x/h}(1 + O_{S^0(\partial\tilde{\mathcal{W}}_{L/R})}(h^2M^2)). \tag{7.32}$$

Rescale by $\tilde{e}_{\pm,L/R} = h^{-\frac{1}{2}} T_h \tilde{e}_{\pm,L/R,h}$ to obtain

$$(-\Delta + h^2M^2)\tilde{e}_{\pm,L/R} = O_{\mathcal{S}(\tilde{\mathcal{W}}^h_{L/R})}(h^\infty),\ (\partial_\nu - \sqrt{\omega^2+h^2M^2})\tilde{e}_{\pm,L/R} = O_{\mathcal{S}(\partial\tilde{\mathcal{W}}^h_{L/R})}(h^\infty), \tag{7.33}$$

$$e_{\pm,L/R}(x; h, \omega, M) = e^{\pm i\omega x}(1 + O_{S^0(\partial\tilde{\mathcal{W}}^h_{L/R})}(h^2M^2)). \tag{7.34}$$

1b. Parametrize $\partial\mathcal{W}^h_\alpha$ by x where $\partial\mathcal{W}^h_\alpha$ is singular at 0, and let (x_0, x) be the normal coordinate in the support of $\chi^h_{L/R} \in S^0(\mathcal{W}^h_\alpha)$ and $x_0 = 0$ on $\partial\mathcal{W}^h_\alpha$. We craft the principal oscillator by

$$\tilde{u}_\omega(x_0, x) = a^o_+\chi^h_R\tilde{e}_{+,R} + a^i_+\chi^h_L\tilde{e}_{+,L} + a^i_-\chi^h_R\tilde{e}_{-,R} + a^o_-\chi^h_L\tilde{e}_{-,L}. \tag{7.35}$$

Let the zero-frequency residue be $\tilde{u}_0 = \bar{u} - \tilde{u}_\omega \in X^l_m(\mathcal{W}^h_\alpha)$ shown in Step 1c below. In this proof, we use bar to denote harmonic extension like $\bar{u}$, and tilde to denote non-harmonic extensions like $\tilde{u}_\omega$. Let $u_0 = \tilde{u}_0|_{\partial\mathcal{W}^h_\alpha}$, $u_\omega = \tilde{u}_\omega|_{\partial\mathcal{W}^h_\alpha}$. Lemma 5.3 implies in $\partial\mathcal{W}^h_\alpha$,

$$\|\langle x\rangle^{\mathfrak{r}_\alpha+\frac{1}{2}-0}(\langle x\rangle\partial_x)^k(1-\chi^h)u_0(x)\|_{H^N} \leq C,\ |\partial^k(1-\chi^h)u_0(x)| \leq C\langle x\rangle^{-\mathfrak{r}_\alpha-1+0-k}. \tag{7.36}$$

Heuristically, u_ω is a mode oscillating at frequencies $\pm\omega$ and at distance at least 1 away from the corner in the semiclassical scale, and u_0 contains modes oscillating at frequency 0 and at distance at least 1 away from the corner and at all frequencies near the corner.

1c. We claim $\tilde{u}_0 \in X^{\mathfrak{r}_\alpha+\frac{1}{2}-0,N}_{\mathfrak{r}_\alpha-0}(\mathcal{W}^h_\alpha)$. We take note of the key relation

$$(-\Delta+h^2M^2)\tilde{u}_0 = -(-\Delta+h^2M^2)\tilde{u}_\omega = a^o_+[\Delta,\chi^h_R]\tilde{e}_{+,R} + a^i_+[\Delta,\chi^h_L]\tilde{e}_{+,L} + a^i_-[\Delta,\chi^h_R]\tilde{e}_{-,R} + a^o_-[\Delta,\chi^h_L]\tilde{e}_{-,L} + O_{\mathcal{S}}(h^\infty), \quad (7.37)$$

since it is supported inside $\{C^{-1}<|x|, x_0>C^{-1}\langle x\rangle - C\}$ in $\mathcal{W}^h_\alpha$. That $\tilde{e}_{\pm,L/R}$ are exponentially decaying in x_0 then implies $(-\Delta+h^2M^2)\tilde{u}_0 \in X^{N,N}_N(\mathcal{W}^h_\alpha) \subset Y^{N,N}_{N-2}(\mathcal{W}^h_\alpha)$ for any $N>0$ large. Note

$$\tilde{u}_0 = \mathcal{H}^{\mathcal{W}^h_\alpha}_{hM}u_0 + \mathcal{R}^{\mathcal{W}^h_\alpha}_{hM}(-\Delta+h^2M^2)\tilde{u}_0 = \mathcal{H}^{\mathcal{W}^h_\alpha}_{hM}u_0 + O(X^{\mathfrak{r}_\alpha+\frac{1}{2}-0,N}_{\mathfrak{r}_\alpha-0}) \in X^{\mathfrak{r}_\alpha+\frac{1}{2}-0,N}_{\mathfrak{r}_\alpha-0}(\mathcal{W}^h_\alpha), \quad (7.38)$$

by applying Lemma 4.12 per (7.36).

2. Our goal in Step 2 and 3, is to use the scattering data of generalized eigenfunctions u on $\partial\mathcal{W}^h_\alpha$, to construct an h-family quasimodes v' on $\partial\tilde{\mathcal{W}}^h_\alpha$. Let $\delta\in(0,1)$ small be a scale to be chosen later. Consider $F_h^*\chi^\delta w$ as a function localized in an $h^{-\delta}$-neighbourhood of 0 in $h^{-1}U\subset\tilde{\mathcal{W}}^h_\alpha$. We confuse the notation by $\chi^\delta F_h^*u = F_h^*\chi^\delta u$ via the pullback F_h^*. We want to extend $\chi^\delta F_h^*u$ by quasimodes $e'_{\pm,L/R}$ on $\tilde{\mathcal{W}}^h_{L/R}$:

$$w_R = a^o_+(h)e'_{+,R} + a^i_-e'_{-,R} = a^o_+(h)\big(1+M^2h^2O(S^0)\big)e^{i\omega x} + a^i_-\big(1+M^2h^2O(S^0)\big)e^{-i\omega x}, \quad (7.39)$$

$$w_L = a^i_+e'_{+,L} + a^o_-(h)e'_{-,L} = a^i_+\big(1+M^2h^2O(S^0)\big)e^{i\omega x} + a^o_-(h)\big(1+M^2h^2O(S^0)\big)e^{-i\omega x}. \quad (7.40)$$

Denote their harmonic extensions by $\bar{w}_{L/R} = \mathcal{H}^{\tilde{\mathcal{W}}^h_{L/R}}_{hM}w_{L/R} \in C^\infty(\tilde{\mathcal{W}}^h_{L/R})$.

We then construct the quasimodes on $\partial\tilde{\mathcal{W}}^h_\alpha$, by

$$v' = \chi^\delta F_h^*u + (\chi-\chi^\delta)F_h^*u_0 + (1-\chi^\delta)(\chi^\delta_R w_R + \chi^\delta_L w_L). \quad (7.41)$$

The rest of Step 2 and Step 3 is devoted to showing that for each $\delta\ll 1$, we have

$$(\Lambda^{\tilde{\mathcal{W}}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2})v' = O_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N,N}}(h^{\min(2\mathfrak{r}_\alpha,k_1-K\delta)-0} + M^2h^{2-K\delta}) \quad (7.42)$$

for some $K>0$ only depending on $k_1, N, \mathfrak{r}_\alpha$, not depending on δ.

2a. We begin with some a priori analysis of (7.41). First note

$$(-\Delta+h^2M^2)F_h^*\bar{u} = F_h^*(-\Delta+h^2M^2+L)\bar{u} = F_h^*L\bar{u}, \quad (7.43)$$

$$\partial_\nu F_h^*\bar{u} = F_h^*DF_h(\partial_\nu)\bar{u} = F_h^*\partial_\nu\bar{u} = \sqrt{\omega^2+h^2M^2}F_h^*\bar{u}, \quad (7.44)$$

where L was defined back in (7.23). We will use them later. The first two terms of (7.41) give

$$\begin{aligned}(\Lambda^{\tilde{\mathcal{W}}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2})(\chi^\delta F_h^*u + (\chi-\chi^\delta)F_h^*u_0) &= (\Lambda^{\tilde{\mathcal{W}}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2})(\chi^\delta F_h^*u_\omega + \chi F_h^*u_0)\\ &= (\partial_\nu - \sqrt{\omega^2+h^2M^2})(\chi^\delta F_h^*\tilde{u}_\omega + \chi F_h^*\tilde{u}_0) - \partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}(-\Delta+h^2M^2)(\chi^\delta F_h^*\tilde{u}_\omega + \chi F_h^*\tilde{u}_0)\\ &= (\partial_\nu - \sqrt{\omega^2+h^2M^2})(\chi^\delta F_h^*\tilde{u}_\omega + \chi F_h^*\tilde{u}_0) - \partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}\chi^\delta(-\Delta+h^2M^2)F_h^*\tilde{u}_\omega\\ &\quad - \partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}\chi(-\Delta+h^2M^2)F_h^*\tilde{u}_0 - \partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}([-\Delta,\chi^\delta]F_h^*\tilde{u}_\omega + [-\Delta,\chi]F_h^*\tilde{u}_0), \quad (7.45)\end{aligned}$$

where the first term, via (7.44), can be further written as

$$(\partial_\nu - \sqrt{\omega^2+h^2M^2})(\chi^\delta F_h^*\tilde{u}_\omega + \chi F_h^*\tilde{u}_0)$$
$$= \chi F_h^*(\partial_\nu - \sqrt{\omega^2+h^2M^2})\bar{u} - (\chi-\chi^\delta)F_h^*(\partial_\nu - \sqrt{\omega^2+h^2M^2})\tilde{u}_\omega = O(h^\infty), \quad (7.46)$$

from the quasimode nature of $\tilde{u}_\omega$ per (7.35), (7.33). On the other hand, $(1-\chi^\delta)\chi^\delta_{L/R}\bar{w}_{L/R}$ is an extension of $(1-\chi^\delta)\chi^\delta_{L/R}w_{L/R}$, and the last term of (7.41) give

$$(\Lambda^{\tilde{\mathcal{W}}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2})\sum_{L/R}(1-\chi^\delta)\chi^\delta_{L/R}w_{L/R} = \sum_{L/R}\Big((\partial_\nu - \sqrt{\omega^2+h^2M^2})(1-\chi^\delta)\chi^\delta_{L/R}\bar{w}_{L/R}$$
$$- \partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}[-\Delta,(1-\chi^\delta)\chi^\delta_{L/R}]\bar{w}_{L/R}\Big) = \sum_{L/R}\partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}[-\Delta,\chi^\delta]\chi^\delta_{L/R}\bar{w}_{L/R}$$
$$- \sum_{L/R}\partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}(1-\chi^\delta)[-\Delta,\chi^\delta_{L/R}]\bar{w}_{L/R} + O(h^\infty\mathcal{S}). \quad (7.47)$$

Thus together we have

$$(\Lambda^{\tilde{\mathcal{W}}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2})v' = -\partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}(\chi-\chi^\delta)(-\Delta+h^2M^2)F_h^*\tilde{u}_0 - \partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}\chi^\delta(-\Delta+h^2M^2)F_h^*\bar{u}$$
$$- \partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}[-\Delta,\chi]F_h^*\tilde{u}_0 - \sum_{L/R}\partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}(1-\chi^\delta)[-\Delta,\chi^\delta_{L/R}]\bar{w}_{L/R}$$
$$+ \partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}[-\Delta,\chi^\delta]\Big(-F_h^*\tilde{u}_\omega + \chi^\delta_L\bar{w}_L + \chi^\delta_R\Big) + O(h^\infty\mathcal{S}). \quad (7.48)$$

We will estimate the above term by term, except $O(h^\infty\mathcal{S})$, in the rest of Step 2 and Step 3.

2b. We estimate the first term in (7.48) by claiming

$$\|\partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}(\chi-\chi^\delta)(-\Delta+h^2M^2)F_h^*\tilde{u}_0\|_{Z^{-\frac12,N,-\frac12}_{-L_0+0,N,N}(\partial\tilde{\mathcal{W}}^h_\alpha)} \le Ch^{\min(2\mathfrak{r}_\alpha-0,k_1)}. \quad (7.49)$$

Note we follow (7.37) for

$$(\chi-\chi^\delta)(-\Delta+h^2M^2)\tilde{u}_\omega = (\chi-\chi^\delta)\Big(a^o_+[\Delta,\chi^h_R]\tilde{e}_{+,R} + a^i_+[\Delta,\chi^h_L]\tilde{e}_{+,L} + a^i_-[\Delta,\chi^h_R]\tilde{e}_{-,R}$$
$$+ a^o_-[\Delta,\chi^h_L]\tilde{e}_{-,L} + O_{\mathcal{S}}(h^\infty)\Big) = O_{\mathcal{S}}(h^\infty), \quad (7.50)$$

since the support of the right hand side is inside $\{C^{-1}h^{-\delta} < |x| < Ch^{-1}, x_0 > C^{-1}\langle x\rangle\}$ in $\mathcal{W}^h_\alpha$. This with $\tilde{u}_0 = \bar{u} - \tilde{u}_\omega$ implies $(\chi-\chi^\delta)(-\Delta+h^2M^2)\tilde{u}_0 = O_{\mathcal{S}}(h^\infty)$ as well. Then

$$(\chi-\chi^\delta)(-\Delta+h^2M^2)F_h^*\tilde{u}_0 = (\chi-\chi^\delta)F_h^*L\tilde{u}_0 + O_{\mathcal{S}}(h^\infty) \in \chi h^{k_1}\mathbf{r}^{k_1}Y^{\mathfrak{r}_\alpha+\frac12-0,N}_{\mathfrak{r}_\alpha-0} \subset h^{\min(2\mathfrak{r}_\alpha-0,k_1)}Y^{-\mathfrak{r}_\alpha-\frac12+0,N}_{-\mathfrak{r}_\alpha+0} \quad (7.51)$$

uniformly as desired, since $\mathbf{r}^{-1}Y^{\mathfrak{r}_\alpha+\frac12-0,N}_{\mathfrak{r}_\alpha-1-0} \subset Y^{\mathfrak{r}_\alpha+\frac12-0,N}_{\mathfrak{r}_\alpha-0}$ and

$$h^{k_1}\mathbf{r}^{k_1}Y^{\mathfrak{r}_\alpha+\frac12-0,N}_{\mathfrak{r}_\alpha-0} \subset h^{k_1}\mathbf{r}^{k_1-(2\mathfrak{r}_\alpha-0)}Y^{-\mathfrak{r}_\alpha-\frac12+0,N}_{-\mathfrak{r}_\alpha+0} \subset h^{\min(2\mathfrak{r}_\alpha-0,k_1)}Y^{-\mathfrak{r}_\alpha-\frac12+0,N}_{-\mathfrak{r}_\alpha+0}, \quad (7.52)$$

Apply $\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}$ on the left to see the estimates we need for the zero frequency,

$$\|\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}(\chi-\chi^\delta)(-\Delta+h^2M^2)F_h^*\tilde{u}_0\|_{H^{-\frac{1}{2},-\mathfrak{r}_\alpha-\frac{1}{2}+0}(\partial\tilde{\mathcal{W}}^h_\alpha)} \le Ch^{\min(2\mathfrak{r}_\alpha-0,k_1)}. \tag{7.53}$$

Let $\gamma^h, \tilde{\gamma}^h \in S^0(\partial\tilde{\mathcal{W}}^h_\alpha)$ supported in the smooth part and $\operatorname{supp}\gamma^h \cap \operatorname{supp}(1-\tilde{\gamma}^h)=\emptyset$. Note (4.114) gives $\tilde{\gamma}^h D^N \gamma^h \partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM} : \dot{Y}^{-\mathfrak{r}_\alpha-\frac{1}{2}+0,N}_{-\mathfrak{r}_\alpha+0} \to H^{-\frac{1}{2},-\mathfrak{r}_\alpha-\frac{1}{2}+N+0}$ bounded, and

$$\tilde{\gamma}^h(1-\psi_0)(D)\gamma^h = \tilde{\gamma}^h(1-\psi_0)(D)D^{-N}\tilde{\gamma}^h D^N\gamma^h. \tag{7.54}$$

We then have

$$\begin{aligned}\|\tilde{\gamma}^h(1-\psi_0)(D)\gamma^h\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}(\chi-\chi^\delta)(-\Delta+h^2M^2)F_h^*\tilde{u}_0\|_{H^{-\frac{1}{2},-\mathfrak{r}_\alpha-\frac{1}{2}+N-0}}&\\ \le C\|(\chi-\chi^\delta)(-\Delta+h^2M^2)F_h^*\tilde{u}_0\|_{Y^{-\mathfrak{r}_\alpha-\frac{1}{2}+0,N}_{-\mathfrak{r}_\alpha+0}} \le Ch^{\min(2\mathfrak{r}_\alpha-0,k_1)}.&\end{aligned} \tag{7.55}$$

Picking large $N>0$ gives desired estimates away from zero frequencies and thus the claim.

2c. We estimate the second term in (7.48) by claiming

$$\|\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}\chi^\delta(-\Delta+h^2M^2)F_h^*\bar{u}\|_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N,N}(\partial\tilde{\mathcal{W}}^h_\alpha)} \le Ch^{k_1-K\delta-0}. \tag{7.56}$$

Note $\mathbf{r}\le Ch^{-\delta}$ over the support of χ^δ. With $(-\Delta+h^2M^2)\bar{u}=0$, we have

$$\chi^\delta(-\Delta+h^2M^2)F_h^*\bar{u} = F_h^*\chi^\delta L\bar{u} = F_h^*\chi^\delta L\tilde{u}_0 + F_h^*\chi^\delta L\tilde{u}_\omega, \tag{7.57}$$

where arguments similar to Step 2b give

$$F_h^*\chi^\delta L\tilde{u}_0 \in \chi^\delta h^{k_1}\mathbf{r}^{k_1}Y^{\mathfrak{r}_\alpha+\frac{1}{2}-0}_{\mathfrak{r}_\alpha-0} \subset \chi^\delta h^{k_1}\mathbf{r}^{k_1-(2\mathfrak{r}_\alpha-0)}Y^{-\mathfrak{r}_\alpha-\frac{1}{2}+0}_{-\mathfrak{r}_\alpha+0} \subset h^{k_1-\delta\max(k_1-(2\mathfrak{r}_\alpha-0),0)}Y^{-\mathfrak{r}_\alpha-\frac{1}{2}+0}_{-\mathfrak{r}_\alpha+0}(\tilde{\mathcal{W}}^h_\alpha). \tag{7.58}$$

For the second term, we have

$$\begin{aligned}F_h^*\chi^\delta L\tilde{u}_\omega \in \chi^\delta h^{k_1}\mathbf{r}^{k_1}H^{-1,-\frac{1}{2}-0} &\subset \chi^\delta h^{k_1}\mathbf{r}^{k_1-\mathfrak{r}_\alpha+0}\mathbf{d}^{\frac{3}{2}+0}(\mathbf{r}^{\mathfrak{r}_\alpha+\frac{1}{2}-0}\mathbf{d}^{-\frac{3}{2}-0}H^{-1})\\ &\subset h^{k_1-\delta\max(k_1-\mathfrak{r}_\alpha+\frac{3}{2}+0,\frac{3}{2}+0)}Y^{-\mathfrak{r}_\alpha-\frac{1}{2}+0}_{-\mathfrak{r}_\alpha+0}(\tilde{\mathcal{W}}^h_\alpha)\end{aligned} \tag{7.59}$$

uniformly. Together we get

$$\|\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}\chi^\delta(-\Delta+h^2M^2)F_h^*\bar{u}\|_{H^{-\frac{1}{2},-\mathfrak{r}_\alpha-\frac{1}{2}+0}(\partial\tilde{\mathcal{W}}^h_\alpha)} \le Ch^{k_1-\delta\max(k_1-\mathfrak{r}_\alpha+\frac{3}{2},\frac{3}{2})-0} \tag{7.60}$$

for the zero frequency. For the nonzero frequencies, note (4.116) gives $\gamma^h(1-\psi_0)(D)\gamma^h\partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM} : \dot{Y}^{-\mathfrak{r}_\alpha-\frac{1}{2}+N+0}_{-\mathfrak{r}_\alpha+0} \to H^{-\frac{1}{2},-\mathfrak{r}_\alpha-\frac{1}{2}+N+0}$ bounded and thus

$$\begin{aligned}\|\gamma^h(1-\psi_0)(D)\gamma^h\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}\chi^\delta(-\Delta+h^2M^2)F_h^*\bar{u}\|_{H^{-\frac{1}{2},-\mathfrak{r}_\alpha-\frac{1}{2}+N-0}}&\\ \le C\|\chi^\delta(-\Delta+h^2M^2)F_h^*\bar{u}\|_{Y^{-\mathfrak{r}_\alpha-\frac{1}{2}+N+0}_{-\mathfrak{r}_\alpha+0}} \le Ch^{k_1-\delta\max(N+k_1-\mathfrak{r}_\alpha+\frac{3}{2},N+\frac{3}{2})-0},&\end{aligned} \tag{7.61}$$

similar to (7.55). This completes the claim.

2d. On the other hand, we have

$$[-\Delta,\chi]F_h^*\tilde{u}_0 = -(\Delta\chi)F_h^*\tilde{u}_0 - 2(\nabla\chi)F_h^*(\nabla+O(h^{k_1}\mathbf{r}^{k_1})\nabla)\tilde{u}_0. \tag{7.62}$$

Noting $\mathbf{r} \sim h^{-1}$ on $\operatorname{supp} \nabla\chi$, we have $|\Delta\chi| \leq Ch^2$, $|\nabla\chi| \leq Ch$ and

$$[-\Delta, \chi]F_h^*\tilde{u}_0 \in h^2 Y^{\mathfrak{r}_\alpha+\frac{1}{2}-0,N}_{\mathfrak{r}_\alpha-2-0} + hY^{\mathfrak{r}_\alpha+\frac{1}{2}-0,N}_{\mathfrak{r}_\alpha-1-0} \subset (h^2\mathbf{r}^{-(2\mathfrak{r}_\alpha-2-0)} + h\mathbf{r}^{-(2\mathfrak{r}_\alpha-1-0)})Y^{-\mathfrak{r}_\alpha-\frac{1}{2}+0,N}_{-\mathfrak{r}_\alpha+0}$$
$$\subset h^{2\mathfrak{r}_\alpha-0}Y^{-\mathfrak{r}_\alpha-\frac{1}{2}+0,N}_{-\mathfrak{r}_\alpha+0}(\tilde{\mathcal{W}}_\alpha^h), \quad (7.63)$$

allowing us to estimate the third term in (7.48) by

$$\|\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}_\alpha^h}_{hM}[-\Delta, \chi]F_h^*\tilde{u}_0\|_{H^{-\frac{1}{2},-\mathfrak{r}_\alpha-\frac{1}{2}+0}(\partial\tilde{\mathcal{W}}_\alpha^h)} \leq Ch^{2\mathfrak{r}_\alpha-0}, \quad (7.64)$$

and thus by estimating the nonzero frequencies similarly to (7.55) we have

$$\|\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}_\alpha^h}_{hM}[-\Delta, \chi]F_h^*\tilde{u}_0\|_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N,N}(\partial\tilde{\mathcal{W}}_\alpha^h)} \leq Ch^{2\mathfrak{r}_\alpha-0}. \quad (7.65)$$

2e. For the fourth term in (7.48), note $\operatorname{supp}(1-\chi^\delta)[-\Delta, \chi^\delta_{L/R}]$ is disjoint from $\partial\tilde{\mathcal{W}}^\delta_{L/R}$ and is in $\{|x| \geq C^{-1}h^{-\delta}\}$ for some $C > 0$, $w_{L/R} = \psi_\omega(D)w_{L/R} + O(\mathcal{S})$. Then (4.117) implies

$$\|(1-\chi^\delta)[-\Delta, \chi^\delta_{L/R}]\mathcal{H}^{\tilde{\mathcal{W}}^h_{L/R}}_{hM} w_{L/R}\|_{H^{N,N}} \leq Ch^N. \quad (7.66)$$

Repeat Step 2c to conclude

$$\|\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}_\alpha^h}_{hM}(1-\chi^\delta)[-\Delta, \chi^\delta_{L/R}]\bar{w}_{L/R}\|_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N,N}(\partial\tilde{\mathcal{W}}_\alpha^h)} \leq Ch^N. \quad (7.67)$$

3. It remains to estimate the last term in (7.48). We claim

$$\partial_\nu \mathcal{R}^{\tilde{\mathcal{W}}_\alpha^h}_{hM}[-\Delta, \chi^\delta]\Big(F_h^*\tilde{u}_\omega - \chi^\delta_L\bar{w}_L - \chi^\delta_R\bar{w}_R\Big) = O_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{0,N,N}}(h^{k_1-K\delta} + M^2h^{2-K\delta}). \quad (7.68)$$

We want to estimate instead on $\tilde{\Omega}^\delta$, the pullback of $F_h^*\tilde{u}_\omega - \chi^\delta_L\bar{w}_L - \chi^\delta_R\bar{w}_R$, that is,

$$f = \tilde{\varphi}^\delta(F^*_{\delta h}\tilde{u}_\omega - \chi^\delta_L\bar{w}_{L,\delta} - \chi^\delta_R\bar{w}_{R,\delta}),\ F_\delta(z) = h^{-1+\delta}F(h^{1-\delta}z),\ F_{\delta h}(z) = h^{-\delta}F_\delta(z) = F_h(h^{-\delta}z). \quad (7.69)$$

Here we rescaled $w_{L/R}$ by $w_{L/R,\delta} = F^*_{\delta h}w_{L/R}$ and $\tilde{e}'_{+,R,\delta} = F^*_{\delta h}\tilde{e}'_{+,R}$, $\tilde{e}_{+,R,\delta} = F^*_{\delta h}\tilde{e}_{+,R}$:

$$w_{R,\delta} = a^o_+(h)e'_{+,R,\delta} + a^i_-e'_{-,R,\delta} = a^o_+(h)\big(1+M^2h^2O(S^0)\big)e^{ih^{-\delta}\omega x} + a^i_-\big(1+M^2h^2O(S^0)\big)e^{-ih^{-\delta}\omega x}, \quad (7.70)$$

$$w_{L,\delta} = a^i_+e'_{+,L,\delta} + a^o_-(h)e'_{-,L,\delta} = a^i_+\big(1+M^2h^2O(S^0)\big)e^{ih^{-\delta}\omega x} + a^o_-(h)\big(1+M^2h^2O(S^0)\big)e^{-ih^{-\delta}\omega x}. \quad (7.71)$$

We inform the reader this process is similar to addressing Lemma 8.2 in the next section. Then f extends $\tilde{\varphi}^\delta(F^*_{\delta h}u_\omega - \chi^\delta_L w_{L,\delta} - \chi^\delta_R w_{R,\delta})$ from $\partial\tilde{\Omega}^\delta$ to $\tilde{\Omega}^\delta$, and thus

$$\varphi^\delta f = \varphi^\delta\mathcal{H}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}\tilde{\varphi}^\delta(F^*_{\delta h}u_\omega - \chi^\delta_L w_{L,\delta} - \chi^\delta_R w_{R,\delta}) + \varphi^\delta\mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}(-\Delta + h^{2-2\delta}M^2)\tilde{\varphi}^\delta(F^*_{\delta h}\tilde{u}_\omega - \chi^\delta_L\bar{w}_{L,\delta} - \chi^\delta_R\bar{w}_{R,\delta}). \quad (7.72)$$

We will estimate it term by term. First note on $\operatorname{supp}\tilde{\varphi}^\delta\chi^\delta_{L/R}$,

$$F_{\delta h}(x_0, x) = (h^{-\delta}x_0, h^{-\delta}x) + h^{k_1-(k_1+1)\delta}O(|(x_0, x)|^{k_1+1}), \quad (7.73)$$

where we understand coordinate x_0 as the normal variable from $\Gamma^\delta_{L/R}$. Note $\tilde\varphi^\delta$ has uniform compact support in the δ-scale. Hence it follows from (7.35) that

$$\tilde\varphi^\delta F^*_{\delta h}\tilde u_\omega(x_0,x)=\tilde\varphi^\delta\Big(a^o_+\chi^h_R\tilde e_{+,R,\delta}+a^i_+\chi^h_L\tilde e_{+,L,\delta}+a^i_-\chi^h_R\tilde e_{-,R,\delta}+a^o_-\chi^h_L\tilde e_{-,L,\delta}\Big)+O_{H^1}(h^{k_1-(k_1+2)\delta}) \tag{7.74}$$

and over the boundary we have

$$\begin{aligned}\tilde\varphi^\delta(F^*_{\delta h}u_\omega-\chi^\delta_L w_{L,\delta}-\chi^\delta_R w_{R,\delta})&=\tilde\varphi^\delta O_{H^{\frac12}_\partial}(h^{k_1-(k_1+\frac32)\delta})+O_{S^0_\partial}(M^2h^{k_1+2-k_1\delta})+O(M^2h^2)\\&=O_{H^{\frac12}_\partial}(h^{k_1-(k_1+\frac32)\delta}+M^2h^{2-\frac\delta2}),\end{aligned} \tag{7.75}$$

where we used $|x|\sim 1$ on $\operatorname{supp}\tilde\varphi^\delta$. Thus

$$\varphi^\delta\mathcal{H}^{\tilde\Omega^\delta}_{h^{1-\delta}M}\tilde\varphi^\delta(F^*_{\delta h}u_\omega-\chi^\delta_L w_{L,\delta}-\chi^\delta_R w_{R,\delta})=O_{H^1}(h^{k_1-(k_1+\frac32)\delta}+M^2h^{2-\frac\delta2}). \tag{7.76}$$

It remains to estimate

$$\begin{aligned}&\varphi^\delta\mathcal{R}^{\tilde\Omega^\delta}_{h^{1-\delta}M}(-\Delta+h^{2-2\delta}M^2)\tilde\varphi^\delta(F^*_{\delta h}\tilde u_\omega-\chi^\delta_L\bar w_{L,\delta}-\chi^\delta_R\bar w_{R,\delta})=\varphi^\delta\mathcal{R}^{\tilde\Omega^\delta}_{h^{1-\delta}M}\tilde\varphi^\delta(-\Delta+h^{2-2\delta}M^2)F^*_{\delta h}\tilde u_\omega\\&-\sum_{L/R}\varphi^\delta\mathcal{R}^{\tilde\Omega^\delta}_{h^{1-\delta}M}\tilde\varphi^\delta[-\Delta,\chi^\delta_{L/R}]\bar w_{L/R,\delta}+\varphi^\delta\mathcal{R}^{\tilde\Omega^\delta}_{h^{1-\delta}M}[-\Delta,\tilde\varphi^\delta](F^*_{\delta h}\tilde u_\omega-\chi^\delta_L\bar w_{L,\delta}-\chi^\delta_R\bar w_{R,\delta}).\end{aligned} \tag{7.77}$$

3a. We show the following a priori elliptic estimates on $\tilde\Omega^\delta$ uniformly: for each $N>0$, there is $C>0$ that

$$\|\kappa^\delta\mathcal{R}^{\tilde\Omega^\delta}_{h^{1-\delta}M}\tau^\delta\|_{H^{-N}(\tilde\Omega^\delta)\to\dot H^N(\tilde\Omega^\delta)}\le C, \tag{7.78}$$

$$\|\varphi^\delta\mathcal{R}^{\tilde\Omega^\delta}_{h^{1-\delta}M}\rho^\delta_{L/R}\|_{Y^{0,-N}_{0,-N}(\tilde\Omega^\delta)\to\dot X^{0,N}_{0,N}(\tilde\Omega^\delta)}\le C, \tag{7.79}$$

uniformly in $h>0$ small. Note although $\tilde\Omega^\delta$ depends on h, $\Omega^\delta=F_\delta(\tilde\Omega^\delta)$ is a fixed compact domain with smooth boundary inside $\mathcal{W}_\alpha$, that does not depend on h.

First note $dF_\delta(z)=\mathrm{I}+O(h^{k_1(1-\delta)}z^{k_1})$ and so is dF_δ^{-1}. We can pullback the massive Laplacian between $\tilde\Omega^\delta$ and Ω^δ by

$$(F_\delta^{-1})^*(-\Delta+h^{2-2\delta}M^2)F^*_\delta=-\Delta+h^{2-2\delta}M^2+L^\delta, \tag{7.80}$$

$$F^*_\delta(-\Delta+h^{2-2\delta}M^2)(F_\delta^{-1})^*=-\Delta+h^{2-2\delta}M^2+\tilde L^\delta, \tag{7.81}$$

where

$$L^\delta=O(h^{k_1(1-\delta)}|z|^{k_1})\nabla^2+O(h^{k_1(1-\delta)}|z|^{k_1-1})\nabla,\ \tilde L^\delta=O(h^{k_1(1-\delta)}|z|^{k_1})\nabla^2+O(h^{k_1(1-\delta)}|z|^{k_1-1})\nabla. \tag{7.82}$$

Now note

$$(-\Delta+h^{2-2\delta}M^2+L^\delta)\mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M}(\mathrm{I}-(\mathrm{I}+L^\delta\mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M})^{-1}L^\delta\mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M})=\mathrm{I}. \tag{7.83}$$

This implies

$$\mathcal{R}^{\tilde\Omega^\delta}_{h^{1-\delta}M}=F^*_\delta\mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M}(\mathrm{I}-(\mathrm{I}+L^\delta\mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M})^{-1}L^\delta\mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M})(F_\delta^{-1})^* \tag{7.84}$$

maps from $H^{-1}(\tilde\Omega^\delta)$ to $\dot H^1(\tilde\Omega^\delta)$ with bounds uniform in h small, and all $M\le M_0$.

By Lemma 4.8, we have $\varphi^\delta \mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M}\rho^\delta_{L/R} : Y^{0,-N}_{0,-N}(\Omega^\delta) \to \dot{X}^{0,N}_{0,N}(\Omega^\delta)$ uniformly bounded with arbitrary gain in the b-regularity because of the disjoint support of φ^δ and $\rho^\delta_{L/R}$. Noting $L^\delta : \dot{X}^{0,N}_{0,N}(\Omega^\delta) \to Y^{0,N}_{0,N}(\Omega^\delta)$ is a local differential operator of size $O(h^{k_1(1-\delta)})$, we have for small h that

$$(F_\delta^{-1})^*\varphi^\delta \mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}\rho^\delta_{L/R}F^*_\delta = \varphi^\delta\Big(\mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M} - \sum_{k=0}^{\infty}\mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M}(-L^\delta\mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M})^k L^\delta \mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M}\Big)\rho^\delta_{L/R} \tag{7.85}$$

is uniformly bounded from $Y^{0,-N}_{0,-N}(\Omega^\delta) \to \dot{X}^{0,N}_{0,N}(\Omega^\delta)$, and thus (7.79). Similarly noting κ^δ and τ^δ have disjoint support both away from $\partial\Omega^\delta$, we have $\kappa^\delta \mathcal{R}^{\Omega^\delta}_{h^{1-\delta}M}\tau^\delta : H^{-N}(\Omega^\delta) \to \dot{H}^N(\Omega^\delta)$ uniformly bounded and thus (7.78).

3b. We estimate (7.77) in the rest of Step 3. By (7.74) we can estimate

$$\tilde{\varphi}^\delta(-\Delta + h^{2-2\delta}M^2)F^*_{\delta h}\tilde{u}_\omega = O_{H^{-1}}(h^{k_1-(k_1+2)\delta}), \tag{7.86}$$

and the first term of (7.77) uniformly by

$$\varphi^\delta \mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}\tilde{\varphi}^\delta(-\Delta + h^{2-2\delta}M^2)F^*_{\delta h}\tilde{u}_\omega = O_{\dot{H}^1}(h^{k_1-(k_1+2)\delta}). \tag{7.87}$$

Note $\tilde{\varphi}^\delta[-\Delta, \chi^\delta_\pm] = \tilde{\varphi}^\delta[-\Delta, \chi^\delta_{L/R}]\kappa^\delta$ and $[-\Delta, \tilde{\kappa}^\delta] = \tau^\delta[-\Delta, \tilde{\kappa}^\delta]$ for the second term of (7.77). We want to understand $\kappa^\delta \bar{w}_{L/R}$. Indeed, we have

$$\tilde{\kappa}^\delta \bar{w}_{L/R,\delta} = \mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}(-\Delta + h^{2-2\delta}M^2)\tilde{\kappa}^\delta \bar{w}_{L/R,\delta} = \mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}\tau^\delta[-\Delta, \tilde{\kappa}^\delta]\tau^\delta \bar{w}_{L/R,\delta} \tag{7.88}$$

and thus $\kappa^\delta \bar{w}_{L/R,\delta} = \kappa^\delta \mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}\tau^\delta[-\Delta, \tilde{\kappa}^\delta]\bar{w}_{L/R,\delta}$. Apply (7.78) and (4.117) to see

$$\|\kappa^\delta \bar{w}_{L/R,\delta}\|_{H^1(\tilde{\Omega}^\delta)} \le C\|\tau^\delta \bar{w}_{L/R,\delta}\|_{H^{-2N}(\bar{\Gamma}^\delta_\pm)} \le C\|w_{L/R,\delta}\|_{H^{-2N+\frac{1}{2},-N}_\partial} \le C(h^{N\delta} + h^{1+k_1-k_1\delta}) = O(h^{1+k_1-k_1\delta}) \tag{7.89}$$

for large $N > 0$, since τ^δ is compactly supported and we have

$$e^{\pm ih^{-\delta}\omega x} = \Big(\frac{\pm h^\delta D}{\omega}\Big)^N e^{\pm ih^{-\delta}\omega x} \tag{7.90}$$

is away from the zero frequency. This implies we can estimate the second term of (7.77) by

$$\|\varphi^\delta \mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}\tilde{\varphi}^\delta[-\Delta, \chi^\delta_{L/R}]\bar{w}_{L/R,\delta}\|_{\dot{H}^1(\tilde{\Omega}^\delta)} = O(h^{1+k_1-k_1\delta}). \tag{7.91}$$

3c. For the second piece of the third term of (7.77), note $\rho^\delta_{L/R}[-\Delta, \tilde{\varphi}^\delta]\chi^\delta_{L/R} = [-\Delta, \tilde{\varphi}^\delta]\chi^\delta_{L/R}$. Similarly to (7.89), we can use Lemma 4.10, (7.79), (4.117) to estimate for any $N > 0$ large

$$\|\varphi^\delta \mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}[-\Delta, \tilde{\varphi}^\delta]\chi^\delta_{L/R}\bar{w}_{L/R,\delta}\|_{H^1(\tilde{\Omega}^\delta)} \le C\|\rho^\delta_{L/R}\bar{w}_{L/R,\delta}\|_{X^{-N,0}_{0,-N}(\tilde{\mathcal{W}}^\delta_{L/R})} \le C\|w_{L/R,\delta}\|_{H^{-2N+\frac{1}{2},-N}_\partial} = O(h^{1+k_1-k_1\delta}) \tag{7.92}$$

for large $N > 0$, since $\rho^\delta_{L/R}$ is again compactly supported.

3d. For the first piece of the last term of (7.77), we want to understand $\rho^\delta_{L/R}\chi^\delta_{L/R}F^*_{\delta h}\tilde{u}_\omega$. On $\tilde{\Omega}^\delta$, by replacing $\tilde{\varphi}^\delta$ by $\rho^\delta_{L/R}\chi^\delta_{L/R}$ in (7.74) we have

$$\rho^\delta_R\chi^\delta_R F^*_{\delta h}\tilde{u}_\omega = \rho^\delta_R\chi^\delta_R\Big(a^o_+ e_{+,R,\delta} + a^i_- e_{-,R,\delta}\Big) + O_{H^1}(h^{k_1-(k_1+2)\delta}) \\ = O_{X^{0,0}_{0,-N}}(h^{N\delta}) + O_{H^1}(h^{k_1-(k_1+2)\delta}), \quad (7.93)$$

and $\rho^\delta_L\chi^\delta_L F^*_{\delta h}\tilde{u}_\omega = O_{X^{0,0}_{0,-N}}(h^{N\delta}) + O_{H^1}(h^{k_1-(k_1+2)\delta})$ accordingly. Similarly to (7.92), apply (7.79) to see

$$\|\varphi^\delta \mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}[-\Delta,\tilde{\varphi}^\delta]\chi^\delta_\pm F^*_{\delta h}\tilde{u}_\omega\|_{\dot{H}^1} \le Ch^{k_1-(k_1+2)\delta}. \quad (7.94)$$

3e. We now finish proving the claim (7.42). Steps 3a–3d imply (7.77) can be estimated by

$$\varphi^\delta \mathcal{R}^{\tilde{\Omega}^\delta}_{h^{1-\delta}M}(-\Delta + h^{2-2\delta}M^2)\tilde{\varphi}^\delta(F^*_{\delta h}\tilde{u}_\omega - \chi^\delta_L\bar{w}_{L,\delta} - \chi^\delta_R\bar{w}_{R,\delta}) = O_{\dot{H}^1(\tilde{\Omega}^\delta)}(h^{k_1-(k_1+2)\delta}), \quad (7.95)$$

and thus (7.72) by

$$\varphi^\delta f = O_{H^1(\tilde{\Omega}^\delta)}(h^{k_1-(k_1+2)\delta} + M^2h^{2-\frac{\delta}{2}}). \quad (7.96)$$

Pulling $\varphi^\delta f$ back to $\tilde{\mathcal{W}}^h_\alpha$, we have

$$\varphi^\delta f = O_{L^2(\tilde{\mathcal{W}}^h_\alpha)}(h^{k_1-(k_1+3)\delta} + M^2h^{2-\frac{\delta}{2}}),\ \nabla\varphi^\delta f = O_{L^2(\tilde{\mathcal{W}}^h_\alpha)}(h^{k_1-(k_1+2)\delta} + M^2h^{2-\frac{\delta}{2}}), \quad (7.97)$$

$$\varphi^\delta f = O_{X^0_0(\tilde{\mathcal{W}}^h_\alpha)}(h^{k_1-(k_1+3)\delta} + M^2h^{2-\frac{\delta}{2}}). \quad (7.98)$$

Revisit (7.48) to give the estimates for the zero frequencies

$$\partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}[-\Delta,\chi^\delta]\Big(F^*_h\tilde{u}_\omega - \chi^\delta_L\bar{w}_{L,\delta} - \chi^\delta_R\bar{w}_{R,\delta}\Big) = \partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}[-\Delta,\chi^\delta]\varphi^\delta f = O_{H^{-\frac{1}{2},0}_\partial}(h^{k_1-(k_1+\frac{7}{2})\delta} + M^2h^{2-\frac{\delta}{2}}). \quad (7.99)$$

For the nonzero frequencies, similarly to (7.61) we have

$$\|\gamma^h(1-\psi_0)(D)\gamma^h\partial_\nu\mathcal{R}^{\tilde{\mathcal{W}}^h_\alpha}_{hM}[-\Delta,\chi^\delta]\Big(F^*_h\tilde{u}_\omega - \chi^\delta_L\bar{w}_{L,\delta} - \chi^\delta_R\bar{w}_{R,\delta}\Big)\|_{H^{-\frac{1}{2},N}} \le C\|\varphi^\delta f\|_{X^{0,N}_0} \\ \le Ch^{k_1-(N+k_1+\frac{7}{2})\delta} + CM^2h^{2-N\delta}, \quad (7.100)$$

similar to (7.55). This gives the claim (7.68) stated at the beginning of Step 3. Together with Step 2 we have (7.42) as desired.

4. In the end, by setting $0 < \delta \ll 1$ and renumbering N, we have

$$(\Lambda^{\tilde{\mathcal{W}}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2})v' = O_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N+1,N+1}}(h^{\min(2\mathfrak{r}_\alpha,k_1)-0} + M^2h^{2-0}),\ \|v'\|_{Z^{\frac{1}{2},N,\frac{1}{2}}_{-L_0+0,-\frac{1}{2}-0,-\frac{1}{2}-0}} \sim 1, \quad (7.101)$$

where near $\pm\omega$ frequencies we have

$$v'(x) \sim (a^i_\mp + O(M^2h^2))e^{-i\omega|x|} + (a^o_\pm + O(M^2h^2))e^{i\omega|x|}, \quad (7.102)$$

as $x \to \pm\infty$. Let $v = v' - (\Lambda^{\tilde{\mathcal{W}}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2} - i0)^{-1}(\Lambda^{\tilde{\mathcal{W}}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2})v'$. Proposition 5.18 implies $(\Lambda^{\tilde{\mathcal{W}}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2} - i0)^{-1} : Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N+1,N+1} \to Z^{\frac{1}{2},N,\frac{1}{2}}_{-L_0+0,N,-\frac{1}{2}-0}$ is bounded, and we

have

$$\|(\Lambda_{hM}^{\tilde{\mathcal{W}}_\alpha^h}-\sqrt{\omega^2+h^2M^2}-i0)^{-1}(\Lambda_{hM}^{\tilde{\mathcal{W}}_\alpha^h}-\sqrt{\omega^2+h^2M^2})v'\|_{Z^{\frac{1}{2},N,\frac{1}{2}}_{-L_0+0,N,-\frac{1}{2}-0}}\leq Ch^{\min(2\mathfrak{r}_\alpha,k_1)-0}+CM^2h^{2-0} \tag{7.103}$$

and $(\Lambda_{hM}^{\tilde{\mathcal{W}}_\alpha^h}-\sqrt{\omega^2+h^2M^2})v=0$. We have

$$v(x)\sim(a^i_\mp+O(M^2h^{2-0})+O(h^{\min(2\mathfrak{r}_\alpha,k_1)-0}))e^{-i\omega|x|}+(a^o_\pm+O(M^2h^{2-0})+O(h^{\min(2\mathfrak{r}_\alpha,k_1)-0}))e^{i\omega|x|} \tag{7.104}$$

near $\pm\omega$ frequencies. This implies

$$\Sigma^{\tilde{\mathcal{W}}_\alpha}_{M,h^{-1}\omega}=\Sigma^{\tilde{\mathcal{W}}_\alpha^h}_{hM,\omega}=\Sigma^{\mathcal{W}_\alpha^h}_{hM,\omega}+O(M^2h^{2-0})+O(h^{\min(2\mathfrak{r}_\alpha,k_1)-0})=\Sigma^{\mathcal{W}_\alpha}_{M,h^{-1}\omega}+O(M^2h^{2-0})+O(h^{\min(2\mathfrak{r}_\alpha,k_1)-0}) \tag{7.105}$$

via rescaling. □

7.4. **Comparison with $M=0$.** We can now work on the straight wedges $\mathcal{W}_{\alpha_j}$. In this subsection, we compare the scattering matrices at high frequencies and fixed intermediate frequencies, on straight wedges.

Lemma 7.5. *Let $M_0>0$, $\omega\in[\frac{1}{2},2]$, $\alpha\in(0,2\pi)$. Then there exists $h_0>0$ that uniformly for all $M\in[0,M_0]$, all $h\in(0,h_0]$, we have*

$$\Sigma^{\mathcal{W}_\alpha}_{M,h^{-1}\omega}=\Sigma^{\mathcal{W}_\alpha}_{0,\omega}+O((hM)^{\min(2\mathfrak{r}_\alpha,2)-0}+h^\infty). \tag{7.106}$$

Proof. 1. Let $a^i=(a^i_+,a^i_-)\in\mathbb{C}^2$ with $|a^i|=1$. In this proof, we adopt the geometry set up in Section 7.1 with $\tilde{\mathcal{W}}_\alpha=\mathcal{W}_\alpha$, and drop the tilde as long as there is no confusion. Proposition 6.3 implies there exists $u\in Z^{\infty,\infty,1-0}_{L_0-0,-\frac{1}{2}-0,-\frac{1}{2}-0}(\partial\mathcal{W}^h_\alpha;\omega)$ that $(\Lambda_{hM}^{\mathcal{W}_\alpha^h}-\sqrt{\omega^2+h^2M^2})u(x)=0$ with $u\sim\tilde{a}^i_\mp(x)e^{-i\omega|x|}+\tilde{a}^o_\pm(x)e^{i\omega|x|}$ away from the zero frequency as $x\to\pm\infty$, where

$$\tilde{a}^i_\pm(x)=a^i_\pm+O(|x|^{-1}),\ \tilde{a}^o_\pm(x)=a^o_\pm+O(|x|^{-1}),\ a^o=(a^o_+,a^o_-)=\Sigma^{\mathcal{W}_\alpha}_{M,h^{-1}\omega}a^i=\Sigma^{\mathcal{W}_\alpha^h}_{hM,\omega}a^i. \tag{7.107}$$

2. We show $\tilde{a}^{i/o}_\pm(x)$ are h^2M^2-close to $a^{i/o}_\pm$ as $|x|\to\infty$. In order to examine the gains in decay at different frequencies, we consider cutoff $\chi_0,\chi,\chi_1,\tilde{\chi}\in C^\infty_c(\mathbb{R})$ with $\chi_0\equiv1$ near 0, $\chi_0\prec\chi\prec\chi_1\prec\tilde{\chi}$. For any of these cutoffs, ψ, set $\psi^h(x)=\psi(h^\delta x)$ for some $\delta>0$ to be chosen arbitrarily small. Let also $\psi,\tilde{\psi},\tilde{\tilde{\psi}}\in C^\infty_c(-2\omega,2\omega)$ with $\operatorname{supp}(1-\psi)\cap\{\pm\omega\}=\emptyset$, $\operatorname{supp}\tilde{\tilde{\psi}}\cap\{0\}=\emptyset$ and $\psi\prec\tilde{\psi}\prec\tilde{\tilde{\psi}}$.

By Lemma 4.18, we can compare $\Lambda_{hM}^{\mathcal{W}_\alpha^h}$ to the model operator

$$\Lambda^{L/R}_{hM}=1_{x>0}\Lambda_{hM}^{\mathcal{W}^h_{\alpha,R}}1_{x>0}+1_{x<0}\Lambda_{hM}^{\mathcal{W}^h_{\alpha,L}}1_{x<0}. \tag{7.108}$$

We have

$$(1-\chi^h_1)\Lambda^{L/R}_{hM}=(1-\chi^h_1)\Lambda^{L/R}_{hM}(1-\chi^h_0)+(1-\chi^h_1)\tilde{\mathcal{E}}_{hM}, \tag{7.109}$$

$$\Lambda^{L/R}_{hM}(1-\chi^h_1)=(1-\chi^h_0)\Lambda^{L/R}_{hM}(1-\chi^h_1)+\tilde{\mathcal{E}}^*_{hM}(1-\chi^h_1), \tag{7.110}$$

where $\tilde{\mathcal{E}}_{hM}$ is the error in Lemma 4.18 given by

$$\tilde{\mathcal{E}}_{hM} = \sum_{L/R} \partial_\nu \mathcal{R}^{\mathcal{W}^h_{\alpha,L/R}}_{hM}[\Delta, \chi_0^h]\mathcal{H}^{\mathcal{W}^h_\alpha}_{hM}. \tag{7.111}$$

Notice that u satisfies

$$(1-\chi_1^h)(\Lambda^{L/R}_{hM}(1-\chi_0^h) + \tilde{\mathcal{E}}_{hM} - \sqrt{\omega^2+h^2M^2})u = 0. \tag{7.112}$$

Therefore, using Lemma C.2 on $\mathcal{W}^h_{\alpha,L/R}$ and then rescaling, we obtain $b_0 \in S^{0,-1}$ such that

$$\begin{aligned}(1-\chi_1^h)\Big(\big(\sqrt{D^2+h^2M^2}\tilde{\psi}(D) + h^3M^2b_0(hx,D)\big)(1-\chi_0^h) - \sqrt{\omega^2+h^2M^2}\Big)\psi(D)u \\ = -(1-\chi_1^h)\big(\Lambda^{L/R}_{hM}(1-\chi_0^h) + \tilde{\mathcal{E}}_{hM} - \sqrt{\omega^2+h^2M^2}\big)(1-\psi(D))u \\ - (1-\chi_1^h)E^{L/R}_{hM}(1-\chi_0^h)\psi(D)u - (1-\chi_1^h)\tilde{\mathcal{E}}_{hM}\psi(D)u. \end{aligned} \tag{7.113}$$

Now, by Proposition 6.3 $(1-\psi(D))u \in Z^{\infty,\infty,1-0}_{L_0-0,\infty,\infty}$. Therefore, using Lemma 4.18 and Proposition 4.22 and the estimate on $E^{L/R}_{hM}$ from Lemma C.2,

$$\begin{aligned}\|\tilde{\psi}(D)(1-\chi_1^h)\Big(\big(\sqrt{D^2+h^2M^2}\tilde{\psi}(D) + h^3M^2b_0(hx,D)\big)(1-\chi_0^h) - \sqrt{\omega^2+h^2M^2}\Big)\psi(D)u\|_{H^{N,N}} \\ \leq C_N h^N \|u\|_{Z^{s,l,s_0}_{L_0-0,-\frac{1}{2}-0,-\frac{1}{2}-0}}. \end{aligned} \tag{7.114}$$

Since we already know that

$$(1-\chi_0^h)\psi(D)u = (1-\chi_0^h)\psi(D)\sum_{\pm}\big(\tilde{a}^i_\pm(x)e^{-i|x|\omega} + \tilde{a}^o_\pm(x)e^{i|x|\omega}\big) + O(\mathcal{S}), \tag{7.115}$$

the calculations in Lemma C.3 imply that

$$(1-\chi^h)(\tilde{a}^{i/o}_\pm(x) - (a^{i/o}_\pm + h^2M^2b^{i/o}_\pm(x))) \in h^\infty\mathcal{S} \tag{7.116}$$

with $b^{i/o}_\pm(x)$ satisfying

$$|\partial_x^\beta b^{i/o}_\pm(x)| \leq C\langle x\rangle^{-2}\|u\|_{Z^{s,l,s_0}_{L_0-0,-\frac{1}{2}-0,-\frac{1}{2}-0}},\ \beta \geq 1. \tag{7.117}$$

3. We consider

$$F_{hM} = \Lambda^{\mathcal{W}^h_\alpha}_{hM} - \sqrt{\omega^2+h^2M^2} - \Lambda^{\mathcal{W}^h_\alpha}_0 + \omega. \tag{7.118}$$

We claim for each $s, l \in \mathbb{R}$, $s_0 \in (0,1)$, we have

$$\|F_{hM}u\|_{Z^{-\frac{1}{2},-\mathfrak{r}_\alpha,\frac{1}{2}}_{-\mathfrak{r}_\alpha-\frac{1}{2}+0,\frac{3}{2}-0,\frac{3}{2}-0}} \leq C((hM)^{\min(2\mathfrak{r}_\alpha,2)-0} + h^\infty)\|u\|_{Z^{\frac{1}{2},\mathfrak{r}_\alpha+\frac{1}{2}-0,\frac{1}{2}}_{\mathfrak{r}_\alpha+\frac{1}{2}-0,-\frac{1}{2}-0,-\frac{1}{2}-0}}. \tag{7.119}$$

Once we have the claim, its application gives

$$\|(\Lambda^{\mathcal{W}^h_\alpha}_0 - \omega - i0)^{-1}(\Lambda^{\mathcal{W}^h_\alpha}_0 - \omega)u\|_{Z^{\frac{1}{2},-\mathfrak{r}_\alpha,\frac{1}{2}}_{-\mathfrak{r}_\alpha-\frac{1}{2}+0,\frac{1}{2}-0,-\frac{1}{2}-0}} \leq C(hM)^{\min(2\mathfrak{r}_\alpha,2)-0} + h^\infty, \tag{7.120}$$

uniformly in $M \in [0, M_0]$ and h small, since $(\Lambda_0^{\mathcal{W}_\alpha^h} - \omega)u = -F_{hM}u$. We then construct a sequence of v by

$$v = u - (\Lambda_0^{\mathcal{W}_\alpha^h} - \omega - i0)^{-1}(\Lambda_0^{\mathcal{W}_\alpha^h} - \omega)u \in Z^{\infty,\infty,1-0}_{-L_0+0,-\frac{1}{2}-0,-\frac{1}{2}-0},\ (\Lambda_0^{\mathcal{W}_\alpha^h} - \omega)v = 0, \tag{7.121}$$

with $u - v = O_{Z^{\frac{1}{2},-\mathfrak{r}_\alpha,\frac{1}{2}}_{-\mathfrak{r}_\alpha-\frac{1}{2}+0,-\frac{1}{2}+0,-\frac{1}{2}-0}}((hM)^{\min(2\mathfrak{r}_\alpha,2)-0} + h^\infty)$. This implies u, v differ by that size in the scattering data, that is

$$\begin{aligned}\Sigma^{\mathcal{W}_\alpha}_{0,h^{-1}\omega}a^i = \Sigma^{\mathcal{W}_\alpha^h}_{0,\omega}a^i &= \big(\Sigma^{\mathcal{W}_\alpha^h}_{hM,\omega} + O((hM)^{\min(2\mathfrak{r}_\alpha,2)-0} + h^\infty)\big)a^i \\ &= \big(\Sigma^{\mathcal{W}_\alpha}_{M,h^{-1}\omega} + O((hM)^{\min(2\mathfrak{r}_\alpha,2)-0} + h^\infty)\big)a^i\end{aligned} \tag{7.122}$$

as desired. The rest of the proof will be devoted to (7.119). We can decompose

$$F_{hM} = \tilde\chi^h F_{hM}\chi^h + (1-\tilde\chi^h)F_{hM}(1-\chi^h) + \tilde\chi^h F_{hM}(1-\chi^h) + (1-\tilde\chi^h)F_{hM}\chi^h, \tag{7.123}$$

which we estimate term by term.

3a. For arbitrary eligible $\chi, \tilde\chi$, Lemma 4.21 implies

$$\|\tilde\chi^h F_{hM}\chi^h\|_{H^{0,-N}\to H^{0,N}} \le C(hM)^{\min(2\mathfrak{r}_\alpha-0,2)}h^{-C\delta}. \tag{7.124}$$

This specifically implies

$$\|\tilde\chi^h F_{hM}\chi^h u\|_{Z^{N,N,0}_{N,N,N}} \le C(hM)^{\min(2\mathfrak{r}_\alpha-0,2)}h^{-C\delta}\|u\|_{Z^{-N,-N,0}_{-N,-N,-N}}, \tag{7.125}$$

which allows us to estimate the first term of (7.123).

3b. We now consider the second term of (7.123). By (7.109) it is enough to estimate

$$\begin{aligned}&(1-\tilde\chi^h)F_{hM}(1-\chi^h) = (1-\tilde\chi^h)(\Lambda^{L/R}_{hM} - \sqrt{\omega^2+h^2M^2} - \Lambda_0^{L/R} + \omega + \tilde{\mathcal{E}}_{hM} - \tilde{\mathcal{E}}_0)(1-\chi^h)\\ &= (1-\tilde\chi^h)(\Lambda^{L/R}_{hM} - \sqrt{\omega^2+h^2M^2} - \Lambda_0^{L/R} + \omega)(1-\chi^h) + (1-\tilde\chi^h)(\tilde{\mathcal{E}}_{hM} - \tilde{\mathcal{E}}_0)(1-\chi^h).\end{aligned} \tag{7.126}$$

We first show that

$$\|\tilde\psi(D)(1-\tilde\chi^h)(\Lambda^{L/R}_{hM} - \sqrt{\omega^2+h^2M^2} - \Lambda_0^{L/R} + \omega)(1-\chi^h)u\|_{Z^{N,N,1}_{N,\frac{3}{2}-0,\frac{3}{2}-0}} \le C\big((hM)^2 + h^N\big)\|u\|_{Z^{s,l,s_0}_{L_0-0,-\frac{1}{2}-0,-\frac{1}{2}-0}}. \tag{7.127}$$

Using Lemma C.2 again and rescaling, we have both

$$\begin{aligned}\tilde\psi(D)(1-\tilde\chi^h)(\Lambda^{L/R}_{hM} - \Lambda_0^{L/R})(1-\chi^h) = \tilde\psi(D)(1-\tilde\chi^h)\Big(&\sqrt{D^2+h^2M^2}\tilde{\tilde\psi}(D) + h^3M^2b_0(hx,D)\\ &- |D|\tilde{\tilde\psi}(D)\Big)(1-\chi^h) + O_{H^{-N,-\mathfrak{r}_\alpha-\frac{1}{2}+0}\to H^{N,N}}(h^\infty)\end{aligned} \tag{7.128}$$

and

$$\begin{aligned}(1-\tilde\chi^h)(\Lambda^{L/R}_{hM} - \Lambda_0^{L/R})(1-\chi^h)\psi(D) = (1-\tilde\chi^h)\Big(&\sqrt{D^2+h^2M^2}\tilde{\tilde\psi}(D) + h^3M^2b_0(hx,D)\\ &- |D|\tilde{\tilde\psi}(D)\Big)(1-\chi^h)\psi(D) + O_{H^{-N,-N}\to H^{N,\mathfrak{r}_\alpha+\frac{1}{2}-0}}(h^\infty).\end{aligned} \tag{7.129}$$

Now, by (7.114) we have

$$\tilde{\psi}(D)(1-\tilde{\chi}^h)(\sqrt{D^2+h^2M^2}\tilde{\tilde{\psi}}(D)+h^3M^2b_0(hx,D)-\sqrt{\omega^2+h^2M^2}-|D|\tilde{\tilde{\psi}}(D)+\omega)(1-\chi^h)u$$
$$=-\tilde{\psi}(D)(1-\tilde{\chi}^h)(|D|\tilde{\tilde{\psi}}(D)-\omega)(1-\chi^h)u+O_{\mathcal{S}}(h^\infty). \quad (7.130)$$

By (7.116) we have

$$\|\tilde{\psi}(D)(1-\tilde{\chi}^h)(|D|\tilde{\tilde{\psi}}(D)-\omega)(1-\chi^h)u\|_{Z^{s-1,l}_{N,\frac{3}{2}-0,\frac{3}{2}-0}}\leq C(hM)^2\|(1-\chi)u\|_{Z^{s,l}_{L_0-0,-\frac{1}{2}-0,-\frac{1}{2}-0}}$$
$$+Ch^N\|u\|_{Z^{s,l,s_0}_{-L_0+0,-N,-N}}, \quad (7.131)$$

since

$$|D|\tilde{\tilde{\psi}}(D)e^{\pm i\omega x}\tilde{a}^{i/o}_{\pm}(x)\mp De^{\pm i\omega x}\tilde{a}^{i/o}_{\pm}(x)\in\mathcal{S}. \quad (7.132)$$

The above implies

$$\|(1-\tilde{\psi}(D))(1-\tilde{\chi}^h)(\Lambda^{L/R}_{hM}-\sqrt{\omega^2+h^2M^2}-\Lambda^{L/R}_0+\omega)(1-\chi^h)\psi(D)u\|_{Z^{N,N,1}_{L_0-0,N,N}}\leq C_Nh^N\|u\|_{Z^{s,l,s_0}_{-L_0+0,-N,-N}}. \quad (7.133)$$

On another hand, we can use Proposition 2.19 and Remark 2.20 to see

$$\|(1-\tilde{\psi}(D))(1-\tilde{\chi}^h)(\Lambda^{L/R}_{hM}-\sqrt{\omega^2+h^2M^2}-\Lambda^{L/R}_0+\omega)(1-\chi^h)(1-\psi(D))u\|_{Z^{-\frac{1}{2},-\mathfrak{r}_\alpha+\frac{1}{2}-0,-\frac{1}{2}}_{-\mathfrak{r}_\alpha+\frac{1}{2}+0,N,N}}$$
$$\leq C\|(1-\tilde{\psi}(D))(1-\tilde{\chi}^h)(\Lambda^{L/R}_{hM}-\sqrt{\omega^2+h^2M^2}-\Lambda^{L/R}_0+\omega)(1-\chi^h)(1-\psi(D))u\|_{H^{-\frac{1}{2},-\mathfrak{r}_\alpha+\frac{1}{2}+0}}$$
$$\leq C(hM)^{\min(2\mathfrak{r}_\alpha,2)-0}\|(1-\chi^h)(1-\psi(D))u\|_{H^{\frac{1}{2},\mathfrak{r}_\alpha+\frac{1}{2}-0}}\leq C(hM)^{\min(2\mathfrak{r}_\alpha,2)-0}\|u\|_{Z^{\frac{1}{2},\mathfrak{r}_\alpha+\frac{1}{2}-0,\frac{1}{2}}_{\mathfrak{r}_\alpha+\frac{1}{2}-0,-N,-N}}. \quad (7.134)$$

This demonstrates (7.127).

For the error terms, one notes also we have from Lemma 4.18 and Lemma 4.19 that

$$(1-\tilde{\chi}^h)(1-\psi_0(D))(1-\tilde{\chi}^h)(\tilde{\mathcal{E}}_{hM}-\tilde{\mathcal{E}}_0)(1-\chi^h)(1-\psi_0(D))(1-\chi^h)=O_{H^{-N,-N}\to H^{N,N}}((hM)^{\min(2\mathfrak{r}_\alpha,2)-0}),$$
$$(1-\tilde{\chi}^h)(1-\psi_0(D))(1-\tilde{\chi}^h)(\tilde{\mathcal{E}}_{hM}-\tilde{\mathcal{E}}_0)(1-\chi^h)=O_{H^{0,L_0-0}\to H^{N,N}}((hM)^{\min(2\mathfrak{r}_\alpha,2)-0}),$$
$$(1-\tilde{\chi}^h)(\tilde{\mathcal{E}}_{hM}-\tilde{\mathcal{E}}_0)(1-\chi^h)(1-\psi_0(D))(1-\chi^h)=O_{H^{-N,-N}\to H^{N,-L_0+0}}((hM)^{\min(2\mathfrak{r}_\alpha,2)-0}),$$
$$(1-\tilde{\chi}^h)(\tilde{\mathcal{E}}_{hM}-\tilde{\mathcal{E}}_0)(1-\chi^h)=O_{H^{0,L_0-0}\to H^{N,-L_0+0}}((hM)^{\min(2\mathfrak{r}_\alpha,2)-0}). \quad (7.135)$$

This implies

$$\|(1-\tilde{\chi}^h)(\tilde{\mathcal{E}}_{hM}-\tilde{\mathcal{E}}_0)(1-\chi^h)u\|_{Z^{N,N,1}_{-L_0+0,N,N}}\leq C(hM)^{\min(2\mathfrak{r}_\alpha,2)-0}\|u\|_{Z^{-N,-N,-1}_{L_0-0,-N,-N}}. \quad (7.136)$$

3c. For the third term of (7.123), we can use (7.110) to see that it is enough to estimate

$$\tilde{\chi}^h(1-\chi^h_0)(\Lambda^{L/R}_{hM}-\sqrt{\omega^2+h^2M^2}-\Lambda^{L/R}_0+\omega)(1-\chi^h_1)u+\tilde{\chi}^h(\tilde{\mathcal{E}}^*_M-\tilde{\mathcal{E}}^*_0)(1-\chi^h_1)u. \quad (7.137)$$

The first term can be estimated as in Step 3b and the last term using Lemma 4.19 as in (7.135).

3d. For the fourth term of (7.123), noting $\operatorname{supp}\chi^h \cap \operatorname{supp}(1-\tilde{\chi}^h) = \emptyset$ and using the representation (7.128) and (7.109), we have

$$(1-\tilde{\chi}^h)F_M\chi^h u = O(h^\infty)_{\Psi^{-\infty,-\infty}} u + (1-\tilde{\chi}^h)(\tilde{\mathcal{E}}_M - \tilde{\mathcal{E}}_0)\chi^h \tag{7.138}$$

and the estimate follows similar to (7.135). Bring all together to observe (7.119) as desired. □

7.5. **Derivatives of scattering matrices.** It will be convenient in Section 10 to know that the scattering matrix on a any curved wedge is $h^{2\pi/\alpha-0}$ close to one with bounded derivatives. (In fact, it is possible to show that the matrices themselves have bounded derivatives, but since it would complicate the proof and we do not need it, we do not pursue this generalization here.)

Throughout this subsection, we work on a curved wedge whose boundary consists of two half-lines outside of a compact set. We then identify the boundary of a curved wedge, $\mathcal{W}_\alpha$ with $\gamma : \mathbb{R} \to \partial\mathcal{W}_\alpha$ such that γ is smooth away from 0, has unit speed, and satisfies $\gamma''(s) = 0$ for $|s| \geq L$.

Lemma 7.6. *Suppose that $\mathcal{W}_\alpha$ is a curved wedge whose boundary consists of two half-lines outside of a compact set. Let $\chi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset (L,\infty)$, $\operatorname{supp}(1-\chi) \subset (-\infty, L+1)$. Then for all $j \geq 0$*

$$(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1}x^j\chi_\pm e^{\pm ix\omega/h} \mp \tfrac{i}{h(j+1)}x^{j+1}\chi_\pm e^{\pm ix\omega/h} =: e_{j,\pm} \in h^{-1}Z^{\infty,\infty,1-0}_{\min(\frac{3}{2},\frac{\pi}{\alpha}+\frac{1}{2})-0,\infty,-\frac{1}{2}-0}, \tag{7.139}$$

where $\chi_\pm(x) = \chi(\pm x)$. Moreover, there are $a_1, a_2 \in \mathbb{C}$ such that for $\psi \in C_c^\infty((-\omega,\omega))$ with $0 \notin \operatorname{supp} 1-\psi$ and $\phi \in C_c^\infty$ with $\operatorname{supp}(1-\phi) \cap [-L,L] = \emptyset$,

$$(1-\psi(hD))\psi(x)e_{\pm,j} = h^{-1}(a_1\chi_+e^{ix\omega/h} + a_2\chi_-e^{-ix/h}) + h^{-1}\mathcal{S}. \tag{7.140}$$

Proof. We consider the case of $+$, the other being similar First, notice that, with $u_j := x^{j+1}\chi_\pm e^{ix\omega/h}$,

$$\|u_j\|_{Z^{\infty,\infty,1}_{\infty,\infty,-\frac{1}{2}-j-0}} \leq C. \tag{7.141}$$

Therefore, we may apply $\Lambda_0^{\mathcal{W}_\alpha}$ to u_j. To prove the lemma, let $\tilde{\chi}_+ \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\tilde{\chi}_+ \subset (L,\infty)$,$\operatorname{supp}\chi_+ \cap \operatorname{supp}(1-\tilde{\chi}_+) = \emptyset$. Then using Lemma 4.18 we compute

$$\begin{aligned}
(h\Lambda_0^{\mathcal{W}_\alpha} - \omega)u_j &= \tilde{\chi}_+(h|D| - \omega)\chi_+x^{j+1}e^{ix\omega/h} + \mathcal{E}_0'\chi_+x^{j+1}e^{ix\omega/h} = \tilde{\chi}_+(h|D|\psi(hD) - \omega)\chi_+x^{j+1}e^{ix\omega/h} \\
&\quad + \tilde{\chi}_+h|D|(1-\psi(hD))\chi_+x^{j+1}e^{ix\omega/h} + \mathcal{E}_0'\chi_+x^{j+1}e^{ix\omega/h} = \chi_+(h|D|\psi(hD) - \omega)x^{j+1}e^{ix\omega/h} \\
&\quad + \tilde{\chi}_+[h|D|\psi(hD), \chi_+]x^{j+1}e^{ix\omega/h} + \tilde{\chi}_+h|D|(1-\psi(hD))\chi_+x^{j+1}e^{ix\omega/h} + \mathcal{E}_0'\chi_+x^{j+1}e^{ix\omega/h} \\
&= -ih\chi_+(j+1)x^je^{ix\omega/h} + \tilde{\chi}_+[h|D|\psi(hD), \chi_+]x^{j+1}e^{ix\omega/h} + \tilde{\chi}_+h|D|(1-\psi(hD))\chi_+x^{j+1}e^{ix\omega/h} \\
&\quad + \mathcal{E}_0'\chi_+x^{j+1}e^{ix\omega/h} =: -ih\chi_+(j+1)x^je^{ix\omega/h} + R_1 + R_2 + R_3.
\end{aligned} \tag{7.142}$$

Now, we estimate the last three terms in (7.142). First, notice that standard scattering calculus estimates imply

$$R_1 = \tilde{\chi}_+[h|D|\psi(hD), \chi_+]x^{j+1}e^{ix\omega/h} \in hZ^{\infty,\infty,\infty}_{\infty,\infty,\infty}. \tag{7.143}$$

Next, since $(1-\psi(hD))\chi_+x^{j+1}e^{ix\omega/h} \in h^\infty\mathcal{S}$,

$$R_2 = \tilde{\chi}_+h|D|(1-\psi(hD))\chi_+x^{j+1}e^{ix\omega/h} \in h^\infty Z^{\infty,\infty,\infty}_{\frac{3}{2}-0,\infty,\infty} \tag{7.144}$$

and, using Lemma 4.18

$$R_3 = \mathcal{E}_0' \chi_+ x^{j+1} e^{ix\omega/h} \in h^\infty Z^{\infty,\infty,\infty}_{\frac{\pi}{\alpha}+\frac{1}{2}-0,\infty,\infty}. \tag{7.145}$$

Hence, since u_j is outgoing

$$(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1} x^j \chi_+ e^{ix\omega/h} = \frac{i}{h(j+1)} \Big(u_j - (h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1}(R_1 + R_2 + R_3) \Big). \tag{7.146}$$

The lemma now follows from Lemmas 7.3 and Lemma 6.1. □

Lemma 7.7. *Suppose that $\mathcal{W}_\alpha$ is a curved wedge whose boundary consists of two half-lines outside of a compact set. Then,*

$$|h^j \partial_\omega^j \Sigma^{\mathcal{W}_\alpha}_{0,h^{-1}\omega}| \leq C_j, \qquad j = 1, 2. \tag{7.147}$$

Proof. Let $\chi_-, \tilde{\chi}_- \in C^\infty(\mathbb{R})$ with $\operatorname{supp} \tilde{\chi}_- \subset (-\infty, -L)$,$\operatorname{supp} \chi_- \cap \operatorname{supp}(1-\tilde{\chi}_-) = \emptyset$, and $\operatorname{supp}(1-\tilde{\chi}_-) \cap (-\infty, -L-1) = \emptyset$. Define $u^- := \chi_-(x) e^{ix\omega/h}$.

Then, observe that by Lemma 4.18,

$$\begin{aligned}
(h\Lambda_0^{\mathcal{W}_\alpha} - \omega) u^- &= (h\Lambda_0^{\mathcal{W}_\alpha} - \omega)\chi_- e^{ix\omega/h} \\
&= \tilde{\chi}_-(h|D| - \omega)\chi_- e^{ix\omega/h} + \mathcal{E}_0' \chi_- e^{ix\omega/h} \\
&= \tilde{\chi}_-(h|D|\psi(hD) - \omega)\chi_- e^{ix\omega/h} + \tilde{\chi}_- h|D|(1-\psi(hD))\chi_- e^{ix\omega/h} + \mathcal{E}_0' \chi_- e^{ix\omega/h} \\
&= \chi_-(h|D|\psi(hD) - \omega) e^{ix\omega/h} + \tilde{\chi_-}[h|D|\psi(hD), \chi_-] e^{ix\omega/h} \\
&\quad \tilde{\chi}_- h|D|(1-\psi(hD))\chi_- e^{ix\omega/h} + \mathcal{E}_0' \chi_- e^{ix\omega/h} \\
&= \tilde{\chi}_-[h|D|\psi(hD), \chi_-] e^{ix\omega/h} + \tilde{\chi}_- h|D|(1-\psi(hD))\chi_- e^{ix\omega/h} + \mathcal{E}_0' \chi_- e^{ix\omega/h}.
\end{aligned}$$

Therefore,

$$(hD_\omega)^\ell (h\Lambda_0^{\mathcal{W}_\alpha} - \omega) u^- = \big(\tilde{\chi}_-[h|D|\psi(hD), \chi_-] + \tilde{\chi}_- h|D|(1-\psi(hD))\chi_- + \mathcal{E}_0' \chi_- \big) x^\ell e^{i\omega x/h}.$$

In particular, arguing as in Lemma 7.6, we have

$$(hD_\omega)^\ell (h\Lambda_0^{\mathcal{W}_\alpha} - \omega) u^- \in h Z^{\infty,\infty,\infty}_{\min(\frac{3}{2}, \frac{\pi}{\alpha}+\frac{1}{2})-0,\infty,\infty}$$

Thus,

$$(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1} (hD_\omega)^\ell (h\Lambda_0^{\mathcal{W}_\alpha} - \omega) u^- = a_{+,\ell} \chi_+ e^{ix\omega/h} + a_{-,\ell} \chi_- e^{-ix\omega/h} + Z^{\infty,\infty,1-0}_{\frac{\pi}{\alpha}+\frac{1}{2}-0,\infty,\infty}$$

Applying Lemma 7.6, we thus obtain

$$\begin{aligned}
(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-j-1} (hD_\omega)^\ell (h\Lambda_0^{\mathcal{W}_\alpha} - \omega) u^- = (ih^{-1})^j a_{+,\ell} \frac{x^j}{j!} \chi_+ e^{ix\omega/h} + (ih^{-1})^j a_{-,\ell} \chi_-(-x)^j e^{-ix\omega/h} \\
+ h^{-j} Z^{\infty,\infty,1-0}_{\frac{\pi}{\alpha}+\frac{1}{2}-0,\infty,\infty}.
\end{aligned} \tag{7.148}$$

Now, consider

$$u = u^- - (h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1} (h\Lambda_0^{\mathcal{W}_\alpha} - \omega) u^-.$$

We study first $e^{-i\omega x/h} \chi_+ u$.

Define $f(\omega) := (h\Lambda_0^{\mathcal{W}_\alpha} - \omega)u^-$. Then, using (7.148) in the last line, we have

$$\begin{aligned}
&hD_\omega e^{-i\omega x}\chi_+ u \\
&= -hD_\omega e^{-i\omega x}\chi_+(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1} f(\omega) \\
&= -(e^{-i\omega x}\chi_+(-x - ih(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1})(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1} f(\omega) \\
&\qquad + e^{-i\omega x}\chi_+(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1} hD_\omega f(\omega) \\
&= b\chi_+^2 + e^{-i\omega x/h} Z^{\infty,\infty,\infty}_{\frac{\pi}{\alpha}+\frac{1}{2}-0,\infty,1-0},
\end{aligned}$$

for some $|b| \leq C$. Together with the same argument but considering $e^{i\omega x/h}\psi(hD)\chi_- u$ for some $\psi \in C_c^\infty(\mathbb{R}\setminus 0)$ with $-\omega \notin \operatorname{supp}(1-\psi)$, instead, we obtain

$$\|hD_\omega \Sigma_0^{\mathcal{W}_\alpha}\| \leq C.$$

Next,

$$\begin{aligned}
&(hD_\omega)^2\Big(e^{-ix\omega/h}(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1} f(\omega)\Big) \\
&= e^{-ix\omega/h}\Big[(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1}(hD_\omega)^2 f(\omega) \\
&\quad - 2x(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1} hD_\omega f(\omega) - 2ih(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-2} hD_\omega f(\omega) \\
&\quad + x^2(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-1} f(\omega) + 2ihx(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-2} f(\omega) \\
&\quad - 2h^2(h\Lambda_0^{\mathcal{W}_\alpha} - \omega - i0)^{-3} f(\omega)\Big].
\end{aligned}$$

Again, using (7.148) (together with a similar argument for $e^{i\omega x/h}\psi(hD)\chi_- u$, this implies the lemma. □

Remark 7.8. All constants in this section can depend on Ω in the following way: the upper bound on uniform Lipschitz characters of $\partial\Omega$, the upper bound on the diameter of $\partial\Omega \setminus \partial\tilde{\Omega}$ in Remark 3.11, for each multi-index α, the upper bounds on the derivatives of the metric g_+ in (3.11), and lower bounds on $|\det g_+|$.

8. Steklov eigenfunctions on curvilinear polygons

In this section, we study the structure of Steklov eigenfunctions, u, on curvilinear polygons, $\mathcal{P}$.

Outline of the strategy We study the structure of u in two steps.

First, we provide a relatively rough description of u on the interior of any smooth edge. More precisely we show in Lemma 8.1 that on the interior of a smooth edge u consists of a sum of (approximate) plane waves and a low frequency function (i.e. a function with uniformly bounded derivatives as the energy tends to infinity). This is done by exploiting the fact that, near smooth points, the DtN map is an elliptic first order pseudodifferential operator and hence that, away from the zero frequency $h\Lambda_M^{\mathcal{P}} - \sqrt{\omega^2 + h^2 M^2}$ is a *semiclassical* psuedodifferential operator of real principal type.

Once the rough structure from Lemma 8.1 is in place. We both improve it and determine the structure of u near corners. The overall strategy is to transplant u near a corner onto a curvilinear

wedge, $\mathcal{W}_\alpha$ that agrees with the curvilinear polygon near a corner. Using a strategy very close to that in the geometric comparison of Section 7, we are able to show that u, or rather an extension and rescaling, v_h' of u of the form (7.4) is nearly a solution to

$$(\Lambda_{hM}^{\mathcal{W}_\alpha^h} - \sqrt{\omega^2 + h^2M^2})v_h' = 0, \tag{8.1}$$

and hence that v_h' and therefore $u_h(x) = u(hx)$ inherits many of the properties of distorted plane waves on the infinite wedge.

8.1. **DtN Eigenfunctions on bounded Lipschitz domains.** We first show we can decompose a quasimode u that $(h\Lambda_M^\Omega - \sqrt{\omega^2 + h^2M^2})u = f$ into plane waves $\sum_\pm \tilde{a}_\pm(x;h)e^{\pm i\omega x/h}$ plus u_e, u_0, pieces at non-characteristic non-zero and zero frequencies respectively.

Lemma 8.1 (Eigenfunction structure on bounded Lipschitz domains)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded uniformly Lipschitz domain. Fix $\omega \in [\frac{1}{2}, 2]$. Let an h-dependent family of $u \in L^2(\partial\Omega)$ with $\|u\|_{L^2} \le C$ satisfy $(h\Lambda_M^\Omega - \sqrt{\omega^2 + h^2M^2})u = f$ with the global decomposition*

$$f = f_0 + f_e + f_\omega,\ f_0 \in \mathcal{D}'(\partial\Omega),\ f_e \in \mathcal{D}_c'(\partial\Omega_{\text{reg}}),\ f_\omega \in C_c^\infty(\partial\Omega_{\text{reg}}), \tag{8.2}$$

where we ask

$$\operatorname{WF}_h f_e \cap (\{\xi = 0\} \cup \{|\xi| = \omega\}) = \operatorname{WF}_h f_\omega \cap \{\xi = 0\} = \emptyset. \tag{8.3}$$

Let $\Gamma \subset \partial\Omega$ be open, smooth and simply connected. Let cutoffs $\gamma, \tilde{\gamma} \in C_c^\infty(\Gamma)$ that $\operatorname{supp}\gamma \cap \operatorname{supp}(1-\tilde{\gamma}) = \emptyset$ and both have connected support. Near $\operatorname{supp}\gamma$, we further ask $f_0 \in S^0$ and and there exist $\tilde{b}_\pm(x;h) \in S^0(\Gamma)$ that

$$f_\omega = h\sum_\pm \tilde{b}_\pm(x;h)e^{\pm i\omega x/h} + O_{S^0}(h^\infty) \text{ near } \operatorname{supp}\gamma. \tag{8.4}$$

Then there exist $\tilde{a}_\pm(x;h) \in S^0(\Gamma)$, $u_e(x;h) \in \mathcal{D}'(\Gamma)$, $u_0(x;h) \in S^0(\Gamma)$ such that

$$\gamma(x)u(x) = \gamma(x)\Big(\sum_\pm \tilde{a}_\pm(x;h)e^{\pm i\omega x/h} + u_e(x;h) + u_0(x;h)\Big) + O_{S^0}(h^\infty), \tag{8.5}$$

in local arclength coordinates x on Γ. Moreover, there exist $a_\pm(h) \in \mathbb{C}$ bounded in h, $a_{\pm,k}(x;h) \in S^0(\Gamma)$ that

$$\tilde{a}_\pm(x;h) \sim a_\pm(h) \pm i\int_{x_0}^x \tilde{b}_\pm(t;h)\,dt + h^2M^2\sum_{k=2}^\infty h^{k-2}a_{\pm,k}(x;h), \tag{8.6}$$

where the values $\tilde{a}_\pm(x_0;h)$ at any point $x_0 \in \operatorname{supp}\gamma$ uniquely determine $\tilde{a}_\pm(x;h)$ for all $x \in \operatorname{supp}\gamma$ modulo $O_{S^0}(h^\infty)$. For all $k, N \ge 0$, we can estimate

$$\|u_e\|_{H_h^k} \le C\|\tilde{\gamma}f_e\|_{H_h^{k-1}} + Ch^N\|u\|_{L^2},\ \operatorname{WF}_h(u_e) \subset \operatorname{WF}_h(f_e), \tag{8.7}$$

$$\|u_0\|_{H^k} \le C\|\tilde{\gamma}f_0\|_{S^0} + Ch\|u\|_{L^2},\ \operatorname{WF}_h(u_0) \subset \{\xi = 0\}. \tag{8.8}$$

Proof. We devote Steps 1 to 4 to show the plane wave structure at the characteristic frequencies, Step 5 to the zero frequencies, and Step 6 to the non-characteristic non-zero frequencies.

1. We make an a priori decomposition. Take $\tilde{\gamma} \in C_c^\infty(\Gamma)$, $\operatorname{supp}\gamma \cap \operatorname{supp}(1-\tilde{\gamma}) = \emptyset$, $\mathrm{WF}_h(f_0 + f_e) \cap (\{|\xi| = \omega\} \cap \operatorname{supp}\tilde{\gamma}) = \emptyset$ and Ω, $\tilde{\Omega}$ share the common smooth boundary near $\operatorname{supp}\tilde{\gamma}$ for some compact smooth domain $\tilde{\Omega}$. Assume $\partial_\nu \tilde{\gamma} = \partial_\nu \gamma = 0$. Then by Lemma 4.17 we have

$$\gamma \Lambda_M^\Omega = \gamma \Lambda_M^{\tilde{\Omega}} \tilde{\gamma} + \gamma \partial_\nu \mathcal{R}_M^{\tilde{\Omega}}[-\Delta, \tilde{\gamma}] \mathcal{H}_M^\Omega. \tag{8.9}$$

Note γ and $[-\Delta, \tilde{\gamma}]$ have disjoint support to see for all $N > 0$ that

$$\gamma \Lambda_M^\Omega = \gamma B \tilde{\gamma} + \gamma \mathcal{E},\ \gamma \mathcal{E} : H_\partial^0 \to H_\partial^N, \tag{8.10}$$

where $B = \Lambda_M^{\tilde{\Omega}} \in \Psi_{\mathrm{phg}}^1(\partial\tilde{\Omega})$ is self-adjoint with the classical principal and subprincipal symbols $\sigma_1(B) = |\xi|$ and $\sigma_0(B) = 0$. See [Tay23, Proposition C.1.] for further details.

2a. Let $Q \in \Psi_h^0(\partial\tilde{\Omega})$ to be fixed later with $\mathrm{WF}_h(Q) \cap (\{|\xi| = 0\} \cup \operatorname{supp}(1-\tilde{\gamma})) = \emptyset$ and $\mathrm{WF}_h(Q) \cap \mathrm{WF}_h(f_0 + f_e) = \emptyset$. We will prove that for any $\gamma, \tilde{\gamma}$ described in Step 1, for all $k \geq 0$, all $A_1, \dots, A_k \in \Psi_h^1(\partial\tilde{\Omega})$ with $\mathrm{WF}_h(A_j) \subset \mathrm{Ell}_h(Q)$, $\sigma_h(A_j)|_{\xi = \pm\omega} = 0$,

$$\|A_k \cdots A_1 \gamma u\|_{L^2} = O(h^k \|u\|_{L^2}) + O(h^k \|\tilde{b}_\pm\|_{S^0}) = O(h^k). \tag{8.11}$$

The nature of f_ω being a Lagrangian state implies

$$\|A_k \cdots A_1 \gamma f_\omega\|_{L^2} = O(h^{k+1} \|\tilde{b}_\pm\|_{S^0}). \tag{8.12}$$

To prove (8.11), we first recall that if $B \in \Psi^m$ and $A \in \Psi_h^l$ with $\mathrm{WF}_h(A) \cap \{\xi = 0\} = \emptyset$, then $Ah^m B \in \Psi_h^{m+l}$ and $\sigma_h(Ah^m B) = \sigma_h(A)(x,\xi)\sigma(B)(x,\xi)$.

2b. We proceed by induction on $k \geq 0$. For $k = 0$, (8.11) is true by the assumption. Suppose that it holds up to the case $k-1$. Now, with $a_1 = \sigma(A_1)$, we may write

$$a_1 \gamma = q_0 \gamma (\sqrt{\xi^2 + h^2 M^2} - \sqrt{\omega^2 + h^2 M^2}) \tilde{\gamma},\ Q_0 = \mathrm{Op}_h(q_0) \tag{8.13}$$

for some $q_0 \in S^0$ with $\operatorname{supp} q_0 \subset \mathrm{Ell}_h(Q)$, $\operatorname{supp} q_0 \cap \operatorname{supp}(1-\tilde{\gamma}) = \emptyset$. In particular,

$$A_1 \gamma = Q_0 \gamma (hB - \sqrt{\omega^2 + h^2 M^2}) \tilde{\gamma} + hR\tilde{\gamma},\ R \in \Psi_h^0,\ \mathrm{WF}_h(R) \subset \mathrm{Ell}_h(Q). \tag{8.14}$$

Then we have

$$\begin{aligned} &A_k \cdots A_1 \gamma u = A_k \cdots A_2 Q_0 \gamma (hB - \sqrt{\omega^2 + h^2 M^2}) \tilde{\gamma} u + h A_k \dots A_2 R \tilde{\gamma} u \\ &= A_k \cdots A_2 Q_0 \gamma (h\Lambda_M^\Omega - \sqrt{\omega^2 + h^2 M^2}) u - h A_k \cdots A_2 Q_0 \gamma \mathcal{E} u + h A_k \cdots A_2 R \tilde{\gamma} u + O_{\Psi^{-\infty}}(h^\infty) u. \end{aligned} \tag{8.15}$$

Noting $\mathrm{WF}_h(Q_0)$ is supported away from $\{\xi = 0\}$, we have for any $N > 0$ large that

$$\|A_k \dots A_2 Q_0 \gamma \mathcal{E} u\|_{L^2} \leq C \|Q_0 \gamma \mathcal{E} u\|_{H_h^{k+N-1}} \leq C \|(hD)^{k+N-1} \gamma \mathcal{E} u\|_{L^2} \leq C h^{k+N-1} \|u\|_{L^2}, \tag{8.16}$$

and thus by noting $A_k \cdots (A_2 R) \tilde{\gamma} u = O_{L^2}(h^{k-1})$ from the case $k-1$ applied to $\tilde{\gamma} u$, we have

$$\|A_k \cdots A_1 \gamma u\|_{L^2} = O(h^k \|u\|_{L^2}) + O(h^k \|\tilde{b}_\pm\|_{S^0}) = O(h^k) \tag{8.17}$$

as desired. The induction on k then gives (8.11).

3. We fix $\psi \in C^\infty(\mathbb{R}, [0,1])$ with $0 \notin \operatorname{supp}\psi$, $\psi \equiv 1$ near $\pm\omega$, and $\mathrm{WF}_h(f_e) \cap \operatorname{supp}\psi \cap \operatorname{supp}\gamma = \emptyset$. We ask $\tilde{\psi}$ to satisfy the same conditions for ψ while $\operatorname{supp}\psi \cap \operatorname{supp}(1-\tilde{\psi}) = \emptyset$. Let $\varphi \in C_c^\infty(\Gamma)$

that $\operatorname{supp}\gamma \cap \operatorname{supp}(1-\varphi) = \operatorname{supp}\varphi \cap \operatorname{supp}(1-\tilde\gamma) = \emptyset$. Fixing Q that $\mathrm{WF}_h(\psi(hD)\gamma) \subset \mathrm{Ell}_h(Q)$, the Lagrangian structure result (8.11) applied to $\tilde\gamma u$ implies there are $\tilde a_\pm(x;h) \in S^0(\Gamma)$ that

$$\psi(hD)\tilde\gamma(x)u(x) = \tilde\gamma(x)(\tilde a_+(x;h)e^{i\omega x/h} + \tilde a_-(x;h)e^{-i\omega x/h}) + O_{C^\infty}(h^\infty). \tag{8.18}$$

See further details in Propositions 25.1.5 and 25.1.3 in [Ḧ85b]. It remains to show that on $\operatorname{supp}\gamma$, (8.6) holds for some $a_\pm(h) \in \mathbb{C}$ and some $a_{\pm,k}(x;h) \in S^0(\Gamma)$. Indeed, note

$$h\sum_\pm \tilde b_\pm \gamma e^{\pm i\omega x/h} \sim \psi(hD)\gamma(h\Lambda_M^\Omega - \sqrt{\omega^2+h^2M^2})u = \psi(hD)\gamma(hB - \sqrt{\omega^2+h^2M^2})\tilde\gamma u \\ + h\psi(hD)\gamma\mathcal{E}u + O_{C^\infty}(h^\infty), \tag{8.19}$$

where $h\psi(hD)\gamma\mathcal{E}u = O_{S^0}(h^\infty\|u\|_{L^2})$. On another hand, we claim

$$\psi(hD)\gamma(hB-\sqrt{\omega^2+h^2M^2})\tilde\gamma u = \psi(hD)\gamma(hB-\sqrt{\omega^2+h^2M^2})\tilde\gamma(\tilde a_+(x;h)e^{i\omega x/h}+\tilde a_-(x;h)e^{-i\omega x/h}) \\ + O_{C^\infty}(h^\infty) \sim \gamma\sum_\pm \Big(\pm\frac{h}{i}\tilde a'_\pm(x;h) + h^2M^2\sum_{k=2}^\infty h^{k-1}L_{\pm,k}\tilde a_\pm(x;h)\Big)e^{\pm i\omega x/h}, \tag{8.20}$$

where $L_{\pm,k}$ are h-uniform differential operators in ∂_x up to order k with smooth, bounded coefficients. Indeed, from Lemma C.2 we have

$$hB\psi(hD) = h\sqrt{D_x^2+M^2}\psi(hD) + O_{S^0}(h^3M^2) + O_{S^0}(h^\infty). \tag{8.21}$$

Note $\mathrm{WF}_h\,\tilde a_+ \subset \{\xi = 0\}$ and we have

$$h\sqrt{D_x^2+M^2}\tilde a_+(x)e^{i\omega x/h} = (\mathcal{F}^{-1}(\sqrt{(h\xi+\omega)^2+h^2M^2}(\mathcal{F}\tilde a_+)(\xi)))e^{i\omega x/h}, \tag{8.22}$$

and near $h\xi = 0$, we have

$$\sqrt{(h\xi+\omega)^2+h^2M^2} - \sqrt{\omega^2+h^2M^2} = h\xi - \frac{h^3M^2\xi}{2\omega(h\xi+\omega)} + O(\frac{h^4M^4}{(h\xi+\omega)^3}). \tag{8.23}$$

This yields (8.20) as claimed. Then we have

$$h\tilde b_\pm(x;h) \sim \pm\frac{h}{i}\tilde a'_\pm(x;h) + h^2M^2\sum_{k=2}^\infty h^{k-1}L_{\pm,k}\tilde a_\pm(x;h) \tag{8.24}$$

The transport equation at size h reads

$$h\tilde b_\pm(x;h) \sim \pm\frac{h}{i}\tilde a'_\pm(x;h) + O(h^3M^2) \tag{8.25}$$

and implies $\tilde a_\pm(x;h) = a_{\pm,0}(h) \pm i\int_{x_0}^x \tilde b_\pm(t;h)\,dt + h^2M^2\tilde a_{\pm,2}(x;h)$ with $\tilde a_{\pm,2}(x;h), a_{\pm,0}(h) \in S^0$. Then at h^3M^2,

$$0 \sim h^3M^2\big(\pm\frac{1}{i}\tilde a'_{\pm,2}(x;h) + L_{\pm,2}(a_{\pm,0}(h) \pm i\int_{x_0}^x \tilde b_\pm(t;h)\,dt)\big) + O(h^4M^2) \tag{8.26}$$

implies $\tilde a'_{\pm,2} = \mp iL_{\pm,2}(a_{\pm,0}(h)\pm i\int_{x_0}^x \tilde b_\pm(t;h)\,dt)+O(h)$ and thus $\tilde a_\pm(x;h) = a_{\pm,0}(h)\pm i\int_{x_0}^x \tilde b_\pm(t;h)\,dt + h^2M^2a_{\pm,2}(x;h) + h^3\tilde a_{\pm,3}(x;h)$ for $a_{\pm,2}, \tilde a_{\pm,3} \in S^0$. Solving recursively for $a_{\pm,k}(x;h)$ and renaming $a_{\pm,0}(h)$ to $a_\pm(h)$ then proves (8.6).

4. We show that the values of $\tilde a_\pm(x_0;h)$ at a point $x_0 \in \operatorname{supp}\gamma$ determine $\tilde a_\pm(x;h)$ uniquely modulo $O(h^\infty)$. This is to show $\tilde a_\pm(x;h) = O(h^\infty)$, if we assume $\tilde a_\pm(x_0;h) = 0$ for some $x_0 \in \operatorname{supp}\gamma$

and $f = 0$. Indeed, (8.24) and $\tilde{a}_\pm(x;h) = O_{S^0}(1)$ implies $\tilde{a}'_\pm(x;h) = \sum_{k=2}^\infty O(h)L_{\pm,k}\tilde{a}_\pm(x;h) = O_{S^0}(h)$, and $\tilde{a}_\pm(x_0;h) = 0$ improves the size to $\tilde{a}_\pm(x;h) \in O_{S^0}(h)$. Now (8.24) implies $\tilde{a}'_\pm(x;h) = O_{S^0}(h^2)$ and one iterates to see $\tilde{a}_\pm(x;h) = O_{S^0}(h^\infty)$ as claimed. This concludes the description of the characteristic frequencies.

5a. Now we turn to the zero frequencies. We show that there is $\epsilon > 0$ small such that for all $k \in \mathbb{N}_0$ and all $\psi_0 \in C_c^\infty((-\epsilon,\epsilon);[0,1])$ with $\psi_0 \equiv 1$ near 0,

$$\Big\|\psi_0(hD_x)(hD_x)^k\gamma u\Big\|_{L^2} \leq Ch^k\big(\|\gamma f_0\|_{H^k} + h\|u\|_{L^2}\big). \tag{8.27}$$

The estimate (8.27) applied to $\tilde{\gamma}u$ instead of γu then gives the desired $u_0(x;h) = \psi_0(hD_x)\tilde{\gamma}u + O_{S^0}(h^\infty)$. Without loss of generality we assume $\{|\xi| \leq \epsilon\} \cap \mathrm{WF}_h(f_e) = \emptyset$, and let $\tilde{\gamma}, \tilde{\tilde{\gamma}} \in C^\infty(\Gamma)$ such that $\operatorname{supp}\gamma \cap \operatorname{supp}(1-\tilde{\gamma}) = \operatorname{supp}\tilde{\gamma} \cap \operatorname{supp}(1-\tilde{\tilde{\gamma}}) = \emptyset$ and $\tilde{\gamma}f = \tilde{\gamma}(f_0 + f_e + f_\omega)$. We start from the base case $k = 0$, let $\tilde{\psi}_0 \in C_c^\infty((-\epsilon,\epsilon),[0,1])$ with $\operatorname{supp}\psi_0 \cap \operatorname{supp}(1-\tilde{\psi}_0) = \emptyset$. We again utilize (8.10) that $\gamma\Lambda_M^\Omega = \gamma B\tilde{\gamma} + \gamma\mathcal{E}$ to see

$$\begin{aligned}
(\tilde{\psi}_0(hD)hB - \sqrt{\omega^2+h^2M^2})\psi_0(hD)\gamma u &= \psi_0(hD)(hB-\sqrt{\omega^2+h^2M^2})\gamma u - \tilde{\psi}_0(hD)[hB,(1-\psi_0(hD))]\gamma u \\
&= \psi_0(hD)\gamma(hB-\sqrt{\omega^2+h^2M^2})\tilde{\gamma}u + \psi_0(hD)[hB,\gamma]\tilde{\gamma}u + O_{\Psi_h^0}(h)\gamma u \\
&= \psi_0(hD)\gamma(h\Lambda_M^\Omega - h\mathcal{E} - \sqrt{\omega^2+h^2M^2})u + O_{\Psi^0}(h)u = O_{L^2}(\|\gamma f_0\|_{L^2} + h\|u\|_{L^2}),
\end{aligned} \tag{8.28}$$

Now for some $C > 0$ independent of h, we have

$$\|\tilde{\psi}_0(hD)hB\|_{L^2\to L^2} \leq h\|\tilde{\psi}_0(hD)\|_{H^{-1}\to L^2}\|B\|_{L^2\to H^{-1}} \leq C\epsilon\|B\|_{L^2\to H^{-1}}. \tag{8.29}$$

Now picking a small ϵ such that $\|(\tilde{\psi}_0(hD)hB - \sqrt{\omega^2+h^2M^2})^{-1}\|$ is uniformly bounded in h small gives

$$\|\psi_0(hD)\gamma u\|_{L^2} \leq C(\|\gamma f_0\|_{L^2} + h\|u\|_{L^2}) \tag{8.30}$$

as desired.

5b. Now we prove the inductive step. Assume (8.27) holds for $k \leq K-1$ and consider the $k = K$ case. Note

$$\begin{aligned}
&(\tilde{\psi}_0(hD)hB - \sqrt{\omega^2+h^2M^2})\psi_0(hD)(hD)^K\gamma u \\
&= (hD)^K(\tilde{\psi}_0(hD)hB - \sqrt{\omega^2+h^2M^2})\psi_0(hD)\gamma u - \sum_{j=0}^{K-1}(hD)^j\tilde{\psi}_0(hD)h[hD,B]\psi_0(hD)(hD)^{K-1-j}\gamma u \\
&= (hD)^K(\tilde{\psi}_0(hD)hB - \sqrt{\omega^2+h^2M^2})\psi_0(hD)\gamma u - \sum_{j=0}^{K-1}\sum_{l=0}^{j}\binom{j}{l}\tilde{\psi}_0(hD)h\operatorname{ad}_{hD}^{j-l+1}(B)\psi_0(hD)(hD)^{K-1-j+l}\gamma u \\
&= (hD)^K(\tilde{\psi}_0(hD)hB - \sqrt{\omega^2+h^2M^2})\psi_0(hD)\gamma u + O_{L^2}(h^K),
\end{aligned} \tag{8.31}$$

the last line of which comes from $\operatorname{ad}_{hD}^k(B) \in h^k\Psi^1$ and the inductive hypothesis, that

$$\|h\operatorname{ad}_{hD}^{j-l+1}(B)\psi_0(hD_x)(hD)^{K-1-j+l}\gamma u\| \leq Ch^{j-l+1}\|\psi_0(hD)(hD)^{K-1-j+l}\gamma u\| \leq Ch^K. \tag{8.32}$$

Now similar to the case $k=0$, we have

$$\begin{aligned}(\tilde\psi_0(hD)hB-\sqrt{\omega^2+h^2M^2})\psi_0(hD)(hD)^K\gamma u&=(hD)^K\psi_0(hD)\gamma(h\Lambda_M^\Omega-\sqrt{\omega^2+h^2M^2})u\\ +(hD)^K\psi_0(hD)h[B,\gamma]u-(hD)^K\tilde\psi_0(hD)[hB,1-\psi_0(hD)]\gamma u&+(hD)^K\psi_0(hD)\gamma h\mathcal{E}u+O_{L^2}(h^K)\\ =(hD)^K\psi_0(hD)\gamma(h\Lambda_M^\Omega-\sqrt{\omega^2+h^2M^2})u&+(hD)^K\psi_0(hD)h[B,\gamma]u\\ -(hD)^K\tilde\psi_0(hD)[hB,1-\psi_0(hD)]\gamma u&+O_{L^2}(h^K).\quad(8.33)\end{aligned}$$

That $\gamma f_0\in S^0$ implies the first term on the right is of size h^K:

$$\|(hD)^K\psi_0(hD)\gamma(h\Lambda_M^\Omega-\sqrt{\omega^2+h^2M^2})u\|\le Ch^K\|\gamma f_0\|_{H^K}+Ch^N\|f_e\|_{H_h^{-N}}+Ch^N\|f_\omega\|_{H_h^{-N}},\tag{8.34}$$

and we will estimate the other terms. Note

$$\begin{aligned}(hD)^K\psi_0(hD)[hB,\gamma]u&=\sum_{l=0}^{K-1}\binom{K-1}{l}hD\psi_0(hD)\operatorname{ad}_{hD}^{K-1-l}([hB,\gamma])(hD)^lu\\ &=\sum_{l=0}^{K-1}\binom{K-1}{l}hD\psi_0(hD)\operatorname{ad}_{hD}^{K-1-l}([hB,\gamma])\Big(\tilde\psi_0(hD)(hD)^l\tilde\gamma u+(hD)^l(1-\tilde\gamma)u\\ &\qquad+(1-\tilde\psi_0(hD))(hD)^l\tilde\gamma u\Big).\quad(8.35)\end{aligned}$$

Here for the first term, by the inductive hypothesis and that $\operatorname{ad}_D^{K-1-l}[B,\gamma]\in\Psi^0$, we have

$$\|\operatorname{ad}_{hD}^{K-1-l}([hB,\gamma])\tilde\psi_0(hD)(hD)^l\tilde\gamma u\|\le Ch^{K-l}\|\tilde\psi_0(hD)(hD)^l\tilde\gamma u\|\le Ch^K(\|\tilde{\tilde\gamma}f_0\|_{H^l}+h\|u\|).\tag{8.36}$$

For the second term, since γ and $1-\tilde\gamma$ have disjoint support, $D\operatorname{ad}_D^{K-1-l}[B,\gamma](1-\tilde\gamma)\in\Psi^{-\infty}$ and

$$\|hD\operatorname{ad}_{hD}^{K-1-l}([hB,\gamma])(hD)^l(1-\tilde\gamma)u\|\le Ch^{K+1}\|u\|_{L^2}.\tag{8.37}$$

For the last term, note that $\mathrm{WF}_h(1-\tilde\psi_0(hD))(hD)^l\tilde\gamma$ is disjoint from $\{\xi=0\}$ and $\operatorname{supp}\psi_0$ to see

$$\|\psi_0(hD)\operatorname{ad}_{hD}^{K-1-l}([hB,\gamma])(1-\tilde\psi_0(hD))(hD)^l\tilde\gamma u\|=O(h^\infty)\|u\|_{L^2}.\tag{8.38}$$

We then obtain in (8.33)

$$(hD_x)^K\psi_0(hD)[hB,\gamma]u=O_{L^2}(h^K).\tag{8.39}$$

We move on to estimate $(hD)^K\tilde\psi_0(hD)[hB,1-\psi_0(hD)]\gamma u$. Let $\tilde{\tilde\psi}_0$ be supported on $(-\epsilon,\epsilon)$ that $\operatorname{supp}\tilde\psi_0\cap\operatorname{supp}(1-\tilde{\tilde\psi}_0)=\emptyset$. On $\operatorname{supp}\gamma$, $[hB,1-\psi_0(hD)]\in h\Psi_h^0$ and $\mathrm{WF}_h(\tilde\psi_0(hD)[hB,1-\psi_0(hD)])$ is supported inside $\{\tilde{\tilde\psi}_0\equiv1\}$. Thus by the elliptic estimate and the inductive hypothesis we have

$$\begin{aligned}\|(hD)^K\tilde\psi_0(hD)[hB,1-\psi_0(hD)]\gamma u\|\le&Ch\|hD\tilde{\tilde\psi}_0(hD)(hD)^{K-1}\gamma u\|+Ch^N\|\gamma u\|_{H_h^{-N}}\\ \le&Ch^K(\|\gamma f_0\|_{H^{K-1}}+h\|u\|_{L^2}).\quad(8.40)\end{aligned}$$

Revisit (8.33) to see

$$\|\psi_0(hD)(hD)^K\gamma u\|\le C\|(\tilde\psi_0(hD)hB-\sqrt{\omega^2+h^2M^2})\psi_0(hD)(hD)^K\gamma u\|\le Ch^K\big(\|\gamma f_0\|_{H^K}+h\|u\|_{L^2}\big),\tag{8.41}$$

which verifies the inductive case $k=K$ of (8.27) and thus gives (8.8).

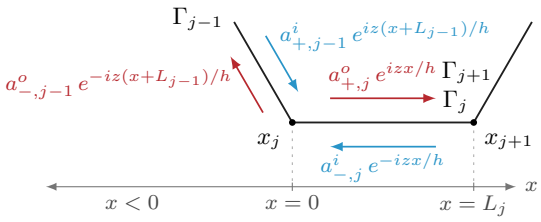


FIGURE 12. Extension of corner x_j of $\mathcal{P}$ into curved wedge $\mathcal{W}_j$

6. It remains to invert f_e at non-characteristic non-zero frequencies. Let $\psi_e = 1-\psi_0-\psi$. Let $\tilde\psi_e \in C^\infty(\mathbb{R}_\xi,[0,1])$ that $\operatorname{supp}\psi_e \cap \operatorname{supp}(1-\tilde\psi_e)=\emptyset$ and $0\notin\operatorname{supp}\tilde\psi_e$. Let $u_e=\tilde\gamma\psi_e(hD)\gamma_1 u$. Let $\gamma_1,\gamma_2\in C_c^\infty(\Gamma)$ with connected support and $\operatorname{supp}\tilde\gamma\cap\operatorname{supp}(1-\gamma_1)=\operatorname{supp}\gamma_1\cap\operatorname{supp}(1-\gamma_2)=\emptyset$. Noting $\mathrm{WF}_h u_e\subset \mathrm{Ell}_h(\tilde\psi_e(hD)\gamma_1(hB-\sqrt{\omega^2+h^2M^2})\gamma_2)$ we have for large $N>0$,

$$\|u_e\|_{H_h^k}\le C\|\tilde\psi_e(hD)\gamma_1(hB-\sqrt{\omega^2+h^2M^2})\gamma_2 u\|_{H_h^{k-1}}+Ch^N\|\gamma_2 u\|_{H_h^{-N}}. \tag{8.42}$$

Note $\psi_e\equiv 1$ over $\mathrm{WF}_h(f_e)$. We have $\gamma_1\Lambda_M^\Omega=\gamma_1 B\gamma_2+\gamma_1\mathcal{E}$ and for large $N>0$,

$$\tilde\psi_e(hD)\gamma_1(hB-\sqrt{\omega^2+h^2M^2})\gamma_2 u=\tilde\psi_e(hD)\gamma_1 f-h\tilde\psi_e(hD)\gamma_1\mathcal{E}u=\gamma_1 f_e+O_{S^0}(h^N\|u\|_{L^2}+h^N\|f_e\|_{H_h^{-N}}), \tag{8.43}$$

since $\tilde\psi_e$ is supported away from 0, similar to (8.16). This gives

$$\|u_e\|_{H_h^k}\le C\|\gamma_1 f_e\|_{H_h^{k-1}}+Ch^N\|u\|_{L^2}. \tag{8.44}$$

Rename γ_1 to $\tilde\gamma$ to see (8.7). □

8.2. **Blow-up near the corners of curvilinear polygons.** Let $\omega\in\mathbb{R}$, $\omega>0$, and $z\in\mathbb{C}$. We first focus on one connected component $\Gamma^l=\cup_{j=1}^J\Gamma_j$ of $\partial\mathcal{P}$, where Γ_j connects x_j to x_{j+1}. We abuse the notation and treat $x_0=x_J$, $x_{J+1}=x_1$ cyclically, and similarly for Γ_0 and Γ_{J+1}.

Near each corner x_j, we build a AC Lipschitz domain $\mathcal{W}_j\subset\mathbb{R}^2$ that $\mathcal{W}_j$ and $\mathcal{P}$ share the common boundary near x_j. Indeed, for small $\epsilon>0$, we truncate $\partial\mathcal{P}\cap\{\operatorname{dist}(x,x_j)<\epsilon\}$ and extend the two truncated endpoint tangentially by two straight lines to obtain $\partial\mathcal{W}_j$, and thus $\mathcal{W}_j$. Such an $\mathcal{W}_j$ has corner angle α_j at x_j and is bi-Lipschitz with small ϵ-dependent error to W_{α_j}, a straight wedge of corner angle α_j. Lemma B.6 then implies for each small $\delta>0$, there is $\epsilon>0$ such that $\mathfrak{r}^0_{\mathcal{W}_j}>\frac{\pi}{\alpha_j}-\delta$. We fix an $\mathcal{W}_j$ with sufficiently small $\epsilon>0$ and abuse the notation to say $\mathfrak{r}^0_{\mathcal{W}_j}>\frac{\pi}{\alpha_j}-0$, through Lemma B.7. See Figure 12.

Now let $\chi_j,\tilde\chi_j\in C_c^\infty(\operatorname{cl}\mathcal{P}\cap\operatorname{cl}\mathcal{W}_j)$ with the vertex $x_j\notin\operatorname{supp}(1-\chi_j)$ and $\chi_j\prec\tilde\chi_j$. Parametrize $\partial\mathcal{W}_j=(-\infty,0)\cup\{0\}\cup(0,\infty)$ by arclength, where $x>0$ on $\Gamma_j\cap\partial\mathcal{W}_j$ and $x<0$ on $\Gamma_{j-1}\cap\partial\mathcal{W}_j$. We extend $(0,\infty)$ tangentially by a straight line on the left to obtain the eventually conic smooth domain $\mathcal{W}_{j,R}\subset\mathbb{R}^2$ that $\mathcal{W}_{j,R}$, $\mathcal{W}_j$ share the common smooth boundary near $[\frac{\epsilon}{2},\infty)$. We similarly obtain $\mathcal{W}_{j,L}\subset\mathbb{R}^2$ that $\mathcal{W}_{j,L}$, $\mathcal{W}_j$ share the common smooth boundary near $(-\infty,-\frac{\epsilon}{2}]$. Similar to Step 1a of Lemma 7.4, Lemma C.3 then yields interior quasimodes $\tilde e_{\pm,j,L/R,h}(x;h,\omega)$ on $\mathcal{W}_{j,L/R}$ that

$$(-\Delta+M^2)\tilde e_{\pm,j,L/R,h}=O_{\mathcal{S}(\mathcal{W}_{j,L/R})}(h^\infty),\ (h\partial_\nu-\sqrt{\omega^2+M^2})\tilde e_{\pm,j,L/R,h}=O_{\mathcal{S}(\partial\mathcal{W}_{j,L/R})}(h^\infty), \tag{8.45}$$

whose boundary restriction $e_{\pm,j,L/R,h}(x;h,\omega,M)=\tilde{e}_{\pm,j,L/R,h}(0,x;h,\omega,M)$ satisfies

$$e_{\pm,j,L/R,h}(x;h,\omega,M)=e^{\pm i\omega x/h}(1+O_{S^0(\partial\mathcal{W}_{j,L/R})}(h^2M^2)). \tag{8.46}$$

Lemma 8.2 (Blow-up near the corner)**.** *Let $\Omega=\mathcal{P}\subset\mathbb{R}^2$ be a curvilinear polygon with angles $\{\alpha_j\}_{j=1}^J$ and edges $\{\Gamma_j\}_{j=1}^J$. Let f,f_0,f_e,f_ω be as defined in Lemma 8.5, and u be that $(h\Lambda_M^\Omega-\sqrt{\omega^2+h^2M^2})u=f$. Let $\gamma_j(x)\in C_c^\infty(\Gamma_j)$ with the vertices disjoint from $\operatorname{supp}\gamma_j$. Let $\chi_j\in C^\infty(\mathcal{P})$ that $x_j\notin\operatorname{supp}(1-\chi_j)$ with $\operatorname{supp}(1-\gamma_j)\cap\Gamma_j\cap\operatorname{supp}\chi_j\cup\operatorname{supp}\chi_{j+1}$. For each j, Lemma 8.5 gives*

$$\gamma_j u=\gamma_j\Big(\sum_{\pm}\tilde{a}_{\pm,j}(x;h)e^{\pm i\omega x/h}+u_e(x;h)+u_0(x;h)\Big)+O_{S^0}(h^\infty). \tag{8.47}$$

It allows us to read off $a_{\pm,j}^{o/i}(h)\in\mathbb{C}$ from $\gamma_j u$ from the right of x_j, by

$$\tilde{a}_{+,j}(x;h)\sim a_{+,j}^o(h)+O_{S^0}(h^2M^2\|u\|_{L^2}),\ \textit{near}\ \operatorname{supp}\gamma_j, \tag{8.48}$$

$$\tilde{a}_{-,j}(x;h)\sim a_{-,j}^i(h)+O_{S^0}(h^2M^2\|u\|_{L^2}),\ \textit{near}\ \operatorname{supp}\gamma_j, \tag{8.49}$$

and also to read off $a_{\mp,j}^{o/i}(h)\in\mathbb{C}$ from the left of x_j, by

$$\tilde{a}_{+,j-1}(x;h)\sim a_{+,j-1}^i(h)+O_{S^0}(h^2M^2\|u\|_{L^2}),\ \textit{near}\ \operatorname{supp}\gamma_{j-1}, \tag{8.50}$$

$$\tilde{a}_{-,j-1}(x;h)\sim a_{-,j-1}^o(h)+O_{S^0}(h^2M^2\|u\|_{L^2}),\ \textit{near}\ \operatorname{supp}\gamma_{j-1}. \tag{8.51}$$

Then we have

$$\begin{pmatrix}a_{-,j-1}^o\\ a_{+,j}^o\end{pmatrix}=\Big(\Sigma_{M,h^{-1}\omega}^{\mathcal{W}_j}+O(M^2h^{2-0})\Big)\begin{pmatrix}a_{+,j-1}^i\\ a_{-,j}^i\end{pmatrix}+O(h^{\frac{\pi}{\alpha_j}-1-0}\|u_0\|_{H_h^{\frac12}})+O(h^\infty)$$
$$+O(h^{-1-0}\|\chi_j f\|_{H_h^{-\frac12}}), \tag{8.52}$$

and

$$\|\chi_j u\|_{H_h^{\frac12}}\le C|a^i|+h^{\frac{\pi}{\alpha_j}-1-0}\|u_0\|_{H_h^{\frac12}}+h^{-1-0}\|\chi_j f\|_{H_h^{-\frac12}}+Ch^N\|u\|_{H_h^{\frac12}}. \tag{8.53}$$

Remark 8.3. Lemma 8.5 below improves the remainder $h^{\frac{\pi}{\alpha_j}-1-0}u_0$ in (8.52) to $h^{\frac{2\pi}{\alpha_+}-0}u$:

$$\begin{pmatrix}a_{-,j-1}^o\\ a_{+,j}^o\end{pmatrix}=\Big(\Sigma_{M,h^{-1}\omega}^{\mathcal{W}_j}+O(M^2h^{2-0})\Big)\begin{pmatrix}a_{+,j-1}^i\\ a_{-,j}^i\end{pmatrix}+O(h^{\frac{2\pi}{\alpha_+}-0}\|u\|_{L^2})+O(h^\infty)$$
$$+O\Big(\sum_{j=1}^J h^{\frac{2\pi}{\alpha_+}-1-0}\|\chi_j f\|_{H_h^{-\frac12}}+h^{\frac{\pi}{\alpha_+}-1-0}\|\tilde{\gamma}f_0\|_{S^0}\Big), \tag{8.54}$$

where $\alpha_+=\max_j(\alpha_j)$.

Proof. To prove the lemma, we write $\mathcal{P}^h=h^{-1}\mathcal{P}$ and define the unitary rescaling operator

$$T_h:L^2(\partial\mathcal{P})\to L^2(\partial\mathcal{P}^h),\ T_hu(x)=h^{\frac12}u(hx). \tag{8.55}$$

Then under the rescaling we have

$$(\Lambda_{hM}^{\mathcal{P}^h}-\sqrt{\omega^2+h^2M^2})T_hu=T_hf. \tag{8.56}$$

We adopt the superscript rescaling notations in Section 7. We now work near each corner x_j. Let $\tilde{\chi}_j, \tilde{\tilde{\chi}}_j$ satisfy the requirement for χ_j with $\chi_j \prec \tilde{\chi}_j \prec \tilde{\tilde{\chi}}_j$. As in Step 1 of Lemma 7.4, we construct the principal oscillator by

$$\tilde{u}_\omega(x_0, x) = \tilde{\tilde{\chi}}_j\Big(a^o_+\chi^h_R\tilde{e}_{+,R,h} + a^i_+\chi^h_L\tilde{e}_{+,L,h} + a^i_-\chi^h_R\tilde{e}_{-,R} + a^o_-\chi^h_L\tilde{e}_{-,L}\Big). \tag{8.57}$$

as in (7.35), and the zero-frequency residue by $\tilde{u}_0 = \tilde{\tilde{\chi}}_j\bar{u} - \tilde{u}_\omega \in X^l_m(\mathcal{W}_j)$. We then construct

$$v'_j = \chi^\delta_j T_h u + (\chi_j - \chi^\delta_j)T_h u_0 + (1-\chi^\delta_j)(\chi^\delta_R w_R + \chi^\delta_L w_L) \tag{8.58}$$

on $\mathcal{W}_j$ with

$$w_R = h^{\frac{1}{2}}\mathbb{1}_{\Gamma^h_j}(a^o_{+,j}e^R_{+,j} + a^i_{-,j}e^R_{-,j}),\ w_L = h^{\frac{1}{2}}\mathbb{1}_{\Gamma^h_{j-1}}(a^o_{+,j-1}e^L_{+,j-1} + a^i_{-,j-1}e^L_{-,j-1}). \tag{8.59}$$

We can then identify (8.58) with the setting (7.41) via

$$\mathcal{W}_\alpha = \tilde{\mathcal{W}}_\alpha = \mathcal{W}_j,\ \tilde{\mathcal{W}}^R_\alpha = \mathcal{W}^R_j,\ \tilde{\mathcal{W}}^L_\alpha = \mathcal{W}^L_j,\ \chi = \chi_j,\ F = \mathrm{I},\ u = T_h u,\ u_0 = T_h u_0,\ v' = v'_j. \tag{8.60}$$

Equation (7.48) now reads

$$\begin{aligned}(\Lambda^{\mathcal{W}^h_j}_{hM} - \sqrt{\omega^2 + h^2M^2})v'_j = {}& \chi_j T_h f - \partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}(\chi_j - \chi^\delta_j)(-\Delta + h^2M^2)T_h\tilde{u}_0 - \partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}\chi^\delta_j(-\Delta+h^2M^2)T_h\bar{u}\\ & - \partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}[-\Delta,\chi_j]T_h\tilde{u}_0 - \sum_{L/R}\partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}(1-\chi^\delta_j)[-\Delta,\chi^\delta_{L/R}]\bar{w}_{L/R}\\ & + \partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}[-\Delta,\chi^\delta_j]\Big(-T_h\tilde{u}_\omega + \sum_{L/R}\chi^\delta_{L/R}\bar{w}_{L/R}\Big) + O(h^\infty\mathcal{S}).\end{aligned} \tag{8.61}$$

Steps 2b, 2c, 2d, 2e and 3 of Lemma 7.4 remain the same:

$$\|\partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}(\chi_j - \chi^\delta_j)(-\Delta + h^2M^2)T_h\tilde{u}_0\|_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N,N}} \le Ch^{\frac{2\pi}{\alpha_j}-0}\|\tilde{\chi}T_h u_0\|_{H^{\frac{1}{2},\frac{\pi}{\alpha_j}+\frac{1}{2}-0}}, \tag{8.62}$$

$$\|\partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}\chi^\delta_j(-\Delta + h^2M^2)T_h\bar{u}\|_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N,N}} \le Ch^N\|\tilde{\chi}^\delta T_h u\|_{H^{\frac{1}{2}}}, \tag{8.63}$$

$$\|\partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}[-\Delta,\chi_j]T_h\tilde{u}_0\|_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N,N}} \le Ch^{\frac{2\pi}{\alpha_j}-0}\|\tilde{\chi}T_h u_0\|_{H^{\frac{1}{2},\frac{\pi}{\alpha_j}+\frac{1}{2}-0}}, \tag{8.64}$$

$$\|\partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}(1-\chi^\delta_j)[-\Delta,\chi^\delta_{L/R}]\bar{w}_{L,R}\|_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N,N}} \le Ch^N(|a^i| + |a^o|), \tag{8.65}$$

$$\|\partial_\nu\mathcal{R}^{\mathcal{W}^h_j}_{hM}[-\Delta,\chi^\delta_j]\Big(-T_h\tilde{u}_\omega + \sum_{L/R}\chi^\delta_{L/R}\bar{w}_{L/R}\Big)\|_{Z^{-\frac{1}{2},N,-\frac{1}{2}}_{-L_0+0,N,N}} \le Ch^{\frac{1}{2}}(M^2h^{2-0} + h^N)(|a^i| + |a^o|). \tag{8.66}$$

Note unitary $T_h : L^2(\partial\mathcal{W}_j) \to L^2(\partial\mathcal{W}^h_j)$ is also an isometry from $H^{\frac{1}{2}}_h(\partial\mathcal{W}_j)$ to $H^{\frac{1}{2}}(\partial\mathcal{W}^h_j)$ for

$$h^{\frac{2\pi}{\alpha_j}-0}\|\tilde{\chi}_j T_h u_0\|_{H^{\frac{1}{2},\frac{\pi}{\alpha_j}+\frac{1}{2}-0}} \le Ch^{\frac{\pi}{\alpha_j}-\frac{1}{2}-0}\|\tilde{\chi}_j T_h u_0\|_{H^{\frac{1}{2}}} \le Ch^{\frac{\pi}{\alpha_j}-\frac{1}{2}-0}\|u_0\|_{H^{\frac{1}{2}}_h}. \tag{8.67}$$

On the other hand, we have

$$\|\chi_j T_h f\|_{Z^{-\frac{1}{2},\frac{1}{2}+0,-\frac{1}{2}}_{\frac{1}{2}+0,\frac{1}{2}+0,\frac{1}{2}+0}} \leq Ch^{-\frac{1}{2}-0}\|\chi_j f\|_{H_h^{-\frac{1}{2}}}. \tag{8.68}$$

The above implies

$$\|(\Lambda_{hM}^{\mathcal{W}_j^h} - \sqrt{\omega^2+h^2M^2})v_j'\|_{Z^{-\frac{1}{2},\frac{1}{2}+0,-\frac{1}{2}}_{-L_0+0,\frac{1}{2}+0,\frac{1}{2}+0}} \leq Ch^{\frac{\pi}{\alpha_j}-\frac{1}{2}-0}\|u_0\|_{H_h^{\frac{1}{2}}} + Ch^{\frac{1}{2}}(M^2h^{2-0}+h^N)(|a^i|+|a^o|)$$
$$+ Ch^N\|u\|_{H_h^{\frac{1}{2}}} + Ch^{-\frac{1}{2}-0}\|\chi_j f\|_{H_h^{-\frac{1}{2}}}. \tag{8.69}$$

We correct the small eigenerror by building generalized eigenfunctions

$$\begin{aligned} v_j &= v_j' - (\Lambda_{hM}^{\mathcal{W}_j^h} - \sqrt{\omega^2+h^2M^2} - i0)^{-1}(\Lambda_{hM}^{\mathcal{W}_j^h} - \sqrt{\omega^2+h^2M^2})v' \\ &= v_j' + O_{Z^{\frac{1}{2},\frac{1}{2}+0,\frac{1}{2}}_{-L_0+0,-\frac{1}{2}+0,-\frac{1}{2}-0}}\Big(h^{\frac{\pi}{\alpha_j}-\frac{1}{2}-0}\|u_0\|_{H_h^{\frac{1}{2}}} + h^{\frac{1}{2}}(M^2h^{2-0}+h^N)(|a^i|+|a^o|) + h^N\|u\|_{H_h^{\frac{1}{2}}} \\ &\qquad + O(h^{-\frac{1}{2}-0}\|\chi_j f\|_{H_h^{-\frac{1}{2}}})\Big). \end{aligned} \tag{8.70}$$

We here have $(\Lambda_{hM}^{\mathcal{W}_j^h} - \sqrt{\omega^2+h^2M^2})v_j = 0$ and the incoming scattering amplitudes of v must agree with $h^{\frac{1}{2}}a^i$. The scattering matrix result Proposition 6.3 implies

$$\Sigma^{\mathcal{W}_j}_{M,h^{-1}\omega}a^i = a^o + O(M^2h^{2-0})(|a^i|+|a^o|)) + O(h^{\frac{\pi}{\alpha_j}-1-0}\|u_0\|_{H_h^{\frac{1}{2}}}) + O(h^\infty) + O(h^{-1-0}\|\chi_j f\|_{H_h^{-\frac{1}{2}}}), \tag{8.71}$$

and

$$a^o = \Big(\Sigma^{\mathcal{W}_j}_{M,h^{-1}\omega} + O(M^2h^{2-0})\Big)a^i + O(h^{\frac{\pi}{\alpha_j}-1-0}\|u_0\|_{H_h^{\frac{1}{2}}}) + O(h^{-1-0}\|\chi_j f\|_{H_h^{-\frac{1}{2}}}), \tag{8.72}$$

as desired. Moreover we have

$$v_j \sim h^{\frac{1}{2}}a^i e^{-i\omega|x|} + h^{\frac{1}{2}}a^o e^{-i\omega|x|} + O_{Z^{N,N,1-0}_{L_0-0,N,N}}(h^{\frac{1}{2}}|a^i|), \tag{8.73}$$

and thus

$$\|v_j\|_{Z^{\frac{1}{2},0,\frac{1}{2}}_{L_0-0,-\frac{1}{2}-0,-\frac{1}{2}-0}} \leq Ch^{\frac{1}{2}}|a^i|, \tag{8.74}$$

$$\chi_j T_h^* v_j \sim a^i e^{-ih^{-1}\omega|x|} + a^o e^{-ih^{-1}\omega|x|} + O_{H_h^{1-0}}(h^{\frac{1}{2}}|a^i|). \tag{8.75}$$

This implies

$$\|v_j'\|_{Z^{\frac{1}{2},0,\frac{1}{2}}_{-L_0+0,-\frac{1}{2}-0,-\frac{1}{2}-0}} \leq Ch^{\frac{\pi}{\alpha_j}-\frac{1}{2}-0}\|u_0\|_{H_h^{\frac{1}{2}}} + Ch^{\frac{1}{2}}|a^i| + Ch^N\|u\|_{H_h^{\frac{1}{2}}} + O(h^{-\frac{1}{2}-0}\|\chi f\|_{H_h^{-\frac{1}{2}}}), \tag{8.76}$$

$$\chi_j T_h^* v_j' \sim a^i e^{-ih^{-1}\omega|x|} + a^o e^{-ih^{-1}\omega|x|} + O_{H_h^{\frac{1}{2}}}(h^{\frac{1}{2}}|a^i| + h^{0-0}\|u_0\|_{H_h^{\frac{1}{2}}} + h^{-\frac{1}{2}-0}\|\chi_j f\|_{H_h^{-\frac{1}{2}}} + h^\infty). \tag{8.77}$$

We record (8.76) for later use. Now note $\chi_j u = \chi_j T_h^* v_j' + O(M^2 h^{2-0})$ to conclude (8.53). □

8.3. **DtN eigenfunctions on curvilinear polygons.** Our final goal in this section is to show, on a curvilinear polygon $\mathcal{P}$, the zero-frequency remainder $u_0(x;h)$ from Lemma 8.1 is actually smaller than one a priori expects, where the improvement depends on the upper bound of angles of $\mathcal{P}$. In this section, we use the results in Section 2 and Section 3 based on $X^l_m(\mathcal{P}^h;\mathcal{V}^h)$ with the set of vertices $\mathcal{V}^h = \{h^{-1}x_j : 1 \le j \le J\} \in \mathcal{P}^h = h^{-1}\mathcal{P}$. Denote unscaled set of vertices by $\mathcal{V} = \{h^{-1}x_j : 1 \le j \le J\} \in \mathcal{P}$ and the distance to vertices by $r(z) = \operatorname{dist}(z;\mathcal{V})$ for $z \in \mathcal{P}$.

Lemma 8.4 (Improved remainders on polygons I)**.** *Let $\Omega = \mathcal{P} \subset \mathbb{R}^2$ be a curvilinear polygon with angles $\{\alpha_j\}_{j=1}^J$. Let $\alpha_+ = \max_j(\alpha_j)$. Then we have*

$$\|u_0\|_{H^k} \le Ch^{0-0}\|\tilde\gamma f_0\|_{H^k} + Ch^{\frac{\pi}{\alpha_+}+\frac12-0}\|\langle h^{-1}r\rangle^{-N}u\|_{L^2}, \tag{8.78}$$

as a further improvement to (8.8)*.*

Proof. Let $\chi, \tilde\chi, \tilde{\tilde\chi} \in S^0(\mathcal{P}^h)$ with $\chi \prec \tilde\chi \prec \tilde{\tilde\chi}$ and $A_0, \tilde A_0 \in \Psi^{0,0}(\partial\mathcal{P}^h)$ with $\mathrm{WF}(A_0)\cap\{|\xi| = \omega\} = \emptyset$ and $\mathrm{WF}\, A_0 \cap \mathrm{WF}(\mathrm{I} - \tilde A_0) = \emptyset$. We have from (8.56),

$$\tilde A_0\tilde\chi(P - \sqrt{\omega^2 + h^2M^2})\tilde{\tilde\chi}T_h u = \tilde A_0\tilde\chi T_h f + \tilde A_0\tilde\chi(\Lambda^{\mathcal{P}^h}_{hM})(1-\tilde{\tilde\chi})T_h u - \tilde A_0\tilde\chi(\Lambda^{\mathcal{P}^h}_{hM} - P)\tilde{\tilde\chi} \tag{8.79}$$

We then use the elliptic parametrix construction in Lemma 5.2 together with Proposition 3.12 and Lemma 4.18 adopted to $X^l_m(\mathcal{P}^h;\mathcal{V}^h)$ setting, to obtain that for $l < \frac{\pi}{\alpha_+} + \frac12$ and any N,

$$\|A_0\chi T_h u\|_{H^{s,l}_k} \le C\|\tilde A_0\tilde\chi T_h f\|_{H^{-N,l}_k} + \|T_h u\|_{H^{0,-N}} \tag{8.80}$$

In particular,

$$\|A_0\chi T_h u\|_{H^{s,0}_k} \le C\|\tilde\chi T_h f_0\|_{H^{-N,0}_k} + Ch^l\|T_h u\|_{H^{0,-N}}. \tag{8.81}$$

Hence,

$$\|u_0\|_{H^{k,0}} \le Ch^{0-0}\|\tilde\chi f_0\|_{H^{k,0}} + Ch^l\|\langle h^{-1}r\rangle^{-N}u\|_{L^2}. \tag{8.82}$$

Making $l = \frac{\pi}{\alpha_+} + \frac12 = 0$ concludes the proof. □

Lemma 8.5 (Improved remainders on polygons II)**.** *Let $\Omega = \mathcal{P} \subset \mathbb{R}^2$ be a curvilinear polygon with angles $\{\alpha_j\}_{j=1}^J$. Let $\alpha_+ = \max_j(\alpha_j)$. Then we have*

$$\|u_0\|_{H^k} \le Ch^{\frac{\pi}{\alpha_+}+1-0}\|u\|_{L^2} + Ch^{0-0}\|\tilde\gamma f_0\|_{S^0} + Ch^{\frac{\pi}{\alpha_+}-0}\sum_j \|\chi_j f\|_{H_h^{-\frac12}}, \tag{8.83}$$

as a further improvement to (8.8)*.*

Proof. Lemma 8.4 implies (8.76) can be improved to

$$\begin{aligned}\|v_j'\|_{Z^{\frac12,0,\frac12}_{-L_0+0,-\frac12-0,-\frac12-0}} &\le Ch^{\frac{\pi}{\alpha_+}-0}\|u_0\|_{H_h^{\frac12}} + Ch^{\frac12}(|a^i| + |a^o|) + Ch^N\|u\|_{H_h^{\frac12}} + Ch^{-\frac12-0}\|\chi_j f\|_{H_h^{-\frac12}} \\ &\le Ch^{\frac{2\pi}{\alpha_+}-\frac12-0}\|\langle h^{-1}r\rangle^{-N}u\|_{L^2} + Ch^{\frac{\pi}{\alpha_+}-0}\|\tilde\gamma f_0\|_{H^{\frac12}} + Ch^{\frac12}\|u\|_{L^2} + Ch^N\|u\|_{H_h^{\frac12}} + Ch^{-\frac12-0}\|\chi_j f\|_{H_h^{-\frac12}}\end{aligned} \tag{8.84}$$

Note $u = T_h^* v'$ in $r \le C^{-1}h$ and thus

$$\|\langle h^{-1}r\rangle^{-N}u\|_{L^2} \le C\sum_{j=1}^{J}\|\langle r\rangle^{-N}v_j'\|_{L^2} + Ch^N\|u\|_{L^2} \le Ch^{\frac{2\pi}{\alpha_+}-\frac12-0}\|\langle h^{-1}r\rangle^{-N}u\|_{L^2} + Ch^{\frac12}\|u\|_{L^2}$$
$$+ Ch^N\|u\|_{H_h^{\frac12}} + Ch^{\frac{\pi}{\alpha_+}-0}\|\tilde\gamma f_0\|_{H^{\frac12}} + Ch^{-\frac12-0}\sum_j\|\chi_j f\|_{H_h^{-\frac12}}. \quad (8.85)$$

Absorption gives

$$\|\langle h^{-1}r\rangle^{-N}u\|_{L^2} \le Ch^{\frac12}\|u\|_{L^2} + Ch^N\|u\|_{H_h^{\frac12}} + Ch^{\frac{\pi}{\alpha_+}-0}\|\tilde\gamma f_0\|_{H^{\frac12}} + Ch^{-\frac12-0}\sum_j\|\chi_j f\|_{H_h^{-\frac12}}, \quad (8.86)$$

as desired. Revisit Lemma 8.4 to conclude. □

Remark 8.6. All constants in this section can depend on $\mathcal{P}$ in the following way: the upper bound on the number of corners J, the upper and lower bounds over the angles α_j, the upper and lower bounds on all $|\Gamma_j|$, the derivatives of the defining functions of Γ_j.

9. The Grushin problem for the DtN map

In this section, we will use the geometric setup in Section 8.2. We now set up a Grushin problem that will allow us to identify the spectrum of $h\Lambda_M^{\mathcal{P}} - z$: see [DZ19a, Appendix C] for further introduction to Grushin problems. Our goal is to relate the invertibility of $h\Lambda_M^{\mathcal{P}} - z$ to the effective Hamiltonian $E_{-+}(z;h)$ defined below. To do this, we augment $h\Lambda_M^{\mathcal{P}} - z$ by observation/control operators $R_\pm$ which determine the amplitudes of waves on various smooth segments of the boundary.

We define the observation operator R_+. Note $\chi_j, \tilde\chi_j \in C_c^\infty(\partial\mathcal{P})$ localize functions near the corner x_j. Within $\operatorname{supp}\tilde\chi_j$ we adopt the arclength coordinates where $x = 0$ at x_j, $x < 0$ on Γ_{j-1}, $x > 0$ on Γ_j. For $1 \le j \le J$, we define $R_+(z) : H_h^{\frac12}(\partial\mathcal{P}) \to \mathbb{C}^{2J}$ by

$$(R_+(z)u)_{2j-1}(z) = \int_{\Gamma_{j-1}} u(x)\chi_j'(x)e^{-iz(x+L_{j-1})/h}\,dx, \quad (9.1)$$

$$(R_+(z)u)_{2j}(z) = -\int_{\Gamma_j} u(x)\chi_j'(x)e^{izx/h}\,dx, \quad (9.2)$$

within the arclength coordinate $x \in [-L_{j-1}, L_j]$ around x_j. Note for $\omega \ge 0$,

$$(R_+(\omega)e_{+,j-1})_{2j-1} = 1 + O(h^2M^2),\ (R_+(\omega)e_{-,j})_{2j} = 1 + O(h^2M^2). \quad (9.3)$$

Here R_+ decomposes the incoming profile u on $\partial\mathcal{P}$ into incoming data in $\mathbb{C}^{2J}$, where $(R_+)_{2j-1}$ and $(R_+)_{2j}$ read off the amplitudes of u incoming towards x_j from Γ_{j-1} and Γ_j respectively.

We define the control operator $R_- : \mathbb{C}^{2J} \to H_h^{-\frac12}(\partial\mathcal{P})$. For $u_- = (u_{-,k})_{k=1}^{2J} \in \mathbb{C}^{2J}$ we let

$$R_-(z)u_- = \sum_{j=1}^{J} -iu_{-,2j-1}\mathbb{1}_{\Gamma_{j-1}}\tilde\chi_j'(x)e^{-iz(x+L_{j-1})/h} + iu_{-,2j}\mathbb{1}_{\Gamma_j}\tilde\chi_j'(x)e^{izx/h}. \quad (9.4)$$

within the arclength coordinate $x \in [-L_{j-1}, L_j]$ around x_j. Here $R_-(z)$ reconstructs outgoing profiles from outgoing data $u_- \in \mathbb{C}^{2J}$, where $u_{-,2j-1}$, $u_{-,2j}$ are the amplitudes of reconstructed waves outgoing from x_j towards Γ_{j-1} and Γ_j respectively. We give a schematic display of what $\mathbb{C}^{2J}$-data means:

$$\begin{array}{c} R_+(z) \\ \text{maps to} \\ \text{amplitudes} \end{array} \begin{pmatrix} \text{incoming to } x_1 \text{ on } \Gamma_J \\ \text{incoming to } x_1 \text{ on } \Gamma_1 \\ \text{incoming to } x_2 \text{ on } \Gamma_1 \\ \text{incoming to } x_2 \text{ on } \Gamma_2 \\ \cdots \\ \text{incoming to } x_J \text{ on } \Gamma_{J-1} \\ \text{incoming to } x_J \text{ on } \Gamma_J \end{pmatrix}, \quad \begin{array}{c} R_-(z) \\ \text{maps from} \\ \text{amplitudes} \end{array} \begin{pmatrix} \text{outgoing from } x_1 \text{ on } \Gamma_J \\ \text{outgoing from } x_1 \text{ on } \Gamma_1 \\ \text{outgoing from } x_2 \text{ on } \Gamma_1 \\ \text{outgoing from } x_2 \text{ on } \Gamma_2 \\ \cdots \\ \text{outgoing from } x_J \text{ on } \Gamma_{J-1} \\ \text{outgoing from } x_J \text{ on } \Gamma_J \end{pmatrix}. \tag{9.5}$$

We now define the scattering matrix and the transfer matrix. Let

$$\Sigma^{\mathcal{P}}_{M,h^{-1}\omega} = \begin{pmatrix} \Sigma^{\mathcal{W}_1}_{M,h^{-1}\omega} & 0 & \dots & 0 \\ 0 & \Sigma^{\mathcal{W}_2}_{M,h^{-1}\omega} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Sigma^{\mathcal{W}_J}_{M,h^{-1}\omega} \end{pmatrix}, \ T_{z/h} = T_- \begin{pmatrix} E^{z/h}_{L_1} & 0 & \dots & 0 \\ 0 & E^{z/h}_{L_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E^{z/h}_{L_J} \end{pmatrix} T_+ \tag{9.6}$$

where $T_\pm$ cyclically shifts up/down the data by one row, and $E^{z/h}_{L_j}$ is the twisted phase shift on Γ_j:

$$(T_+\mathbf{a})_k = (\mathbf{a})_{k+1}, \ (T_-\mathbf{a})_k = (\mathbf{a})_{k-1}, \ E^{z/h}_{L_j} = \begin{pmatrix} 0 & e^{-iL_jz/h} \\ e^{-iL_jz/h} & 0 \end{pmatrix}. \tag{9.7}$$

for vector $\mathbf{a} \in \mathbb{C}^{2J}$.

The above construction are on each individual component Γ^l labeled by l. We can then glue up them in l to obtain the global construction on all components:

$$R_+(z) = \bigoplus_{l=1}^{L} R^l_+(z) : H^{\frac{1}{2}}_h(\partial\mathcal{P}) \to \mathbb{C}^{2J}, \ R_-(z) = \bigoplus_{l=1}^{L} R^l_-(z) : \mathbb{C}^{2J} \to H^{-\frac{1}{2}}_h(\partial\mathcal{P}), \tag{9.8}$$

$$\Sigma^{\mathcal{P}}_{M,h^{-1}\omega} = \operatorname{diag}(\Sigma^{\mathcal{P},l}_{M,h^{-1}\omega}; 1 \le l \le L), \ T_{z/h} = \operatorname{diag}(T^l_{z/h}; 1 \le l \le L). \tag{9.9}$$

Note here J is now the total number of vertices on all components of $\partial\mathcal{P}$.

In this paper, $z \mapsto \sqrt{z}$ has the branch cut along the negative real axis. Here is our main result of this section:

Proposition 9.1 (The Grushin problem for $\Lambda^{\mathcal{P}}_M$)**.** *There exists $h_0 < \frac{1}{2}$, $r_1 > 0$ such that for $z \in [\frac{1}{2}, 2] + i[-r_1h, r_1h]$, and $h \in (0, h_0]$, the Grushin problem $\mathcal{G}(z;h) : H^{\frac{1}{2}}(\partial\mathcal{P}) \oplus \mathbb{C}^{2J} \to H^{-\frac{1}{2}}(\partial\mathcal{P}) \oplus \mathbb{C}^{2J}$ defined by*

$$\mathcal{G}(z;h) := \begin{pmatrix} \Lambda^{\mathcal{P}}_M - \sqrt{h^{-2}z^2 + M^2} & R_-(z) \\ R_+(z) & 0 \end{pmatrix} \tag{9.10}$$

is well-posed with inverse

$$\mathcal{G}^{-1}(z;h)=\begin{pmatrix}E(z;h) & E_+(z;h)\\ E_-(z;h) & E_{-+}(z;h)\end{pmatrix}, \tag{9.11}$$

where

$$\|\mathcal{G}^{-1}(z;h)\|_{H_h^{-\frac12}(\partial\mathcal{P})\oplus\mathbb{C}^{2J}\to H_h^{\frac12}(\partial\mathcal{P})\oplus\mathbb{C}^{2J}}\le C, \tag{9.12}$$

and

$$E_{-+}(z;h)=\Sigma^{\mathcal{P}}_{M,h^{-1}\omega}-T_{z/h}+O(h^{\frac{2\pi}{\alpha_+}-0}+M^2h^{2-0})+O(|\Im z/h|) \tag{9.13}$$

is analytic in $z\in[\frac12,2]+i[-r_1h,r_1h]$.

Remark 9.2. This implies for $z\in\mathbb{C}$ with $\Re z>0$ sufficiently large and $|\Im z|\le r_1$, we can define

$$\mathcal{Q}^{\Lambda}_z=E_{-+}(z)=\Sigma^{\mathcal{P}}_{M,\Re z}-T_z+O(|\Re z|^{\frac{2\pi}{\alpha_+}-0}+M^2|\Re z|^{2-0})+O(|\Im z|), \tag{9.14}$$

analytic in z, by picking a small h and set $E_{-+}(z)=E_{-+}(hz;h)$.

Proof. 1. For readability, we only address the case $\partial\mathcal{P}$ has only one connected component. The same proof works for cases when $\partial\mathcal{P}$ has more than one connected component. In Steps 1 to 7, we prove the invertibility of $\mathcal{G}(\omega;h)$ in the case $z=\omega\in[\frac12,2]$. Suppose that

$$\mathcal{G}(z;h)\begin{pmatrix}u\\ u_-\end{pmatrix}=\begin{pmatrix}v\\ v_+\end{pmatrix}. \tag{9.15}$$

Since $\Lambda^{\mathcal{P}}_M:H^{\frac12}\to H^{-\frac12}$ is Fredholm with index 0, the same holds for $\mathcal{G}(\omega;h)$. Therefore to show invertibility of $\mathcal{G}(\omega;h)$, it is enough to check its injectivity.

2a. We set up some analytic tools we need later. Let $\gamma_{j,0},\gamma_j,\tilde\gamma_j,\gamma_{j,0.5},\tilde{\tilde\gamma}_j,\gamma_{j,1},\gamma_{j,2}\in C_c^\infty(\Gamma_j)$ be cutoffs that $\gamma_{j,0}\prec\gamma_j\prec\tilde\gamma_j\prec\gamma_{j,0.5}\prec\tilde{\tilde\gamma}_j\prec\gamma_{j,1}\prec\gamma_{j,2}$ such that $\chi_j'|_{\Gamma_j}\prec\gamma_{j,0}$, $\chi_{j+1}'|_{\Gamma_j}\prec\gamma_{j,0}$, $\operatorname{supp}(1-\gamma_{j,0})\cap\Gamma_j\subset\operatorname{supp}\chi_j\cup\operatorname{supp}\chi_{j+1}$ on each edge Γ_j. Let $\tilde\gamma_j=\tilde\gamma_j(X_\omega+X_0+X_\infty)$, where $X_\omega\in\Psi_h^{\mathrm{comp}}(\Gamma_j)$ with $\mathrm{WF}_h(X_\omega)\cap\{\xi=0\}=\emptyset$ and $\mathrm{WF}_h(I-X_\omega)\cap\{|\xi|=\omega\}\cap(\operatorname{supp}\chi_j\cup\operatorname{supp}\chi_{j+1})=\emptyset$, $X_0\in\Psi_h^{\mathrm{comp}}$ with $\mathrm{WF}_h(X_0)\subset\{|\xi|<\frac\epsilon4\}$ and $\mathrm{WF}_h(I-X_0)\cap\{|\xi|=0\}\cap(\operatorname{supp}\chi_j\cup\operatorname{supp}\chi_{j+1})=\emptyset$, for some $\epsilon>0$ to be chosen small. Let the standard semiclassical parametrix $E_\infty\in\Psi_h^{-1}$ be that $(hB-\sqrt{\omega^2+h^2M^2})\tilde{\tilde\gamma}_jE_\infty=\tilde\gamma_jX_\infty+h^\infty\Psi^{-\infty}$, where $B\in\Psi^1$ is the local structure of $\Lambda^{\mathcal{P}}_M$ in the sense of

$$\Lambda^{\mathcal{P}}_M\gamma_{j,1}=\gamma_{j,2}B\gamma_{j,1}+\mathcal{E}\gamma_{j,1},\ \mathcal{E}:H^{-N}\to H^0 \tag{9.16}$$

by taking the adjoint of (8.10).

2b. The rest of Step 2 is devoted to show the existence of $E_0(h)\in\Psi^0$ uniformly bounded for small $h>0$, with $(hB-\sqrt{\omega^2+h^2M^2})\tilde{\tilde\gamma}_jE_0(h)=\tilde\gamma_jX_0+O(h\Psi^{-\infty})$. For the rest of this proof, we may abbreviate $\omega_{hM}=\sqrt{\omega^2+h^2M^2}$ if it makes the proof easier to read. Indeed, let $\tilde X_0\in\Psi_h^{\mathrm{comp}}$ be that $\tilde X_0X_0=X_0+O(h^\infty\Psi^{-\infty})$ with $\mathrm{WF}_h(\tilde X_0)\subset\{|\xi|_g<\frac\epsilon2\}$. Then, since $B:H^1\to L^2$ and

$$\|h\tilde X_0\|_{L^2\to H^1}\le C(h\|\tilde X_0\|_{L^2\to L^2}+C\|h\nabla\tilde X_0\|_{L^2\to L^2})\le C\epsilon, \tag{9.17}$$

we have $\|hB\tilde{X}_0\|_{L^2\to L^2} \leq C\epsilon$. Thus, choosing $\epsilon > 0$ small enough, uniformly for small h, $(hB\tilde{X}_0 - \omega)$ is invertible on L^2 and

$$(hB\tilde{X}_0 - \omega_{hM})^{-1} = -\omega_{hM}^{-1}\Big(\mathrm{I} + \sum_{k=1}^{\infty}(\omega_{hM}^{-1}hB\tilde{X}_0)^k\Big). \tag{9.18}$$

2c. We next show that for $k \in \mathbb{Z}$, there is C such that

$$\|(hB\tilde{X}_0 - \omega_{hM})^{-1}\|_{H^k\to H^k} \leq C, \tag{9.19}$$

uniformly in h small. For this, observe that for $A \in \Psi^k$,

$$[A, (hB\tilde{X}_0 - \omega_{hM})^{-1}] = -(hB\tilde{X}_0 - \omega_{hM})^{-1}[A, hB\tilde{X}_0](hB\tilde{X}_0 - \omega_{hM})^{-1}, \tag{9.20}$$

where

$$[A, hB\tilde{X}_0] = h[A, B]\tilde{X}_0 + hB[A, \tilde{X}_0] \in \Psi^{k-1}, \tag{9.21}$$

and $h\tilde{X}_0 \in \Psi^{-1}$, $[A, B] \in \Psi^k$, and $[A, h\tilde{X}_0] \in \Psi^{k-2}$. Hence, for any $s \in \mathbb{R}$ we have uniformly in $h > 0$ small that

$$\|[A, hB\tilde{X}_0]\|_{H^{s+k-1}\to H^s} \leq C. \tag{9.22}$$

In particular, using this with $A = \langle D\rangle$ implies

$$\langle D\rangle(hB\tilde{X}_0-\omega_{hM})^{-1}\langle D\rangle^{-1} = (hB\tilde{X}_0-\omega_{hM})^{-1}-(hB\tilde{X}_0-\omega_{hM})^{-1}[\langle D\rangle, hB\tilde{X}_0](hB\tilde{X}_0-\omega_{hM})^{-1}\langle D\rangle^{-1} \tag{9.23}$$

is bounded on L^2 and hence $(hB\tilde{X}_0 - \omega_{hM})^{-1} : H^1 \to H^1$ is uniformly bounded. Now take $A = \langle D\rangle^2$ gives the boundedness of $(hB\tilde{X}_0 - \omega_{hM})^{-1} : H^2 \to H^2$. Iterative arguments and duality show (9.19).

2d. Next, we claim that for $A_1, \cdots, A_N \in \Psi^0$, for any $k \in \mathbb{Z}$,

$$\|\operatorname{ad}_{A_N}\operatorname{ad}_{A_{N-1}}\cdots\operatorname{ad}_{A_1}(hB\tilde{X}_0 - \omega_{hM})^{-1}\|_{H^k\to H^{k+N-1}} \leq Ch. \tag{9.24}$$

To do this, we argue as above. This time using (9.21) for $A \in \Psi^0$, $[A, B] \in \Psi^0$, and $[A, \tilde{X}_0] \in \Psi^{-1}$, we have

$$[A, hB\tilde{X}_0] \in h\Psi^0 \tag{9.25}$$

Together with (9.19), this implies (9.24) with $N = 1$. We now claim by induction that for any $N \geq 1$ and $A_1, \ldots, A_N \in \Psi^0$,

$$\operatorname{ad}_{A_N}\operatorname{ad}_{A_{N-1}}\ldots\operatorname{ad}_{A_1}(hB\tilde{X}_0-\omega_{hM})^{-1} = h\sum_{l=1}^{L_N}(hB\tilde{X}_0-\omega_{hM})^{-1}\prod_{m=1}^{M_l}Q_{lm}(hB\tilde{X}_0-\omega_{hM})^{-1}, \tag{9.26}$$

for some $L_N \in \mathbb{N}$, $Q_{lm} \in \Psi^{n_{lm}}$, $n_{lm} \leq 0$ and $\sum_{m=1}^{M_l} n_{lm} = -N+1$. Notice that (9.24) then follows from (9.26) and (9.19) so it remains to prove (9.26). The equality (9.26) with $N = 1$ follows from (9.20) and (9.25). Suppose by induction (9.26) holds for some $N \geq 1$. Let $A_1, \cdots, A_N, A_{N+1} \in \Psi^0$.

Then,

$$\begin{aligned}\operatorname{ad}_{A_{N+1}}\operatorname{ad}_{A_N}\dots\operatorname{ad}_{A_1}(hB\tilde X_0-\omega_{hM})^{-1} &= h\sum_{l=1}^{L_N}[A_{N+1},(hB\tilde X_0-\omega_{hM})^{-1}\prod_{m=1}^{M_l}Q_{lm}(hB\tilde X_0-\omega_{hM})^{-1}]\\ &= -h\sum_{l=1}^{L_N}(hB\tilde X_0-\omega_{hM})^{-1}[A_{N+1},hB\tilde X_0](hB\tilde X_0-\omega_{hM})^{-1}\prod_{m=1}^{M_l}Q_{lm}(hB\tilde X_0-\omega_{hM})^{-1}\\ &+h\sum_{l=1}^{L_N}(hB\tilde X_0-\omega_{hM})^{-1}\sum_{k=1}^{M_l}\Big(\prod_{m=1}^{k-1}Q_{lm}(hB\tilde X_0-\omega_{hM})^{-1}\Big)\Big([A_{N+1},Q_{lk}](hB\tilde X_0-\omega_{hM})^{-1}\\ &-Q_{lk}(hB\tilde X_0-\omega_{hM})^{-1}[A_{N+1},hB\tilde X_0](hB\tilde X_0-\omega_{hM})^{-1}\Big)\prod_{m=k+1}^{M_l}Q_{lm}(hB\tilde X_0-\omega_{hM})^{-1}. \quad (9.27)\end{aligned}$$

The equality (9.26) with N replaced by $N+1$ now follows from the fact that $[A_{N+1},hB\tilde X_0]\in\Psi^{-1}$ and $[A_{N+1},Q_{lm}]\in\Psi^{n_{lm}-1}$.

2e. We can now complete Step 2b. Let $E_0=(hB\tilde X_0-\omega_{hM})^{-1}\tilde\gamma_jX_0\in\Psi^0$. We then have by (9.24) that for any N,

$$(1-\tilde{\tilde\gamma}_j)(hB\tilde X_0-\omega_{hM})^{-1}\tilde\gamma_jX_0=(1-\tilde{\tilde\gamma}_j)\operatorname{ad}^N_{(1-\gamma_{j,0.5})}(hB\tilde X_0-\omega_{hM})^{-1}\tilde\gamma_jX_0\in h\Psi^{-N+1}, \quad (9.28)$$

Similarly, using that $\mathrm{WF}_h(1-\tilde X_0)\cap\mathrm{WF}_h(X_0)=\emptyset$, and letting $X_0'\in\Psi_h^0$ with $\mathrm{WF}_h(I-\tilde X_0)\cap\mathrm{WF}_h(X_0')=\mathrm{WF}_h(1-X_0')\cap\mathrm{WF}_h(X_0)=\emptyset$, we obtain for any $N>0$ that

$$\begin{aligned}(hB-\omega_{hM})\tilde{\tilde\gamma}_jE_0&=\tilde\gamma_jX_0+hB(1-\tilde X_0)(hB\tilde X_0-\omega_{hM})^{-1}\tilde\gamma_jX_0+O(h\Psi^{-N})\\ &=\tilde\gamma_jX_0+hB(1-\tilde X_0)(\operatorname{ad}^N_{(1-X_0')}(hB\tilde X_0-\omega_{hM})^{-1})\tilde\gamma_jX_0+O(h\Psi^{-N})=\tilde\gamma_jX_0+O(h\Psi^{-N}),\end{aligned} \quad (9.29)$$

as desired. We also have

$$\begin{aligned}(1-\tilde X_0)E_0=(1-\tilde X_0)(hB\tilde X_0-\omega_{hM})^{-1}\tilde\gamma_jX_0&=(1-\tilde X_0)(\operatorname{ad}^N_{(1-X_0')}(hB\tilde X_0-\omega_{hM})^{-1})\tilde\gamma_jX_0\\ &\quad+O(h^\infty\Psi^{-\infty})=O(h\Psi^{-N}),\end{aligned} \quad (9.30)$$

which gives $\|(1-\tilde X_0)E_0u\|_{H^N}\le Ch\|u\|_{H_h^{-\frac12}}$. On the other hand, we have

$$\|\tilde X_0E_0u\|_{H_h^N}\le C\|E_0u\|_{L^2}\le C\|\tilde\gamma_jX_0u\|_{L^2}\le C\|u\|_{H_h^{-\frac12}}. \quad (9.31)$$

Hence, for any $N>0$, there is $C>0$ that for all $u\in H_h^{-\frac12}$,

$$\|E_0u\|_{H_h^N}\le C\|u\|_{H_h^{-\frac12}}. \quad (9.32)$$

3. We now approximate u on edges away from the corners. Let $T>\max_j L_j$. Our strategy is to construct quasimodes $\tilde u$ that characterize the influence of v, and $u-\tilde u$ will then behave like

plane waves. Indeed, we can construct $\tilde{u}_j$ localised on each edge Γ_j, that approximate u, via

$$\tilde{u}_j := -i\tilde{\gamma}_j \int_0^T e^{it(B-\omega_{hM}/h)} X_\omega \tilde{\gamma}_j v dt + h\tilde{\tilde{\gamma}}_j (E_\infty + E_0)\tilde{\gamma}_j v,\ \ \tilde{u} = \sum_{j=1}^J \tilde{u}_j, \tag{9.33}$$

with E_∞ and E_0 given in Step 2. We can estimate that

$$\|\tilde{u}_j\|_{H_h^{-\frac{1}{2}}} \leq CT\|\langle hD\rangle^{\frac{1}{2}} X_\omega \tilde{\gamma}_j v\|_{H_h^{-\frac{1}{2}}} + Ch\|\tilde{X}_\infty \tilde{\gamma}_j v\|_{H_h^{-\frac{1}{2}}} + Ch\|\tilde{X}_0 \tilde{\gamma}_j v\|_{H_h^{-\frac{1}{2}}} \leq CT\|\tilde{\gamma}_j v\|_{H_h^{-\frac{1}{2}}}, \tag{9.34}$$

for some $\tilde{X}_\infty \in \Psi_h^0$ with $\mathrm{WF}_h(1-\tilde{X}_\infty) \cap \mathrm{WF}_h X_\infty = \emptyset$, since $e^{it(B-\omega_{hM}/h)}$ is unitary on L^2 and we have (9.32). Note that from (9.16), we have

$$\begin{aligned}(h\Lambda_M^{\mathcal{P}} - \sqrt{\omega^2+h^2M^2})\tilde{u}_j &= -i\gamma_{j,2}[hB,\tilde{\gamma}_j]\int_0^T e^{it(B-\omega_{hM}/h)} X_\omega \tilde{\gamma}_j v\, dt + h\tilde{\gamma}_j X_\omega \tilde{\gamma}_j v \\ -h\tilde{\gamma}_j e^{iT(B-\omega_{hM}/h)} X_\omega \tilde{\gamma}_j v + \gamma_{j,2}(hB-\omega_{hM})h\tilde{\tilde{\gamma}}_j E_\infty \tilde{\gamma}_j v + \gamma_{j,2}(hB-\omega_{hM})h\tilde{\tilde{\gamma}}_j E_0 \tilde{\gamma}_j v + O_{H^0}(h\|\tilde{\gamma}_j v\|_{H^{-N}}) \\ &= -i\gamma_{j,2}[hB,\tilde{\gamma}_j]\int_0^T e^{it(B-\omega_{hM}/h)} X_\omega \tilde{\gamma}_j v\, dt + h\tilde{\gamma}_j^2 v + O_{H^0}(h\|\tilde{\gamma}_j v\|_{H^{-N}}),\end{aligned} \tag{9.35}$$

since $\mathrm{WF}_h(\tilde{\gamma}_j e^{iT(B-\omega_{hM}/h)}\tilde{\gamma}_j X_\omega) = \emptyset$ via propagation. It is interpreted that $\tilde{u}_j$ is a good quasi-mode on Γ_j that $(h\Lambda_M^{\mathcal{P}} - \sqrt{\omega^2+h^2M^2})\tilde{u}_j$ is close to $h\tilde{\gamma}_j^2 v$. Here,

$$\|e^{it(B-\omega_{hM}/h)} X_\omega \tilde{\gamma}_j v\|_{H_h^{-\frac{1}{2}}} \leq \|e^{it(B-\omega_{hM}/h)} X_\omega \tilde{\gamma}_j v\|_{L^2} \leq C\|X_\omega \tilde{\gamma}_j v\|_{L^2} \leq C\|\tilde{\gamma}_j v\|_{H_h^{-\frac{1}{2}}}, \tag{9.36}$$

and we can estimate

$$\|(h\Lambda_M^{\mathcal{P}} - \sqrt{\omega^2+h^2M^2})\tilde{u}_j\|_{H_h^{-\frac{1}{2}}} \leq CTh\|\tilde{\gamma}_j v\|_{H_h^{-\frac{1}{2}}} \tag{9.37}$$

and then by that $\Lambda_M^{\mathcal{P}}$ is classically elliptic, we have

$$\|\tilde{u}_j\|_{H_h^{\frac{1}{2}}} \leq C\|\tilde{u}_j\|_{H_h^{-\frac{1}{2}}} + C\|h\Lambda_M^{\mathcal{P}}\tilde{u}_j\|_{H_h^{-\frac{1}{2}}} \leq C(1+\omega)T(1+h)\|\tilde{\gamma}_j v\|_{H_h^{-\frac{1}{2}}} \leq C\|\tilde{\gamma}_j v\|_{H_h^{-\frac{1}{2}}}. \tag{9.38}$$

Now

$$\begin{aligned}(h\Lambda_M^{\mathcal{P}} - \sqrt{\omega^2+h^2M^2})(u-\tilde{u}) = \sum_j i\gamma_{j,2}[hB,\tilde{\gamma}_j]\int_0^T e^{it(B-\omega_{hM}/h)} X_\omega \tilde{\gamma}_j v\ dt + h(1-\sum_j \tilde{\gamma}_j^2)v \\ - hR_-(\omega)u_- + O_{H^0}(h\|v\|_{H^{-N}}) = f.\end{aligned} \tag{9.39}$$

Note here we have the global expression of f on $\partial\mathcal{P}$:

$$f = -hR_-(\omega)u_- + O_{H_h^{-\frac{1}{2}}}(h\|v\|_{H_h^{-\frac{1}{2}}}) = \sum_j \Big(-ih(u_-)_{2j}\tilde{\chi}_j' e_{+,j} + ih(u_-)_{2j+1}\tilde{\chi}_{j+1}' e_{-,j}\Big) + O_{H_h^{-\frac{1}{2}}}(h\|v\|_{H_h^{-\frac{1}{2}}}). \tag{9.40}$$

We have

$$\|u-\tilde{u}\|_{H_h^{\frac{1}{2}}} \leq C\|u-\tilde{u}\|_{H_h^{-\frac{1}{2}}} + C\|f\|_{H_h^{-\frac{1}{2}}} \leq C\|u\|_{H_h^{-\frac{1}{2}}} + C\|v\|_{H_h^{-\frac{1}{2}}} + Ch|u_-|, \tag{9.41}$$

$$\|u\|_{H_h^{\frac{1}{2}}} \leq C\|u\|_{H_h^{-\frac{1}{2}}} + C\|v\|_{H_h^{-\frac{1}{2}}} + Ch|u_-|. \tag{9.42}$$

4. We want to understand how the scattering data near two ends of the same edge communicate via propagation. In this step, we are in the global coordinates on Γ_j where $x=0$ at x_j, $x=L_j$ at x_{j+1}. See Figure 13. Lemma 8.5 applied to (9.39) give

$$\gamma_j(u-\tilde{u})=\gamma_j\Big(\tilde{a}_{+,j}(x;h)e_{+,j,h}+\tilde{a}_{-,j}(x;h)e_{-,j,h}\Big)+O_{S^0}(h^{\frac{\pi}{\alpha_+}+1-0}\|u\|_{L^2}+h^{1-0}\|v\|_{H_h^{-\frac{1}{2}}}),\quad(9.43)$$

and allows us to read off $a^{o/i}_{\pm,j}(h)\in\mathbb{C}$ from $\gamma_j(u-\tilde{u})$ from the left of Γ_j, by

$$\tilde{a}_{+,j}(x;h)\sim a^o_{+,j}(h)+(u_-)_{2j}(\tilde{\chi}_j(x)-1)+O_{S^0}(h^2M^2\|\tilde{a}_{+,j}\|_{S^0}),\ \text{near supp}\,\gamma_j,\quad(9.44)$$

$$\tilde{a}_{-,j}(x;h)\sim a^i_{-,j}(h)+(u_-)_{2j+1}\tilde{\chi}_{j+1}(x)+O_{S^0}(h^2M^2\|\tilde{a}_{-,j}\|_{S^0}),\ \text{near supp}\,\gamma_j.\quad(9.45)$$

We can also use Lemma 8.5 to read off $a^{o/i}_{\mp,j}(h)\in\mathbb{C}$ from the right of Γ_j, by

$$\tilde{a}_{+,j}(x;h)\sim e^{-i\omega L_j/h}a^i_{+,j}(h)+e^{-i\omega L_j/h}(u_-)_{2j}\tilde{\chi}_j(x)+O_{S^0}(h^2M^2\|\tilde{a}_{+,j}\|_{S^0}),\ \text{near supp}\,\gamma_j,\quad(9.46)$$

$$\tilde{a}_{-,j}(x;h)\sim e^{i\omega L_j/h}a^o_{-,j}(h)+e^{i\omega L_j/h}(u_-)_{2j+1}(\tilde{\chi}_{j+1}(x)-1)+O_{S^0}(h^2M^2\|\tilde{a}_{-,j}\|_{S^0}),\ \text{near supp}\,\gamma_j.\quad(9.47)$$

Here the phase shifts come from the fact that $a^{o/i}_{\mp,j}(h)$ are read near x_{j+1} but normalized near x_j, in bases $e^{\mp i\omega(x+L_j)}$ in coordinates where $x=0$ at x_j. We remind the reader that $v_+=R_+(\omega)u=R_+(\omega)(u-\tilde{u})+R_+(\omega)\tilde{u}$, and

$$|R_+(\omega)\tilde{u}|_{\mathbb{C}^{2J}}\le C\|\tilde{u}\|_{H_h^{\frac{1}{2}}}\le C\|v\|_{H_h^{-\frac{1}{2}}}\quad(9.48)$$

from (9.38) implies

$$(v_+)_{2j-1}=(R_+(\omega)(u))_{2j-1}\sim a^i_{+,j-1}+O(\|v\|_{H_h^{-\frac{1}{2}}}+h^2M^2\|\tilde{a}_{-,j-1}\|_{S^0}),\quad(9.49)$$

$$(v_+)_{2j}=(R_+(\omega)u)_{2j}\sim a^i_{-,j}+O(\|v\|_{H_h^{-\frac{1}{2}}}+h^2M^2\|\tilde{a}_{-,j}\|_{S^0}).\quad(9.50)$$

Note the unique determination of $\tilde{a}_{\pm,j}$ in Lemma 8.5 then gives

$$\|\tilde{a}_{\pm,j}\|_{S^0}\le C\|\tilde{a}_{\pm,j}\|_{L^\infty}\le C|v_+|+C|u_-|+C\|v\|_{H_h^{-\frac{1}{2}}}.\quad(9.51)$$

We then use $O(h^2M^2)$ to abbreviate $O(h^2M^2(|v_+|+|u_-|+\|v\|_{H_h^{-\frac{1}{2}}}))$. Evaluating the above reading at some $x\in\Gamma_j$ where $\tilde{\chi}_j(x)=\tilde{\chi}_{j+1}(x)=0$ gives

$$a^o_{+,j}=e^{-i\omega L_j/h}a^i_{+,j}+(u_-)_{2j}+O(h^2M^2)+O(h^\infty),\quad(9.52)$$

$$a^o_{-,j}=e^{-i\omega L_j/h}a^i_{-,j}+(u_-)_{2j+1}+O(h^2M^2)+O(h^\infty)\quad(9.53)$$

5. In order to understand how the scattering data communicate through the corner x_j, we use Lemma 8.2 to blow up near the corner x_j. See Figure 13. Note

$$\chi_j f=O_{H_h^{-\frac{1}{2}}}(h\|v\|_{H_h^{-\frac{1}{2}}}),\ \tilde{\gamma}f_0=O_{S^0}(h\|v\|_{H_h^{-\frac{1}{2}}}).\quad(9.54)$$

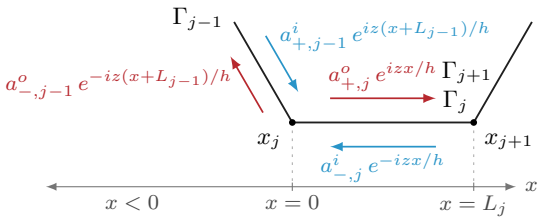


FIGURE 13. Scattering and propagation near vertex x_j on $\partial\mathcal{P}$

Lemma 8.2 then implies

$$\begin{pmatrix} a^o_{-,j-1} \\ a^o_{+,j} \end{pmatrix} = \left(\Sigma^{\mathcal{W}_j}_{M,h^{-1}\omega} + O(M^2h^{2-0})\right) \begin{pmatrix} a^i_{+,j-1} \\ a^i_{-,j} \end{pmatrix} + O(h^{\frac{2\pi}{\alpha_+}-0}\|u\|_{L^2}) + O(h^\infty) + O(h^{\frac{\pi}{\alpha_+}-0}\|v\|_{H_h^{-\frac{1}{2}}}), \tag{9.55}$$

where (8.53) and (9.51) imply

$$\|\chi_j u\|_{H_h^{\frac{1}{2}}} \le Ch^{\frac{2\pi}{\alpha_+}-0}\|u\|_{L^2} + C|v_+| + C|u_-| + C\|v\|_{H_h^{-\frac{1}{2}}}. \tag{9.56}$$

6. We now glue up the scattering relations. Now note Steps 4 and 5 imply the scattering around corners and propagation on edges must agree:

$$\begin{pmatrix} a^o_{-,J} \\ a^o_{+,1} \\ a^o_{-,1} \\ \vdots \\ a^o_{+,J} \end{pmatrix} = (\Sigma^{\mathcal{P}}_{M,h^{-1}\omega} + O(M^2h^{2-0})) \begin{pmatrix} a^i_{+,J} \\ a^i_{-,1} \\ a^i_{+,1} \\ \vdots \\ a^i_{-,J} \end{pmatrix} + O(h^{\frac{2\pi}{\alpha_+}-0}\|u\|_{L^2}) + O(h^\infty) + O(h^{\frac{\pi}{\alpha_+}-0}\|v\|_{H_h^{-\frac{1}{2}}})$$

$$= T_{\omega/h} \begin{pmatrix} a^i_{+,J} \\ a^i_{-,1} \\ a^i_{+,1} \\ \vdots \\ a^i_{-,J} \end{pmatrix} + u_- + O(h^2M^2) + O(h^\infty), \tag{9.57}$$

respectively. Thus we have

$$(\Sigma^{\mathcal{P}}_{M,h^{-1}\omega} - T_{\omega/h})v_+ + O(h^{\frac{2\pi}{\alpha_+}-0}\|u\|_{H_h^{-\frac{1}{2}}} + h^{\frac{\pi}{\alpha_+}-0}\|v\|_{H_h^{-\frac{1}{2}}}) + O(M^2h^{2-0}) + O(h^\infty) = (1 + O(h^{\frac{2\pi}{\alpha_+}+1-0}))u_-. \tag{9.58}$$

7. We now show $\mathcal{G}^{-1}(\omega;h)$ is uniformly bounded and obtain $E_{-+}(\omega;h)$. Note $\Sigma^{\mathcal{P}}_{M,h^{-1}\omega} - T_{\omega/h}$ is bounded and (9.58) implies

$$|u_-| \le C|v_+| + Ch^{\frac{2\pi}{\alpha_+}-0}\|u\|_{H_h^{-\frac{1}{2}}} + Ch^{\frac{\pi}{\alpha_+}-0}\|v\|_{H_h^{-\frac{1}{2}}}. \tag{9.59}$$

Now note by (9.43) we have

$$\|\gamma_j(u-\tilde{u})\|_{H_h^{\frac{1}{2}}} \leq C\|\tilde{a}_{+,j}(x;h)\|_{S^0} + C\|\tilde{a}_{-,j}(x;h)\|_{S^0} + Ch^{\frac{\pi}{\alpha_+}+1-0}\|u\|_{H_h^{-\frac{1}{2}}} + Ch^{1-0}\|v\|_{H_h^{-\frac{1}{2}}}$$
$$\leq C|v_+| + Ch^{\max(\frac{2\pi}{\alpha_+},\frac{\pi}{\alpha_+}+1)-0}\|u\|_{H_h^{-\frac{1}{2}}} + C\|v\|_{H_h^{-\frac{1}{2}}} + O(h^\infty), \quad (9.60)$$

where we used (9.51) and (9.59) to estimate $\tilde{a}_{\pm,j}(x;h)$. Now note that by (9.56), (9.38) we have

$$\|u\|_{H_h^{\frac{1}{2}}} \leq \sum_{j=1}^J \|\chi_j u\|_{H_h^{\frac{1}{2}}} + \|\gamma_j\tilde{u}\|_{H_h^{\frac{1}{2}}} + \|\gamma_j(u-\tilde{u})\|_{H_h^{\frac{1}{2}}} \leq Ch^{\frac{\pi}{\alpha_+}}\|u\|_{L^2} + C\|v\|_{H_h^{-\frac{1}{2}}} + C|v_+|, \quad (9.61)$$

the first term on the right of which can be absorbed. This gives

$$\|u\|_{H_h^{\frac{1}{2}}} \leq C\|v\|_{H_h^{-\frac{1}{2}}} + C|v_+|. \quad (9.62)$$

Then (9.58) reduces to

$$u_- = (\Sigma^{\mathcal{P}}_{M,h^{-1}\omega} - T_{\omega/h})v_+ + O(h^{\frac{2\pi}{\alpha_+}-0}|v_+|) + O(h^{\frac{\pi}{\alpha_+}-0}\|v\|_{H_h^{-\frac{1}{2}}}) + O(M^2h^{2-0}) + O(h^\infty)$$
$$= (\Sigma^{\mathcal{P}}_{M,h^{-1}\omega} - T_{\omega/h} + O(h^{\frac{2\pi}{\alpha_+}-0} + M^2h^{2-0}))v_+ + O(h^{\frac{\pi}{\alpha_+}-0} + M^2h^{2-0})\|v\|_{H_h^{-\frac{1}{2}}}) \quad (9.63)$$

Here we used (9.59). We then can see $\mathcal{G}^{-1}(\omega;h)$ given by

$$\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} O_{H_h^{\frac{1}{2}}}\Big(\|v\|_{H_h^{-\frac{1}{2}}} + |v_+|\Big) \\ O\Big(h^{\frac{\pi}{\alpha_+}-0} + M^2h^{2-0}\Big)\|v\|_{H_h^{-\frac{1}{2}}} + \Big(\Sigma^{\mathcal{P}w}_{M,h^{-1}\omega} - T_{\omega/h} + O(h^{\frac{2\pi}{\alpha_+}-0} + M^2h^{2-0})\Big)|v_+| \end{pmatrix} \quad (9.64)$$

is bounded and

$$E_{-+}(\omega;h)v_+ = \Big(\Sigma^{\mathcal{P}}_{M,h^{-1}\omega} - T_{\omega/h} + O(h^{\frac{2\pi}{\alpha_+}-0} + M^2h^{2-0})\Big)v_+ = u_- \quad (9.65)$$

as desired. This gives the invertibility of $\mathcal{G}(\omega;h)$ for all $\omega \in [\frac{1}{2},2]$.

8. We now show the invertibility of $\mathcal{G}(z;h)$ holds in a complex neighbourhood of $\omega \in [\frac{1}{2},2]$. Note one can check for real μ close to 0 we have

$$\mathcal{G}(z;h) = \mathcal{G}(\omega;h) - \mathcal{A}(z;h),\ \mathcal{A}(z;h) = O_{H_h^{\frac{1}{2}}\oplus\mathbb{C}^{2J}\to H_h^{-\frac{1}{2}}\oplus\mathbb{C}^{2J}}(|\mu|), \quad (9.66)$$

from the definition of $R_\pm(z)$ and

$$\mathcal{A}(z;h) = \begin{pmatrix} \sqrt{h^{-2}(\omega+ih\mu)^2+M^2} - \sqrt{h^{-2}\omega^2+M^2} & R_-(\omega) - R_-(z) \\ R_+(\omega) - R_+(z) & 0 \end{pmatrix}. \quad (9.67)$$

Now (9.64) implies $\|\mathcal{G}^{-1}(\omega;h)\|$ is bounded uniformly in $\omega \in [\frac{1}{2},2]$, $M \in [0,M_0]$, $h \in (0,h_0]$. This implies there is $r_1 > 0$ small that uniformly for any $z = \omega + ih\mu \in [\frac{1}{2},2] + i[-r_1h, r_1h]$ we have

$\|\mathcal{A}(z;h)\mathcal{G}^{-1}(\omega;h)\| < 1$. Now we invert $\mathcal{G}(z;h)$ with the Neumann series:

$$\mathcal{G}^{-1}(z;h) = \mathcal{G}^{-1}(\omega;h)\Big(\mathrm{I} + \sum_{k=1}^{\infty}(\mathcal{A}(z;h)\mathcal{G}^{-1}(\omega;h))^k\Big) = \mathcal{G}^{-1}(\omega;h) + O_{H_h^{-\frac{1}{2}}\oplus\mathbb{C}^{2J}\to H_h^{\frac{1}{2}}\oplus\mathbb{C}^{2J}}(|\mu|). \tag{9.68}$$

This implies

$$E_{-+}(z;h) = E_{-+}(\omega;h) + O(|\mu|) = \Sigma^{\mathcal{P}}_{M,h^{-1}\omega} - T_{\omega/h} + O(h^{\frac{2\pi}{\alpha_+}-0} + M^2h^{2-0}) + O(|\mu|). \tag{9.69}$$

That $E_{-+}(z;h)$ is analytic Fredholm in z follows from $\mathcal{G}(z;h)$ is so. □

Remark 9.3 (Scattering matrices of curvature order $k_0 \geq 0$)**.** Let $\mathcal{W}^{k_0}_{\alpha_j}$ be some 2D curved wedge of corner angle α_j centered at x_j, with first (k_0+1) derivatives of the boundary at x_j agreeing with those for $\mathcal{P}$ at x_j. Proposition 6.3 implies the existence of scattering matrices $\Sigma^{\mathcal{W}^{k_0}_{\alpha_j}}_{M,h^{-1}\omega}$. If $\mathcal{P}$ has all derivatives of order $2 \leq k \leq (k_0+1)$ of the boundary at x_j vanishes, then

$$\mathcal{W}^k_{\alpha_j} = W_{\alpha_j},\ \Sigma^{\mathcal{W}^{k_0}_{\alpha_j}}_{M,h^{-1}\omega} = \Sigma^{W_{\alpha_j}}_{M,h^{-1}\omega} = \Sigma^{W_{\alpha_j}}_{hM,\omega},\ \text{for each } 2 \leq k \leq (k_0+1) \tag{9.70}$$

by the scaling invariance of the straight wedge W_{α_j}.

Proposition 7.1 allows us to only use fixed scattering matrices on $\mathcal{W}^{k_0}_{\alpha_j}$ to define the effective Hamiltonian. Indeed, let for $\omega > 0$ large.

$$\Sigma^{k_0}(\omega) = \begin{pmatrix} \Sigma^{\mathcal{W}^{k_0}_{\alpha_1}}_{0,\omega} & 0 & \dots & 0 \\ 0 & \Sigma^{\mathcal{W}^{k_0}_{\alpha_2}}_{0,\omega} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Sigma^{\mathcal{W}^{k_0}_{\alpha_J}}_{0,\omega} \end{pmatrix},\ \Sigma^0 = \begin{pmatrix} \Sigma^{\mathcal{W}_{\alpha_1}}_{0,1} & 0 & \dots & 0 \\ 0 & \Sigma^{\mathcal{W}_{\alpha_2}}_{0,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Sigma^{\mathcal{W}_{\alpha_J}}_{0,1} \end{pmatrix}. \tag{9.71}$$

Corollary 9.4. *There exist $C, h_0 > 0$ such that uniformly for $\omega \in [\frac{1}{2}, 2]$, and $h \in (0, h_0]$, we have* $\dim\ker(\Lambda^{\mathcal{P}}_M - h^{-1}\omega) = \dim\ker \mathcal{Q}^{\Lambda}_{h^{-1}\omega}$, *where*

$$\mathcal{Q}^{\Lambda}_{h^{-1}\omega} = E_{-+}(\omega;h) = \Sigma^{k_0}(h^{-1}\omega) - T_{\omega/h} + O\Big(h^{\min(\frac{2\pi}{\alpha_+},1+k_0)-0} + (Mh)^{\min(\frac{2\pi}{\alpha_+},2)-0}\Big). \tag{9.72}$$

Proof. Proposition 9.1 implies that $\dim\ker(\Lambda^{\mathcal{P}}_M - h^{-1}z) = \dim\ker E_{-+}(z;h)$. It suffices to note Proposition 7.1 implies

$$\Sigma^{k_0}(h^{-1}\omega) - \Sigma^{\mathcal{P}}_{M,h^{-1}\omega} = O\Big(h^{\min(\frac{2\pi}{\alpha_+},1+k_0)-0} + (Mh)^{\min(\frac{2\pi}{\alpha_+},2)-0}\Big) \tag{9.73}$$

and thus

$$\begin{aligned} \mathcal{Q}^{\Lambda}_{h^{-1}\omega} = E_{-+}(\omega;h) &= \Sigma^{\mathcal{P}}_{M,h^{-1}\omega} - T_{\omega/h} + O(h^{\frac{2\pi}{\alpha_+}-0}) + O(h^2M^2) \\ &= \Sigma^{k_0}(h^{-1}\omega) - T_{\omega/h} + O\Big(h^{\min(\frac{2\pi}{\alpha_+},1+k_0)-0} + (Mh)^{\min(\frac{2\pi}{\alpha_+},2)-0}\Big), \end{aligned} \tag{9.74}$$

as desired. □

Corollary 9.5. *There exist $h_0, r_1 > 0$ such that for all $\epsilon > 0$, $\ell \geq 0$, there is $C > 0$ such that the uniformly for $z \in [\frac{3}{4}, \frac{3}{2}]$, and $h \in (0, h_0]$,*

$$|\partial_z^\ell (Q^\Lambda_{h^{-1}z} + T_{z/h})| \leq C h^{2-\ell-\epsilon}$$

Proof. Observe that $\Sigma^0(z) = \Sigma^0$ is independent of z and thus, by Corollary 9.4,

$$|f(h,z)| \leq Ch^{1-0} + C|\Im z|/h, \qquad f(h,z) := Q^\Lambda_{h^{-1}z} + T_{z/h} - \Sigma^0$$

In particular since f is analytic, for $z \in [\frac{1}{2}, 2] + i[-r_1 h, r_1 h]$, letting

$$\Gamma_\epsilon = 2 + [-\epsilon h, \epsilon h] \cup [2, \tfrac{1}{2}] + i\epsilon h \cup \tfrac{1}{2} + i[\epsilon h, -\epsilon h] \cup [\tfrac{1}{2}, 2] - i\epsilon h$$

$$\begin{aligned}
\big|\partial_z^\ell f(z)\big| &= \frac{\ell!}{2\pi}\Big| \int_{\Gamma_\epsilon} \frac{f(s)}{(s-z)^{\ell+1}} ds\Big| \\
&\leq C\int_{-\epsilon h}^{\epsilon h} (h^{1-0} + s/h) ds + C\int_{-2}^{2} \frac{h^{1-0}+\epsilon}{(\sqrt{s^2+\epsilon^2 h^2})^{\ell+1}} ds. \\
&\leq C(\epsilon h^{2-0} + \epsilon^2 h^{1-0}) + \begin{cases} C(h^{1-0}+\epsilon)\log h & \ell = 1 \\ C(h^{1-0}+\epsilon)h^{1-\ell} & \ell \geq 2. \end{cases}
\end{aligned}$$

Hence, choosing $\epsilon = h$, we obtain

$$|\partial_z^\ell f(z)| \leq C h^{2-\ell-0},$$

which implies the corollary. □

Remark 9.6. All constants in this section can depend on $\mathcal{P}$ in the following way: the upper bound on the number of corners J, the upper and lower bounds over the angles α_j, the upper and lower bounds on all $|\Gamma_j|$, the derivatives of the defining functions of Γ_j.

10. The Steklov Dirac operator and approximation on curvilinear polygons

Using the Grushin problem from Section 9, we prove Theorem 1.3 in this section. The key remaining issues are i) to show that the values of ω that $\mathcal{Q}^\Lambda_\omega = E_{-+}(\omega)$ from (9.13) is not invertible are close, with multiplicity, to those of $\mathcal{Q}^{\mathscr{D}}(\omega) = \Sigma^{k_0}(\omega) - T_{\omega/h}$ from Corollary 9.4 (which in turn is the Steklov Driac operator with matching from Theorem 1.3 and ii) to prove that the labeling of the eigenvalues for the Dirac operator is identical to that for the Steklov problem.

Outline of the strategy

Closeness of the spectra:

To prove that the spectra of $\mathcal{Q}^\Lambda_\omega$ and $\mathcal{Q}^{\mathscr{D}}_\omega$ are close, with multiplicity, we extend $\mathcal{Q}^{\mathscr{D}}_\omega$ off of the real axis so that $\bar\partial \mathcal{Q}^{\mathscr{D}}_z|_{|\Im z|=0} = 0$ and use a version of Rouche's theorem valid for smooth functions. The key features that make this possible are that Σ^{k_0} is unitary and has small derivatives. This allows us to show that the eigenvalues of $(\Sigma^{k_0}(\omega))^* T_\omega$ wind monotonically and with strictly positive speed around the unit. Thus has two important consequences. First, there are at most $2J$ roots of $\det \mathcal{Q}^{\mathscr{D}}_\omega$ in fixed size intervals and hence the spectrum has many gaps, and second, that the extension of $\mathcal{Q}^{\mathscr{D}}_\omega$ off the real axis is well behaved. The closeness of $\mathcal{Q}^{\mathscr{D}}_z$ and the extension of $\mathcal{Q}^\Lambda_\omega$ then persists off the real axis, making the Rouché argument possible.

Eigenvalue labeling:

After proving that the roots of $\det \mathcal{Q}_z^\Lambda$ are close to the eigenvalues of the Steklov problem, we have established that, up to a possible shift of the index, the eigenvalues, λ_n of the Dirac operator are close to those of the Steklov problem, σ_n i.e. there is $N \in \mathbb{Z}$ such that

$$|\sigma_n - \lambda_{n+N}| \leq C\sigma_n^{-\min(\frac{2\pi}{\alpha_+}, 1+k_0)+\epsilon} + CM^2\sigma_n^{-2+\epsilon}, \qquad \sigma_n > \sigma_0. \tag{10.1}$$

It thus remains to show that $N = 0$.

To prove that $N = 0$ we make a deformation argument. Specifically, we observe that σ_0 is uniform over $M \in [0, M_0]$, for curvilinear polygons whose angles are uniformly bounded away from 0 and 2π, and for which the C^k norms of the the edges of the polygon are uniformly bounded. Thus, we may deform any curvilinear polygon with $M \geq 0$ to a smooth domain with $M = 0$ whose boundary is linear near each (fictitious) corner and without changing σ_0. This implies that L is constant among all domains with the same topology and for all $M \geq 0$. It thus remains to observe that the results of [GPPS14] (see also [Roz86] for the simply connected case) imply $L = 0$ for smooth domains.

10.1. **The graph Dirac operator on curvilinear polygons.** Given $\mathcal{P} \Subset \mathbb{R}^2$, a connected Lipschitz domain with connected piecewise smooth boundary, we parametrize $\partial\mathcal{P}$ by $\mathbb{S}^1$ and the vertices by $x_j \in \mathbb{S}^1$ for $1 \leq j \leq J$: here we understand it as a graph with edges Γ_j connecting x_j to x_{j+1} of length L_j. Let $\mathcal{W}_{\alpha_j}^{k_0}$ be defined in Remark 9.3. Let $\psi_0 \in C_c^\infty(\mathbb{R})$ be even and $\psi_0(\omega) \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, and $\psi_0(\omega) \equiv 1$ on $|\omega| \geq 1$. Let $\delta > 0$ to be chosen small and define the regularized scattering matrix

$$\Sigma_j^{k_0}(\omega) = \exp\Big((1-\psi_0)(\delta\omega)\log\big(\Sigma_{0,\omega}^{\mathcal{W}_{\alpha_j}^{k_0}}(\Sigma_{0,1}^{W_{\alpha_j}})^*\big)\Big)\Sigma_{0,1}^{W_{\alpha_j}}, \ \omega \geq 0, \tag{10.2}$$

and $\Sigma_j^{k_0}(\omega) = -\sigma_3\overline{\Sigma_j^{k_0}(-\omega)}\sigma_3$ for $\omega < 0$, where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.3}$$

is the third Pauli matrix. Note $\Sigma_j^{k_0}(\omega)$ is unitary on all of $\omega \in \mathbb{R}$, and $\Sigma_j^0(\omega) = \Sigma_{0,1}^{W_{\alpha_j}}$ on $\omega \in \mathbb{R}$, where

$$\Sigma_{0,1}^{W_{\alpha_j}} = \begin{pmatrix} -i\cos(\frac{\pi^2}{2\alpha}) & \sin(\frac{\pi^2}{2\alpha}) \\ \sin(\frac{\pi^2}{2\alpha}) & -i\cos(\frac{\pi^2}{2\alpha}) \end{pmatrix}, \tag{10.4}$$

given by Lemma D.8. One can check $\Sigma_{0,1}^{W_{\alpha_j}} = -\sigma_3\overline{\Sigma_{0,1}^{W_{\alpha_j}}}\sigma_3$ and thus $\Sigma_j^{k_0}(\omega)$ is identically $\Sigma_{0,1}^{W_{\alpha_j}}$ near $\omega = 0$.

Consider the Steklov Dirac operator $\mathscr{D}_\omega^{k_0} = \begin{pmatrix} D_x & 0 \\ 0 & -D_x \end{pmatrix}$ with domain

$$\mathrm{Dom}(\mathscr{D}_\omega^{k_0}) = \Big\{ u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \in H^1(\partial\mathcal{P}; \mathbb{C}^2) \,:\, \Sigma_j^{k_0}(\omega)\begin{pmatrix} u_+(x_j - 0) \\ u_-(x_j + 0) \end{pmatrix} = \begin{pmatrix} u_-(x_j - 0) \\ u_+(x_j + 0) \end{pmatrix} \Big\} \subset L^2(\partial\mathcal{P}; \mathbb{C}^2), \tag{10.5}$$

where the Dirichlet-type boundary condition records the scattering happening at vertices x_j's. We may drop the index ω and k_0 whenever there is no confusion.

For $\omega \in \mathbb{C}$, we define the effective Hamiltonian $\mathcal{Q}_\omega^{\mathscr{D}}$ by,

$$\mathcal{Q}_\omega^{\mathscr{D}} = \Sigma^{k_0}(\omega) - T_\omega,\ \Sigma^{k_0}(\omega) = \begin{pmatrix} \Sigma_1^{k_0}(\omega) & 0 & \dots & 0 \\ 0 & \Sigma_2^{k_0}(\omega) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Sigma_J^{k_0}(\omega) \end{pmatrix}, \tag{10.6}$$

where $\Sigma^{k_0}(\omega)$ tracks the scattering at vertices and the transfer operator T_z for $z \in \mathbb{C}$ tracks the propagation of states at complex frequency z along each edge,

$$T_z = T_- \begin{pmatrix} E_{L_1}^z & 0 & \dots & 0 \\ 0 & E_{L_2}^z & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E_{L_J}^z \end{pmatrix} T_+,\ E_{L_j}^z = \begin{pmatrix} 0 & e^{-izL_j} \\ e^{-izL_j} & 0 \end{pmatrix}. \tag{10.7}$$

We remind the readers that $\Sigma^{k_0}(\omega)$ and $\mathcal{Q}_\omega^{\mathscr{D}}$ are not necessarily analytic in ω unless $k_0 = 0$.

Set $S_3 = \mathrm{diag}(\sigma_3, \dots, \sigma_3) \in \mathbb{C}^{2J\times 2J}$. Note

$$S_3 T_+ S_3 = -T_+,\ S_3 T_- S_3 = -T_-,\ S_3 T_z S_3 = -T_z. \tag{10.8}$$

The above implies

$$S_3 \overline{\mathcal{Q}_\omega^{\mathscr{D}}} S_3 = S_3 (\overline{\Sigma^{k_0}(\omega) - T_\omega}) S_3 = -(\Sigma^{k_0}(-\omega) - T_{-\omega}) = -\mathcal{Q}_{-\omega}^{\mathscr{D}}. \tag{10.9}$$

Taking the determinant gives

$$\overline{\det \mathcal{Q}_\omega^{\mathscr{D}}} = \det S_3 \overline{\mathcal{Q}_\omega^{\mathscr{D}}} S_3 = \det(-\mathcal{Q}_{-\omega}^{\mathscr{D}}) = \det \mathcal{Q}_{-\omega}^{\mathscr{D}}. \tag{10.10}$$

This implies the zeros of $\det \mathcal{Q}_\omega^{\mathscr{D}}$ are symmetric about $\omega = 0$, and if $\det \mathcal{Q}_0^{\mathscr{D}} = 0$, $\omega = 0$ is a zero with even multiplicity. We define the zero set of $\det \mathcal{Q}_\omega^{\mathscr{D}}$ and $\det \mathcal{Q}_\omega^{\Lambda}$ by

$$\mathcal{Z}^{\mathscr{D}} := \{\omega \in \mathbb{R} : \det \mathcal{Q}_\omega^{\mathscr{D}} = 0\},\ \mathcal{Z}^{\Lambda} := \{\omega \in \mathbb{R} : \det \mathcal{Q}_\omega^{\Lambda} = 0\} \tag{10.11}$$

and their associated counting functions by

$$\mathcal{N}_I^{\mathscr{D}} := \sum_{\omega\in I} \dim\ker \mathcal{Q}_\omega^{\mathscr{D}},\ \mathcal{N}_I^{\Lambda} := \sum_{\omega\in I} \dim\ker \mathcal{Q}_\omega^{\Lambda},\ I \subset \mathbb{R}. \tag{10.12}$$

In the following lemma, we show that $\ker \mathcal{Q}_\omega^{\mathscr{D}}$ characterizes $\mathrm{char}(\mathscr{D}_\omega^{k_0})$.

Lemma 10.1 (The Steklov Dirac operator)**.** *The Steklov Dirac operator $\mathscr{D}_\omega^{k_0}$ is self-adjoint. Moreover, let $\omega \in \mathbb{R}$ and*

$$\mathbf{a} = (a_{+,J}^i, a_{-,1}^i, a_{+,1}^i, \cdots, a_{-,J}^i)^T \in \mathbb{C}^{2J}. \tag{10.13}$$

Then we have $\mathcal{Q}_\omega^{\mathscr{D}}\mathbf{a} = 0$ if and only if $u_{\mathbf{a}} \in H^1(\partial\mathcal{P}; \mathbb{C}^2)$ defined by

$$u_{\mathbf{a}}|_{(x_j, x_{j+1})} = \begin{pmatrix} a_{+,j}^i e^{iz(x-x_{j+1})} \\ a_{-,j}^i e^{-iz(x-x_j)} \end{pmatrix} \tag{10.14}$$

is a solution to $(\mathscr{D}_\omega^{k_0} - \omega)u_{\mathbf{a}} = 0$.

Proof. 1. Let $z = \omega + i\mu$ and we formally solve $(\mathscr{D} - z)u = f$. Given $f \in L^2(\partial\mathcal{P}; \mathbb{C}^2)$, we want to find $u \in \mathrm{Dom}(\mathscr{D})$ that

$$(\mathscr{D} - z)u = (\mathscr{D} - z)\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} (D_x - z)u_+ \\ (-D_x - z)u_- \end{pmatrix} = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = f. \tag{10.15}$$

We will denote the end data of u near vertex x_j by

$$a^i_{+,j-1} = u_+(x_j - 0),\ a^o_{-,j-1} = u_-(x_j - 0),\ a^o_{+,j} = u_+(x_j + 0),\ a^i_{-,j} = u_-(x_j + 0) \tag{10.16}$$

and will solve for those data that matches the scattering relation governed by the boundary condition. Note that on edge Γ_j, we have two ordinary differential equations for $u_\pm$ and they are solved by

$$u_+(x) = a^i_{+,j} e^{iz(x - x_{j+1})} - i\int_x^{x_{j+1}} e^{iz(x-s)} f_+(s)\, ds, \tag{10.17}$$

$$u_-(x) = a^i_{-,j} e^{-iz(x - x_j)} - i\int_{x_j}^{x} e^{-iz(x-s)} f_-(s)\, ds, \tag{10.18}$$

for $x \in \Gamma_j = (x_j, x_{j+1})$, where $a^i_{\pm,j}$ unique determines $u_\pm$. Notably we have

$$a^i_{+,j} = e^{izL_j}(a^o_{+,j} - F_{+,j}),\ a^o_{-,j} = e^{-izL_j} a^i_{-,j} + F_{-,j}, \tag{10.19}$$

where

$$F_{+,j} = -i\int_{x_j}^{x_{j+1}} e^{iz(x_j - s)} f_+(s)\, ds,\ F_{-,j} = -ie^{-izL_j}\int_{x_j}^{x_{j+1}} e^{-iz(x_j - s)} f_-(s)\, ds \tag{10.20}$$

are purely determined by the data f. Now note $u \in \mathrm{Dom}(\mathscr{D})$ if and only if the scattering relations per (10.5) holds, that is

$$\begin{pmatrix} a^o_{-,J} \\ a^o_{+,1} \\ a^o_{-,1} \\ \vdots \\ a^o_{+,J} \end{pmatrix} = \Sigma^{k_0}(\omega)\begin{pmatrix} a^i_{+,J} \\ a^i_{-,1} \\ a^i_{+,1} \\ \vdots \\ a^i_{-,J} \end{pmatrix} = T_z \begin{pmatrix} a^i_{+,J} \\ a^i_{-,1} \\ a^i_{+,1} \\ \vdots \\ a^i_{-,J} \end{pmatrix} + \begin{pmatrix} F_{-,J} \\ F_{+,1} \\ F_{-,1} \\ \vdots \\ F_{+,J} \end{pmatrix} = T_z\mathbf{a} + \mathbf{F}. \tag{10.21}$$

That is, $(\mathscr{D} - z)u = f$ for $u \in \mathrm{Dom}(\mathscr{D})$, if and only if, u is given by (10.17) and (10.18) and the scattering relation $(\Sigma^{k_0}(\omega) - T_z)\mathbf{a} = \mathbf{F}$ holds. For $z = \omega \in \mathbb{R}$, this means $\mathcal{Q}^{\mathscr{D}}_\omega \mathbf{a} = (\Sigma^{k_0}(\omega) - T_\omega)\mathbf{a} = \mathbf{F}$.

2. We show $(\Sigma^{k_0}(\omega) - T_z)^{-1}$ exists for $z = \omega + i\mu \in \mathbb{C} \setminus \mathbb{R}$. Note $\Sigma^{k_0}(\omega)$ is unitary. When $\mu > 0$, we have $\|T_z^{-1}\| = e^{-\mu \min_j\{L_j\}} < 1$, and thus for each $\mathbf{a} \in \mathbb{C}^{2J}$, we have

$$|(\Sigma^{k_0}(\omega) - T_z)\mathbf{a}| \geq (e^{\mu \min_j\{L_j\}} - 1)|\mathbf{a}| \geq C^{-1}\min(\mu \min_j(L_j), 1)e^{\mu \min_j(L_j)}|\mathbf{a}| \tag{10.22}$$

as desired. When $\mu < 0$, we have $\|T_z\| = e^{\mu \min_j\{L_j\}} < 1$ and then

$$|(\Sigma^{k_0}(\omega) - T_z)\mathbf{a}| \geq (1 - e^{\mu \min_j\{L_j\}})|\mathbf{a}| \geq C^{-1}\min(|\mu| \min_j(L_j), 1)|\mathbf{a}|. \tag{10.23}$$

The above shows for $z \in \mathbb{C} \setminus \mathbb{R}$, $(\Sigma^{k_0}(\omega) - T_z)^{-1}$ exists and by Step 1, $\mathscr{D} - z$ is surjective.

3. We show $\mathscr{D}$ is self-adjoint. First note $\mathscr{D}$ is symmetric, since for $u, v \in \mathrm{Dom}(\mathscr{D})$,

$$\langle \mathscr{D}u, v\rangle - \langle u, \mathscr{D}v\rangle = -i\sum_{j=1}^{J}\Big(u_+(x_{j+1}-0)\overline{v_+(x_{j+1}-0)} - u_+(x_j+0)\overline{v_+(x_j+0)} - u_-(x_{j+1}-0)\overline{v_-(x_{j+1}-0)}$$

$$+u_-(x_j+0)\overline{v_-(x_j+0)}\Big) = -i\sum_{j=1}^{J}\Big(u_+(x_j-0)\overline{v_+(x_j-0)} - u_+(x_j+0)\overline{v_+(x_j+0)} - u_-(x_j-0)\overline{v_-(x_j-0)}$$

$$+u_-(x_j+0)\overline{v_-(x_j+0)}\Big) = -i\sum_{j=1}^{J}\Big(\Big\langle \begin{pmatrix} u_+(x_j-0)\\ u_-(x_j+0)\end{pmatrix}, \begin{pmatrix} v_+(x_j-0)\\ v_-(x_j+0)\end{pmatrix}\Big\rangle - \Big\langle \begin{pmatrix} u_-(x_j-0)\\ u_+(x_j+0)\end{pmatrix}, \begin{pmatrix} v_-(x_j-0)\\ v_+(x_j+0)\end{pmatrix}\Big\rangle\Big)$$

$$= -i\sum_{j=1}^{J}\Big(\Big\langle \begin{pmatrix} u_+(x_j-0)\\ u_-(x_j+0)\end{pmatrix}, \begin{pmatrix} v_+(x_j-0)\\ v_-(x_j+0)\end{pmatrix}\Big\rangle - \Big\langle \Sigma_j^{k_0}(\omega)\begin{pmatrix} u_+(x_j-0)\\ u_-(x_j+0)\end{pmatrix}, \Sigma_j^{k_0}(\omega)\begin{pmatrix} v_+(x_j-0)\\ v_-(x_j+0)\end{pmatrix}\Big\rangle\Big)$$

$$= 0, \quad (10.24)$$

the last line of which follows from $\Sigma_j^{k_0}(\omega)$ being unitary. By Step 2, $\mathrm{Ran}(\mathscr{D}\pm i) = L^2(\partial\mathcal{P};\mathbb{C}^2)$ and thus $\mathscr{D}$ is self-adjoint.

4. Let $\mathcal{Q}_z^{\mathscr{D}}\mathbf{a} = 0$, then Step 1 immediately implies that $u_{\mathbf{a}}$ is a solution to $(\mathscr{D}-z)u_{\mathbf{a}} = 0$. Let $u_{\mathbf{a}} \in \mathrm{Dom}(\mathscr{D})$ be that $(\mathscr{D}-z)u_{\mathbf{a}} = 0$, then by (10.17) and (10.18),

$$u_+ = a^i_{+,j}e^{iz(x-x_{j+1})},\ u_- = a^i_{-,j}e^{-iz(x-x_j)}, \quad (10.25)$$

and $a^i_{\pm,j}$ must satisfy the scattering relation (10.21), that is, $\mathcal{Q}_z^{\mathscr{D}}\mathbf{a} = 0$. □

10.2. **Properties of $\mathcal{Q}_\omega^{\mathscr{D}}$.** In this subsection, we collect some important facts about $\mathcal{Q}_\omega^{\mathscr{D}}$ that will be used in the proof of Theorem 1.3. All estimates in this section are uniform in $0 < \delta < 1$ unless explicitly specified.

By Proposition 7.1, we have

$$\Sigma^{k_0}(\omega) - \Sigma^0 = O(|\omega|^{-1+0}), \quad (10.26)$$

and, by Lemma 7.7, we have

$$\|\partial_\omega^2\Sigma^{k_0}\| + \|\partial_\omega\Sigma^{k_0}\| \le C. \quad (10.27)$$

Therefore,

$$\|\partial_\omega\Sigma^{k_0}\| \le C(|\omega|^{-\frac{1}{2}+0}). \quad (10.28)$$

Denote for convenience

$$U(\omega) = \Sigma^{k_0}(\omega),\ T(\omega) = T_\omega,\ W(\omega) = U^*(\omega)T(\omega),\ A_W(\omega) = -iW^*(\omega)W'(\omega). \quad (10.29)$$

Here U, T, W are unitary matrices in $\mathbb{C}^{2J\times 2J}$. We then have

$$\mathcal{Q}_\omega^{\mathscr{D}} = U(\omega)(I - W(\omega)). \quad (10.30)$$

Specifically we have

$$A_T(\omega) := -iT^*(\omega)T'(\omega) = -iT_-\,\mathrm{diag}_j((E^\omega_{L_j})^*(E^\omega_{L_j})')T_+ = -T_-\,\mathrm{diag}_j\begin{pmatrix} L_j & 0\\ 0 & L_j\end{pmatrix}T_+, \quad (10.31)$$

since $(E^\omega_{L_j})' = -iL_jE^\omega_{L_j}$. Then we have

$$-L_+\,\mathrm{I} \leq A_T(\omega) \leq -L_-\,\mathrm{I},\ L_- = \min_j(L_j),\ L_+ = \max_j(L_j). \tag{10.32}$$

Note that for ω large,

$$-2L_+\,\mathrm{I} \leq A_W = A_T - iT^*U(U')^*T \leq -\frac{L_-}{2}\,\mathrm{I}, \tag{10.33}$$

since $U'(\omega) = o(1)$ by (10.28).

Since $W(\omega)$ is unitary, it has on orthonormal basis of eigenvectors $\{v_j(\omega)\}_{j=1}^{2J}$, with eigenvalues $\{e^{i\theta_j(\omega)}\}_{j=1}^{2J}$, for $\theta_j(\omega) \in \mathbb{R}$ and, moreover, (10.33) implies that $-2L_+ \leq \theta_j'(\omega) \leq -\frac{L_-}{2}$. In particular, there is $c > 0$ that

$$\theta_j(\omega_0) \in 2\pi\mathbb{Z} \quad \Rightarrow \quad \operatorname{dist}(\theta_j(\omega), 2\pi\mathbb{Z}) \geq c|\omega - \omega_0| \text{ for } |\omega - \omega_0| < c. \tag{10.34}$$

Lemma 10.2 (Root density for $\det \mathcal{Q}^{\mathscr{D}}_z$)**.** *There is $r_0 > 0$ such that for any $\omega_0 \in \mathbb{R}$,*

$$\mathcal{N}^{\mathscr{D}}_{(\omega_0 - r_0, \omega_0 + r_0)} = \#\{\omega \in (\omega_0 - r_0, \omega_0 + r_0) : \det \mathcal{Q}^{\mathscr{D}}_\omega = 0\} \leq 2J, \tag{10.35}$$

counting multiplicities.

Proof. Observe that, since $\mathcal{Q}^{\mathscr{D}}_\omega = \Sigma^{k_0}(\omega)(I - W(\omega))$,

$$\mathcal{N}^{\mathscr{D}}_{(\omega_0 - r_0, \omega_0 + r_0)} = \#\{(\omega, j) \in (\omega_0 - r_0, \omega_0 + r_0) \times \{1, \dots, 2J\} \,:\, \theta_j(x) \in 2\pi\mathbb{Z}\}. \tag{10.36}$$

Thus, the lemma follows from (10.34). □

Lemma 10.3 (Control on $(\mathcal{Q}^{\mathscr{D}}_\omega)^{-1}$)**.** *There is $C > 0$ such that for all*

$$\|(\mathcal{Q}^{\mathscr{D}}_\omega)^{-1}\| \leq C \operatorname{dist}\left(\omega, \mathcal{Z}^{\mathscr{D}}\right)^{-1}. \tag{10.37}$$

Proof. Observe that

$$(\mathcal{Q}^{\mathscr{D}}_\omega)^{-1} = (\mathrm{I} - W(\omega))^{-1}U(\omega)^*. \tag{10.38}$$

Therefore,

$$\|(\mathcal{Q}^{\mathscr{D}}_\omega)^{-1}\| \leq C \inf_j \operatorname{dist}(\theta_j, 2\pi\mathbb{Z})^{-1}. \tag{10.39}$$

The lemma now follows from (10.34). □

We build $\tilde{\mathcal{Q}}^{\mathscr{D}}_z$, the almost analytic extension of $\mathcal{Q}^{\mathscr{D}}_\omega$ of order 1 to $z \in \mathbb{C}$ by

$$\tilde{\mathcal{Q}}^{\mathscr{D}}_{\omega + i\mu} := \mathcal{Q}^{\mathscr{D}}_\omega + i\mu\partial_\omega \mathcal{Q}^{\mathscr{D}}_\omega. \tag{10.40}$$

Notice, in particular, that $\bar\partial_{\omega+i\mu}\mathcal{Q}^{\mathscr{D}}_\omega = 0$ for $\omega \in \mathbb{R}$.

Lemma 10.4 (Structure of $\tilde{\mathcal{Q}}^{\mathscr{D}}_z$ near a root)**.** *Let $\omega_0 \in \mathbb{R}$ and define $m := \dim\ker \mathcal{Q}^{\mathscr{D}}_{\omega_0}$. Then, there is $a \in \mathbb{C}$, $a \neq 0$ such that*

$$\det \tilde{\mathcal{Q}}^{\mathscr{D}}_z = a(z - \omega_0)^m + O(|z - \omega_0|^{m+1}). \tag{10.41}$$

Proof. Observe that for $|\mu| \le C$,

$$\|(\Sigma^{k_0}(\omega) + i\mu\partial_\omega\Sigma^{k_0}(\omega))^{-1}\| \le C. \tag{10.42}$$

Extend $W(\omega)$ by

$$\tilde W(\omega + i\mu) = (\Sigma^{k_0}(\omega) + i\mu\partial_\omega\Sigma^{k_0}(\omega))^{-1}(T_\omega + i\mu\partial_\omega T_\omega), \tag{10.43}$$

so that $\tilde W(\omega) = W(\omega)$ and

$$\det \tilde{\mathcal{Q}}^{\mathscr{D}}_{\omega+i\mu} = \det(\Sigma^{k_0}(\omega) + i\mu\partial_\omega\Sigma^{k_0}(\omega)) \det(\mathrm{I} - \tilde W(\omega + i\mu)) = e(\omega,\mu)\det(\mathrm{I} - \tilde W), \tag{10.44}$$

where $e(\omega,\mu)$ is smooth and nonvanishing. Let $V := \ker(W(\omega_0) - \mathrm{I})$. Then, decomposing $\mathbb{C}^{2J} = V \oplus V^\perp$, we have

$$W(\omega_0) - \mathrm{I} = \begin{pmatrix} 0 & 0 \\ 0 & W_{22}(\omega_0) - \mathrm{I} \end{pmatrix}, \tag{10.45}$$

where $W_{22}(\omega_0) - \mathrm{I}$ is invertible. Then, with $z = \omega + i\mu$, we write

$$\tilde W(z,\bar z) - \mathrm{I} = \begin{pmatrix} \tilde W_{11}(z,\bar z) - \mathrm{I} & \tilde W_{12}(z,\bar z) \\ W_{21}(z,\bar z) & \tilde W_{22}(z,\bar z) - \mathrm{I} \end{pmatrix}, \tag{10.46}$$

with $\tilde W_{ij}(z,\bar z) \in C^2$. Notice that

$$-i\tilde W^*\partial_\omega\tilde W|_{\omega_0} = A_W(\omega_0) < 0,\ -i\tilde W^*\partial_\mu\tilde W|_{\omega_0} = iA_W(\omega_0). \tag{10.47}$$

Hence, since $W_{11}(\omega_0) = \mathrm{I}$,

$$\tilde W_{11}(z,\bar z) = \mathrm{I} + i(A_W(\omega_0))_{11}(z - \omega_0) + O(|z - \omega_0|^2) \tag{10.48}$$

with $(A_W(\omega_0))_{11} < 0$. Thus,

$$\tilde W(z,\bar z) - \mathrm{I} = \begin{pmatrix} i(z-\omega_0)(A_W(\omega_0))_{11} + O(|z-\omega_0|^2) & O(|z-\omega_0|) \\ O(|z-\omega_0|) & \tilde W_{22}(z,\bar z) - \mathrm{I} \end{pmatrix}. \tag{10.49}$$

The Schur complement formula then implies

$$\begin{aligned} &\det(\tilde W(z,\bar z) - \mathrm{I}) = \det(\tilde W_{22}(z,\bar z) - \mathrm{I}) \det\Big(i(z-\omega_0)(A_W(\omega_0))_{11} + O(|z-\omega_0|^2) \\ &- O(|z-\omega_0|)(\tilde W_{22}(z,\bar z) - \mathrm{I})^{-1}O(|z-\omega_0|)\Big) = e_1(z,\bar z)(z-\omega_0)^m \det((A_W(\omega)_0)_{11} + O(|z-\omega_0|)) \end{aligned} \tag{10.50}$$

from which the lemma follows. □

For $\epsilon > 0$, we set up the target spectral scale

$$\gamma_{\mathcal{P},M}(\omega) := \omega^{-\min(\frac{2\pi}{\alpha_+}, 1+k_0)+\epsilon} + M^2\omega^{-2+\epsilon}, \tag{10.51}$$

abbreviated $\gamma(\omega)$ when there is no confusion.

Lemma 10.5 (Counting zeros finely)**.** *Let $r > 0$, $\delta > 0$, $\epsilon > 0$. There is $\omega_0 > 0$ (depending on r, ϵ, δ) such that for any $\omega \ge \omega_0$ and any interval $[a,b] \subset [\frac{1}{2}\omega, \infty)$ such that*

$$\operatorname{dist}\big(\{a,b\}, \mathcal{Z}^{\mathscr{D}}\big) > r\gamma(\omega) = r\big(\omega^{-\min(\frac{2\pi}{\alpha_+}, 1+k_0)+\epsilon} + M^2\omega^{-2+\epsilon}\big) \tag{10.52}$$

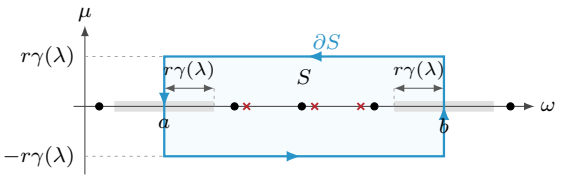


FIGURE 14. Contour integral used in Lemma 10.5. Zeros of $\mathcal{Q}^{\mathscr{D}}_\omega$ and $\mathcal{Q}^{\Lambda}_\omega$ are marked as black dots and red crosses respectively.

we have

$$\mathcal{N}^{\mathscr{D}}_{[a,b]} = \sum_{\omega\in[a,b]} \dim\ker \mathcal{Q}^{\mathscr{D}}_\omega = \sum_{\omega\in[a,b]} \dim\ker \mathcal{Q}^{\Lambda}_\omega = \mathcal{N}^{\Lambda}_{[a,b]}, \tag{10.53}$$

where $\mathcal{Q}^{\Lambda}_z = E_{-+}(z)$ *for* $z\in\mathbb{C}$ *from Proposition 9.1.*

Proof. Recall that for $\omega_0 > 0$ large enough $\mathcal{Q}^{\Lambda}_z$ is analytic in z in $[\omega_0,\infty)+i[-r_1,r_1]$ for some $r_1>0$. Moreover, by Corollary 9.5, for $\beta\geq 0$,

$$\|\partial^\beta_\omega(\mathcal{Q}^{\Lambda}_\omega + T_\omega)(\omega)\| \leq C|\omega|^{-2+0}. \tag{10.54}$$

Recall also the smooth extension (10.40) of $\mathcal{Q}^{\mathscr{D}}_\omega$.

Observe that

$$\tilde{\mathcal{Q}}^{\mathscr{D}}_{\omega+i\mu} = \Sigma^{k_0}(\omega)(\mathrm{I}-W(\omega)(\mathrm{I}-\mu A_W(\omega))) + i\mu\partial_\omega\Sigma^{k_0}(\omega)(\mathrm{I}-W(\omega)). \tag{10.55}$$

Note that the second term is of size $o(|\mu|)$ as $\omega\to\infty$. Since $A_W(\mu)\leq -\frac{L_-}{2}\mathrm{I}$, we have either

$$\mathrm{I}-\mu A_W(\omega) \geq (1+\frac{L_-\mu}{2})\mathrm{I} \text{ for } \mu>0, \text{ or } \mathrm{I}-\mu A_W(\omega)\leq (1-\frac{L_-|\mu|}{2})\mathrm{I} \text{ for } \mu<0. \tag{10.56}$$

Since $W(\omega)$ is unitary, this implies for large enough ω_0,

$$\|(\tilde{\mathcal{Q}}^{\mathscr{D}}_{\omega+i\mu})^{-1}\| \leq \frac{4}{L_-|\mu|}, \quad \omega+i\mu\in[\omega_0,\infty)+i[r_1,r_1] \tag{10.57}$$

Let the box $S := [a,b]+i[-r,r]\gamma(\omega)$, where $\gamma(\omega)$ was defined in (10.51). See Figure 14. Let $z=\omega+i\mu\in\overline{S}$. By (10.37), (10.52), (10.57), and the fact that $\|\nabla\tilde{\mathcal{Q}}^{\mathscr{D}}_{\omega+i\mu}\|\leq C$ we have

$$\|(\tilde{\mathcal{Q}}^{\mathscr{D}}_z)^{-1}|_{\partial S}\| \leq Cr^{-1}\gamma(\omega)^{-1} \tag{10.58}$$

In addition, for $\omega\in[a,b]$ we have

$$\|\mathcal{Q}^{\Lambda}_\omega - \mathcal{Q}^{\mathscr{D}}_\omega\| \leq C\gamma(\omega)\omega^{-\frac{\epsilon}{2}}, \tag{10.59}$$

and by (10.28) and (10.54) we have

$$\|\partial_\omega(\mathcal{Q}^{\Lambda}_\omega - \mathcal{Q}^{\mathscr{D}}_\omega)\| \leq C\omega^{-\frac{1}{2}+0}. \tag{10.60}$$

Hence, for $z=\omega+i\mu\in\overline{S}$, we have

$$\|\mathcal{Q}^{\Lambda}_{\omega+i\mu} - \tilde{\mathcal{Q}}^{\mathscr{D}}_{\omega+i\mu}\| \leq C\gamma(\omega)\omega^{-\frac{\epsilon}{2}} + C\omega^{-\frac{1}{2}+\epsilon}|\mu|. \tag{10.61}$$

Therefore, for $z \in \partial S$,

$$\begin{aligned}\det \mathcal{Q}_z^\Lambda = \det \tilde{\mathcal{Q}}_z^{\mathscr{D}} \det(I + (\tilde{\mathcal{Q}}_z^{\mathscr{D}})^{-1}(\mathcal{Q}_z^\Lambda - \tilde{\mathcal{Q}}_z^{\mathscr{D}})) &= \det \tilde{\mathcal{Q}}_z^{\mathscr{D}} \det(I + (\tilde{\mathcal{Q}}_z^{\mathscr{D}})^{-1} O(\gamma(\omega)\omega^{-\frac{\epsilon}{2}})) \\ &= \det \tilde{\mathcal{Q}}_z^{\mathscr{D}} \det(I + O(\omega^{-\frac{\epsilon}{2}})) = \det \tilde{\mathcal{Q}}_z^{\mathscr{D}} (1 + O(\omega^{-\frac{\epsilon}{2}})). \quad (10.62)\end{aligned}$$

Now, by homotopy invariance of the winding number [Hat02, p. 134 property (c)], this implies

$$\operatorname{wind}_{\partial S} \det \mathcal{Q}_z^\Lambda = \operatorname{wind}_{\partial S} \det \tilde{\mathcal{Q}}_z^{\mathscr{D}} = \sum_{\omega \in [a,b]} \dim \ker \mathcal{Q}_\omega^{\mathscr{D}} = \mathcal{N}_{[a,b]}^{\mathscr{D}}, \quad (10.63)$$

where $\operatorname{wind}_{\partial S}$ denotes the winding number along ∂S and the second equality follows from Lemma 10.4 and (10.57) implies that all zeros of $\det \tilde{\mathcal{Q}}_z^{\mathscr{D}}$ are real.

Finally, the fact that $\|(\mathcal{Q}_z^\Lambda)^{-1}\| \le C|\Im z|^{-1}$ implies that $\mathcal{Q}_z^\Lambda$ has no Jordan blocks with eigenvalue 0 and that all zeros of $\det \mathcal{Q}_z^\Lambda$ are real and hence that

$$\operatorname{wind}_{\partial S} \det \mathcal{Q}_z^\Lambda = \#\{\text{zeros of } \det \mathcal{Q}_z^\Lambda \text{ in } \Omega\} = \sum_{\omega \in [a,b]} \dim \ker \mathcal{Q}_\omega^\Lambda = \mathcal{N}_{[a,b]}^\Lambda, \quad (10.64)$$

since $\det \mathcal{Q}_z^\Lambda$ is analytic in S. This completes the proof. □

An almost identical proof yields the following lemma.

Lemma 10.6 (Counting zeros loosely but uniformly in δ)**.** *Let $r > 0$, $\epsilon > 0$. There is $\omega_0 > 0$ (depending on r, ϵ) such that for any $0 < \delta < 1$, any $\omega \ge \omega_0$ and any interval $[a, b] \subset [\frac{1}{2}\omega, \infty)$ such that*

$$\operatorname{dist}\left(\{a, b\}, \mathcal{Z}^{\mathscr{D}}\right) > r\omega^{-1+\epsilon} \quad (10.65)$$

we have

$$\mathcal{N}_{[a,b]}^{\mathscr{D}} = \sum_{\omega \in [a,b]} \dim \ker \mathcal{Q}_\omega^{\mathscr{D}} = \sum_{\omega \in [a,b]} \dim \ker \mathcal{Q}_\omega^\Lambda = \mathcal{N}_{[a,b]}^\Lambda, \quad (10.66)$$

We emphasize that the key to uniformity in δ is that $(\Sigma^{k_0}(\omega) - \Sigma^0) = O(\omega^{-1+\epsilon})$ while omitting the proof.

10.3. **Proof of Theorem 1.3.**

Proof of Theorem 1.3. 1. We remind the reader that our goal is to show first that, for each $\epsilon > 0$, there is $\omega_0 > 0$ such that eigenvalues in $\operatorname{Spec}(\Lambda^{\mathcal{P}}) \cap [\omega_0, \infty)$ and $\operatorname{Spec}(\mathscr{D}^{k_0}) \cap [\omega_0, \infty)$ are related pairwise, and each pair is separated by small distance. Specifically, in Steps 2–5, we show pairing happens at the scale

$$\gamma_{\mathcal{P},M}(\omega) = \omega^{-\min(\frac{2\pi}{\alpha_+}, 1+k_0)+\epsilon} + M^2\omega^{-2+\epsilon}, \quad (10.67)$$

defined in (10.51), and then in Step 6 we show that the pairs of labeling of each pair agrees. To do the pairing, we will show $[\omega_0, \infty)$ can be written as a union of intervals of length $\sim \gamma(\omega)$, so that the numbers of eigenvalues must agree in each such interval.

2. We show for each $r \le \frac{1}{4}$, there is $\omega_0 > 0$ such that for any $\omega \ge \omega_0$ with $\det \mathcal{Q}_\omega^\Lambda = 0$, we have

$$\mathcal{Z}^{\mathscr{D}} \cap [\omega - 4r\gamma(\omega), \omega + 4r\gamma(\omega)] = \{\omega' \in [\omega - 4r\gamma(\omega), \omega + 4r\gamma(\omega)] : \det \mathcal{Q}_{\omega'}^{\mathscr{D}} = 0\} \neq \emptyset. \quad (10.68)$$

Assume this is not true, then $\operatorname{dist}(\{\omega \pm 2r\gamma(\omega)\}, \mathcal{Z}^{\mathscr{D}}) \geq 2r\gamma(\omega)$, and the fine counting Lemma 10.5 implies

$$\mathcal{N}^{\mathscr{D}}_{[\omega-2r\gamma(\omega),\omega+2r\gamma(\omega)]} = \mathcal{N}^{\Lambda}_{[\omega-2r\gamma(\omega),\omega+2r\gamma(\omega)]} \geq 1, \tag{10.69}$$

yielding a contradiction. This argument specifically implies that if (10.68) is empty, then

$$\{\omega \in [\omega - r\gamma(\omega), \omega + r\gamma(\omega)] : \det \mathcal{Q}^{\Lambda}_{\omega} = 0\} = \emptyset. \tag{10.70}$$

3. By Lemma 10.2 there are at most $2J$ zeros of $\det \mathcal{Q}^{\mathscr{D}}_{\omega}$ within intervals of length $2r_0$. Fix $r = \frac{1}{64(8J+1)}$, there is ω_0 that allows the use of Step 2. Without loss of generality, let $\gamma(\omega_0) \leq \frac{1}{4r_0}$. We claim for each $\omega > \omega_0$, there exists two intervals $I_{\pm}$ with center $a_{\pm}$ that

$$I_{\pm} = \bar{B}(a_{\pm}, \frac{\gamma(\omega)}{32(8J+1)}) \cap \mathbb{R},\ I_{-} \subset [\omega - \gamma(\omega), \omega - \frac{3}{4}\gamma(\omega)],\ I_{+} \subset [\omega + \frac{3}{4}\gamma(\omega), \omega + \gamma(\omega)], \tag{10.71}$$

such that $I_{\pm} \cap (\mathcal{Z}^{\mathscr{D}} \cup \mathcal{Z}^{\Lambda}) = \emptyset$. Indeed, the $2J$ upper bound on the zeros of $\det \mathcal{Q}^{\mathscr{D}}_{\omega}$ within

$$[\omega - \gamma(\omega), \omega - \frac{3}{4}\gamma(\omega)] = \cup_{j=0}^{8J} I_j,\ I_j = [\omega - (1 - \frac{j}{4(8J+1)})\gamma(\omega), \omega - (1 - \frac{j+1}{4(8J+1)})\gamma(\omega)]. \tag{10.72}$$

Thus there must be one of those intervals centered at some a_{-} that contains no zeros of $\det \mathcal{Q}^{\mathscr{D}}_{\omega}$, and Step 2 implies $I_{-} = \bar{B}(a_{\pm}, \frac{1}{32(8J+1)}\gamma(\omega)) \cap \mathbb{R}$ contains neither zeros of $\det \mathcal{Q}^{\mathscr{D}}_{\omega}$ nor zeros of $\det \mathcal{Q}^{\Lambda}_{\omega}$. Similar argument gives I_{+}.

4. We show $\det \mathcal{Q}^{\mathscr{D}}_{\omega}$ and $\det \mathcal{Q}^{\Lambda}_{\omega}$ have the same number of zeros in $\omega \in [a_{-}, a_{+}]$, where

$$\omega - \gamma(\omega) \leq a_{-} \leq \omega - \frac{3}{4}\gamma(\omega) \leq \omega + \frac{3}{4}\gamma(\omega) \leq a_{+} \leq \omega + \gamma(\omega). \tag{10.73}$$

Note that $I_{\pm} \cap \mathcal{Z}^{\mathscr{D}} = \emptyset$ implies $\operatorname{dist}\left(\{a_{-}, a_{+}\}, \mathcal{Z}^{\mathscr{D}}\right) > \frac{\gamma(\omega)}{64(8J+1)}$. Apply the fine counting Lemma 10.5 to see

$$\mathcal{N}^{\mathscr{D}}_{[a_{-},a_{+}]} = \mathcal{N}^{\Lambda}_{[a_{-},a_{+}]}. \tag{10.74}$$

5. Now we show $\mathcal{Z}^{\mathscr{D}}$ and $\mathcal{Z}^{\Lambda}$ have one-to-one correspondence after large $\omega \geq \omega_0$. Start with ω_0, where $\mathcal{N}^{\mathscr{D}}_{[a_{-},a_{+}]} = \mathcal{N}^{\Lambda}_{[a_{-},a_{+}]}$. Let $\omega_1 > \omega_0$ such that

$$\omega_0 + \frac{3}{4}\gamma(\omega_0) \leq \omega_1 - (1 - \frac{1}{32(8J+1)})\gamma(\omega_0) \leq a_{+} \leq \omega_1 - (\frac{3}{4} + \frac{1}{32(8J+1)})\gamma(\omega_0). \tag{10.75}$$

Note $\operatorname{dist}\left(a_{+}, \mathcal{Z}^{\mathscr{D}}\right) > \frac{\gamma(\omega_1)}{32(8J+1)}$ satisfies the requirement in Step 3. Thus we can repeat Step 3 and get some $a'_{+} \in [\omega_1 + \frac{3}{4}\gamma(\omega_1), \omega_1 + \gamma(\omega_1)]$ where we still have

$$\mathcal{N}^{\mathscr{D}}_{[a_{+},a'_{+}]} = \mathcal{N}^{\Lambda}_{[a_{+},a'_{+}]}. \tag{10.76}$$

We can iterate this process to cover $\mathbb{R}_{\geq \omega_0}$.

6. We now show that the labels of the eigenvalues for the Dirac operator and the Steklov problem agree. For a curvilinear polygon $\mathcal{P} \subset \mathbb{R}^2$ and $M \geq 0$, we denote the Steklov eigenvalues by

$$0 = \sigma_0(\mathcal{P}, M) \leq \sigma_1(\mathcal{P}, M) \leq \sigma_2(\mathcal{P}, M) \leq \cdots, \tag{10.77}$$

and the eigenvalues of $\mathscr{D}$ by

$$\cdots \lambda_{-2}(\mathcal{P}) \leq \lambda_{-1}(\mathcal{P}) \leq 0 \leq \lambda_0(\mathcal{P}) \leq \lambda_1(\mathcal{P}) \leq \lambda_2(\mathcal{P}) \leq \cdots. \tag{10.78}$$

By the symmetry of $\mathcal{Q}_\omega^{\mathscr{D}}$, we may choose λ_n so that $\lambda_n(\mathcal{P}) = -\lambda_{-n-1}(\mathcal{P})$ for $n \in \mathbb{N}_0$. Let $M_0 > 0$, $J \geq 1$, $\{C_j\}_{j=0}^\infty$ be positive constants.

Using the argument above in the previous steps, but appealing to Lemma 10.6 instead of Lemma 10.5, there is $\omega_0 > 0$ such that for all $0 < \delta < 1$, all curvilinear polygons $\mathcal{P}$ with J vertices with angles $\alpha_j \in (\epsilon, 2\pi - \epsilon)$ for all $1 \leq j \leq J$, and with edges

$$\epsilon \leq |\Gamma_j| \leq \epsilon^{-1}, \ \|\Gamma_j\|_{C^k} \leq C_k, \ k \in \mathbb{N}_0, \tag{10.79}$$

and all $M \in [0, M_0]$ there is $N = N(\mathcal{P}, M)$ such that for all n with $\sigma_n(\mathcal{P}, M) > \omega_0$,

$$|\lambda_{n+N}(\mathcal{P}) - \sigma_n(\mathcal{P}, M)| \leq C\sigma_n(\mathcal{P}, M)^{-1+\epsilon}. \tag{10.80}$$

Now, let $\mathscr{D}^0$ denote the basic Steklov Dirac operator on $\mathcal{P}$ and let $\delta < \frac{1}{2}\omega_0^{-1}$. Then, if $n \geq 0$, $\lambda_n(\mathscr{D}^0) \leq 2\omega_0$ we have $\lambda_n(\mathscr{D}^0) = \lambda_n(\mathscr{D})$. In particular, the value of N in (10.80) does not depend on k_0. Thus, from now on, we argue with $k_0 = 0$.

6a. We show that $N(\mathcal{P}, M)$ in (10.80) is unique. Indeed, suppose that (10.80) holds with both N and N' instead of N with $N \neq N'$ for $n = n_0, n_0 + 1, \dots n_0 + 2J$. Suppose without loss of generality that $N' > N$. Then for all $0 \leq k \leq 2J + 1$ and n_0 large enough,

$$\begin{aligned} |\lambda_{n_0+k(N'-N)}(\mathcal{P}) - \lambda_{n_0}(\mathcal{P})| \leq \sum_{m=0}^{k-1} |\lambda_{n_0+m(N'-N)}(\mathcal{P}) - \sigma_{n_0+m(N'-N)-N}(\mathcal{P})| \\ + \sum_{m=0}^{k-1} |\sigma_{n_0+m(N'-N)-N}(\mathcal{P}) - \lambda_{n_0+(m+1)(N'-N)}(\mathcal{P})| \leq Ck\sigma_{n_0-N}^{-1+\epsilon}. \end{aligned} \tag{10.81}$$

Let $k = 2J + 1$. In particular since $\sigma_j \to \infty$, this implies that for any $\epsilon > 0$ there is large $C > 0$ such that

$$\#\{\lambda_j \in [C, C + \epsilon]\} \geq k = 2J + 1, \tag{10.82}$$

which contradicts Lemma 10.2. In particular, for any n_0 large enough, there is a unique N such that (10.80) holds for $n \in \{n_0, n_0 + 1, \dots, n_0 + 2J\}$.

6b. We next claim that if $\partial\mathcal{P}$ is smooth, $M = 0$, and $\partial\mathcal{P}$ is flat near the (fictitious) corners, then $N(\mathcal{P}, 0) = 0$. Let $\partial\mathcal{P}$ be smooth, flat near (fictitious) corners with connected components of lengths $\{L^1, \dots, L^L\}$. One can check that the Dirac operator orthogonally decomposes into $\mathscr{D} = \oplus_{l=1}^L \mathscr{D}^l$ where $\mathscr{D}^l$ is the Dirac operator on the l-th connected component of the boundary. Then since

$$\Sigma_{0,1}^{W_\pi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \lambda_{2n}(\mathscr{D}^l) = \lambda_{2n-1}(\mathscr{D}^l) = \frac{2\pi n}{L^l}, \ n \geq 0. \tag{10.83}$$

Then, [GPPS14, Theorem 1.4] (see also Rozenblum [Roz86] for the simply connected version) proves the claim.

6c. To prove that $N(\mathcal{P}, M) = 0$ for all M and curvilinear polygons $\mathcal{P}$, we use a deformation argument. Fix $M > 0$ and $\mathcal{P}$ a curvilinear polygon with J vertices with angles $\alpha_j \in (2\epsilon, 2\pi - 2\epsilon)$ for all $1 \leq j \leq J$, and with edges

$$2\epsilon \leq |\Gamma_j| \leq \frac{1}{2}\epsilon^{-1}, \ \|\Gamma_j\|_{C^k} \leq \frac{1}{2}C_k, \ k \in \mathbb{N}_0. \tag{10.84}$$

Then there is a continuous family of curvilinear polygons $t \mapsto \mathcal{P}_t$ such that $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_1$ has smooth boundary and is flat near the corners x_j, such that $\mathcal{P}_t$ has the same set of edges and vertices, with angles

$$\epsilon < \inf_{t\in[0,1]} (\alpha_j(\mathcal{P}_t)) \leq \sup_{t\in[0,1]} (\alpha_j(\mathcal{P}_t)) < 2\pi - \epsilon, \tag{10.85}$$

and

$$\epsilon \leq \inf_{t\in[0,1]} |\Gamma_j(\mathcal{P}_t)| \leq \sup_{t\in[0,1]} |\Gamma_j(\mathcal{P}_t)| \leq \epsilon^{-1}, \ \sup_{t\in[0,1]} \|\Gamma_j(\mathcal{P}_t)\|_{C^k} \leq C_k, \tag{10.86}$$

for each $1 \leq j \leq J$. In particular, there is $\lambda_0 > 0$ such that for all $t, t' \in [0,1]$, there is a unique $N_{t,t'} \in \mathbb{Z}$ such that for $\sigma_n(\mathcal{P}_t, (1-t')M) > \lambda_0$,

$$|\lambda_{n+N_{t,t'}}(\mathcal{P}_t) - \sigma_n(\mathcal{P}_t, (1-t')M)| < Cr_{\mathcal{P}_t,(1-t')M}(\sigma_n(\mathcal{P}_t, (1-t')M)). \tag{10.87}$$

Now, observe that since $\Sigma_{0,1}^{\mathcal{W}_{\alpha_t}}$ is continuous in t, $t \mapsto \lambda_n(\mathcal{P}_t)$ is continuous. Moreover, the variational characterization of $\sigma_n(\mathcal{P}_t, (1-t')M)$ implies that $t' \mapsto \sigma_n(\mathcal{P}_t, (1-t')M)$ is continuous and, by [BGT20, Theorem 4.1], $t \mapsto \sigma_n(\mathcal{P}_t, 0)$ is also continuous.

6d. We show that $N_{0,1} = 0$. Since $\mathcal{P}_1$ is smooth, we have $N_{1,1} = 0$. Let

$$t_0 = \inf\{t \in [0,1] \,:\, N_{s,1} = 0,\, s \in [t,1]\}. \tag{10.88}$$

If $t_0 = 0$, then by continuity of $t \mapsto \lambda_n(\mathcal{P}_t)$ and $t \mapsto \sigma_n(\mathcal{P}_t, 0)$, $N_{0,1} = 0$. Assume instead $t_0 > 0$. Then, by construction there are $s_k \to t_0 - 0$ and $t_k \to t_0 + 0$ such that

$$N_{s_k,1} \neq 0,\ N_{t,1} = 0,\ t \in [t_k, 1]. \tag{10.89}$$

Hence, using continuity of $t \mapsto \lambda_n(\mathcal{P}_t)$ and $t \mapsto \sigma_n(\mathcal{P}_t, 0)$ again, together with the fact that $N_{t_0,1}$ is determined by finitely many values of λ_n and σ_n, $N_{t_0,1} = \inf_{t>t_0} N_{t,1} = 0$ and $N_{t_0,1} = \lim_{k\to\infty} N_{s_k,1} \neq 0$, yielding the desired contradiction. Note that the limit $\lim_{k\to\infty} N_{s_k,1}$ exists since $N_{s_k,1}$ is integer valued.

6e. Now, we conclude with showing that $N_{0,0} = 0$. To do this, we use that $N_{0,1} = 0$. Let

$$t_0' = \inf\{t' \in [0,1] \,:\, N_{0,s} = 0,\, s \in [t',1]\}. \tag{10.90}$$

If $t_0' = 0$, then by continuity of $s \mapsto \sigma_n(\mathcal{P}_0, (1-s)M)$, $N_{0,0} = 0$. Assume instead $t_0' > 0$. Then, by construction there are $s_k \to t_0' - 0$ and $t_k' \to t_0' + 0$ such that

$$N_{0,s_k} \neq 0,\ N_{0,t'} = 0,\ t' \in [t_k', 1]. \tag{10.91}$$

Hence, using continuity of $s \mapsto \sigma_n(\mathcal{P}_0, (1-s)M)$, $N_{0,t_0'} = \inf_{t'>t_0'} N_{0,t'} = 0$ and $N_{0,t_0'} = \lim_{k\to\infty} N_{0,s_k} \neq 0$, yielding the desired contradiction. □

Part 3. Appendices

Appendix A. A second-microlocal pseudodifferential calculus

Given smooth scattering manifold $\partial\Omega$, consider its half-space collar $\Omega_+ = \{(x_0, x) : x_0 \in [0,\infty), x \in \partial\Omega\}$. We denote $\bar{x} = (x_0, x)$. We define $(T^*\partial\Omega)_+ = \{(x_0, x, \xi) : x_0 \in [0,\infty), (x,\xi) \in T^*\partial\Omega\}$ similarly. For $\delta \in [0,1)$ we consider the following symbol class:

Definition A.1 (Symbols in $S^{s,l}_{k,\delta}((T^*\partial\Omega)_+)$)**.** Let $M_0 > 0$. Let $h_0 > 0$ small. We say $a(x_0, x, \xi; M, h) \in C^\infty([0,\infty)_{x_0} \times \partial\Omega_x \times \mathbb{R}^{d-1}_\xi \times [0, M_0]_M \times (0, h_0]_h)$ is in $S^{s,l}_{k,\delta}((T^*\partial\Omega)_+)$, if for every multi-indices α, β we have

$$\left|(\langle x\rangle\partial_{\bar{x}})^\alpha(\langle x\rangle^{-\delta}\langle\langle x\rangle^\delta\sqrt{|\xi|^2+M^2}\rangle\partial_\xi)^\beta a(x_0, x, \xi; M, h)\right| \le C_{\alpha\beta}\langle\xi\rangle^s\langle x\rangle^l\langle\langle x\rangle^\delta\sqrt{|\xi|^2+M^2}\rangle^k, \quad \text{(A.1)}$$

uniformly for all $x_0 \in [0,\infty)$, $x \in \partial\Omega$, $\xi \in \mathbb{R}^{d-1}$, $h \in (0, h_0]$ and $M \in [0, M_0]$.

For brevity, we may omit the dependency on M, h whenever there is no confusion. Our goal in this section is to build a pseudodifferential calculus on Ω_+, associated with the symbol class $S^{s,l}_{l,\delta}((T^*\partial\Omega)_+)$. To do it we need to look at the half-space case $\Omega_+ = \mathbb{R}^d_+ = \{(x_0, x) : x_0 \in [0,\infty), x \in \mathbb{R}^{d-1}\}$ with $\partial\Omega = \partial\mathbb{R}^d_+ = \mathbb{R}^{d-1}$, $(T^*\partial\Omega)_+ = \mathbb{R}_{x_0} \times T^*\mathbb{R}^{d-1}$, and then later transplant them to a general Ω_+.

Definition A.2 (A second-microlocal calculus on $\mathbb{R}^d_+$)**.** Let $a(x_0, x, \xi) \in S^{s,l}_{k,\delta}((T^*\mathbb{R}^{d-1})_+)$. We define its left quantization as an operator bounded from $\mathcal{S}(\mathbb{R}^d_+)$ to $\mathcal{S}'(\mathbb{R}^d_+)$ whose kernel is given by

$$\mathrm{Op}_h(a)(x_0, x; y_0, y) = \frac{\delta(y_0 - x_0)}{(2\pi h)^{d-1}}\int_{\mathbb{R}^{d-1}_\xi} e^{\frac{i}{h}\langle x-y,\xi\rangle}a(x_0, x, \xi)\, d\xi. \quad \text{(A.2)}$$

We define the residual class by $h^\infty\Psi^{-\infty,-\infty}_{-\infty,\delta}(\mathbb{R}^d_+) = h^\infty\Psi^{-\infty,-\infty}(\mathbb{R}^d_+)$, consisting of all operators with kernel in $h^\infty\delta(x_0-y_0)C^\infty_b([0,\infty)_{x_0}; \mathcal{S}(\mathbb{R}^{d-1}_x\times\mathbb{R}^{d-1}_y)$. We define $\Psi^{s,l}_{k,\delta}(\mathbb{R}^d_+) = \mathrm{Op}_h(S^{s,l}_{k,\delta}((T^*\mathbb{R}^{d-1})_+))$.

A.1. **Change of coordinates.** Now we study the invariance of the calculus under change of coordinates. We start from proving the invariance of $S^{s,l}_{k,\delta}((T^*\mathbb{R}^{d-1})_+)$.

Lemma A.3 (Change of variables for symbols)**.** *Let $\Phi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ be a diffeomorphism that*

$$|\partial^\alpha\Phi(x)| \le C_\alpha\langle x\rangle^{1-|\alpha|}, \ |\partial^\alpha(\Phi^{-1}(x'))| \le C_\alpha\langle x'\rangle^{1-|\alpha|} \quad \text{(A.3)}$$

for every multi-index α. Then, for any $a(x_0, x, \xi) \in S^{s,l}_{k,\delta}((T^*\mathbb{R}^{d-1})_+)$, we have

$$(\Phi^*a)(x_0, x', \xi') = a(x_0, \Phi^{-1}(x'), D\Phi|^T_{\Phi^{-1}(x')}\xi') \in S^{s,l}_{k,\delta}((T^*\mathbb{R}^{d-1})_+). \quad \text{(A.4)}$$

Proof. We follow the approach of [Zwo12, Theorem 9.4]. Let $x = \Phi^{-1}(x')$ with $\xi = D\Phi|^T_{\Phi^{-1}(x')}\xi'$ given by the symplectomorphism induced by Φ. Compute

$$\nabla_{x'} = D\Phi|^{-T}_x(\nabla_x + ((\nabla_x(D\Phi|^T_x))\xi')^T\nabla_\xi) = O(S^0)\nabla_x + O(S^{-1})|\xi|\nabla_\xi, \ \nabla_{\xi'} = D\Phi|_x\nabla_\xi = O(S^0)\nabla_\xi. \quad \text{(A.5)}$$

Then iteratively for all multi-indices α, β and $j \ge 0$ we have

$$\begin{aligned}\partial^j_{x_0}\partial^\alpha_{x'}\partial^\beta_{\xi'}(\Phi^*a)(x_0, x', \xi') &= \sum_{\alpha',\beta'} O(S^{|\alpha'|-|\alpha|})|\xi|^{|\beta'|-|\beta|}(\partial^j_{x_0}\partial^{\alpha'}_x\partial^{\beta'}_\xi a)(x_0, x, \xi)\\ &= O(\langle\xi\rangle^s\langle x\rangle^{l-j-|\alpha|+\delta|\beta|}\langle\langle x\rangle^\delta\sqrt{|\xi|^2+M^2}\rangle^{k-|\beta|}), \quad \text{(A.6)}\end{aligned}$$

summing over $|\alpha'| \le |\alpha|$, $|\beta'| \ge |\beta|$ and $|\beta'|+|\alpha'| \le |\beta|+|\alpha|$. Note here we used $\langle\langle x\rangle^\delta\sqrt{|\xi|^2+M^2}\rangle \ge \langle x\rangle^\delta|\xi|$. Note $\langle x\rangle \sim \langle x'\rangle$, $|\xi| \sim |\xi'|$ to conclude the proof. □

Lemma A.4. *Let $\Phi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ be a diffeomorphism that*

$$|\partial^\alpha \Phi(x)| \leq C_\alpha \langle x\rangle^{1-|\alpha|},\ |\partial^\alpha(\Phi^{-1}(x'))| \leq C_\alpha \langle x'\rangle^{1-|\alpha|} \tag{A.7}$$

for every multi-index α. Then for all $a \in S^{s,l}_{k,\delta}((T^*\mathbb{R}^{d-1})_+)$, there exists $a' \in S^{s,l}_{k,\delta}((T^*\mathbb{R}^{d-1})_+)$ such that $\operatorname{Op}_h(a') = \Phi^{-*}\operatorname{Op}_h(a)\Phi^*$ and

$$a'(x_0, x', \xi') = a(x_0, \Phi^{-1}(x'), D\Phi|^T_{\Phi^{-1}(x')}\xi') + O(hS^{s,l-1+\delta}_{k-1,\delta}((T^*\mathbb{R}^{d-1})_+)). \tag{A.8}$$

Proof. 1. Let $x' = \phi(x)$ and $y' = \phi(y)$. We follow the approach in the proof of [Zwo12, Theorem 9.3]. Assume for now $a \in \mathcal{S}$.

$$\operatorname{Op}_h(a)(u) = \int K_a(\bar{x}, \bar{y})u(\bar{y})d\bar{y},$$

with $K_a(\bar{x}, \bar{y}) := \frac{\delta(x_0-y_0)}{(2h\pi)^{d-1}} \int a(\bar{x}, \xi)e^{\frac{i}{h}\langle x-y,\xi\rangle}\, d\xi$. Notice that the tangential kernel

$$k_a := \frac{1}{(2h\pi)^{d-1}} \int a(\bar{x}, \xi)e^{\frac{i}{h}\langle x-y,\xi\rangle}\, d\xi \in \mathcal{S}$$

since $a \in \mathcal{S}$. Moreover, the tangential kernel of $(\Phi^{-1})^* \operatorname{Op}_h(a)\Phi^* = \operatorname{Op}_h(\tilde{a})$ is

$$\tilde{k}(x_0, x', y') = k_a(x_0, \Phi^{-1}(x'), \Phi^{-1}(y'))|\det \partial(\Phi^{-1})(y')|.$$

One can show by a direct estimate using the hypothesis on the growth of Φ and Φ^{-1} that $\tilde{k} \in \mathcal{S}$ Hence, by Fourier inversion there is $\tilde{a} \in \mathcal{S}$ such that $(\Phi^{-1})^* \operatorname{Op}_h(a)\Phi^* = \operatorname{Op}_h(\tilde{a})$. Thus,

$$\operatorname{Op}_h(\tilde{a})(u') = \int K_{\tilde{a}}(\bar{x}', \bar{y}')u(\bar{y}')d\bar{y}',$$

with $K_{\tilde{a}}(\bar{x}', \bar{y}') := \frac{\delta(x_0-y_0)}{(2h\pi)^{d-1}} \int \tilde{a}(\bar{x}', \xi')e^{\frac{i}{h}\langle x'-y',\xi'\rangle}\, d\xi'$. Namely,

$$K_a(x_0, x, y) = K_{\tilde{a}}(x_0, \Phi(x), \Phi(y))|\det \partial\Phi(y)|.$$

Now,

$$\begin{aligned}\int K_{\tilde{a}}(x', y')u(\bar{y}')d\bar{y}' &= \int K_{\tilde{a}}(x_0, \Phi(x), y_0, \Phi(y))u(y_0, \Phi(y))dy_0 d\Phi(y) \\ &= \int \tilde{a}(x_0, \Phi(x), \xi')e^{\frac{i}{h}\langle \Phi(x)-\Phi(y),\xi'\rangle}u(x_0, \Phi(y))|\det \partial\Phi(y)|\, dyd\xi'.\end{aligned}$$

2. We can assume that $|x-y|$ is small. In fact, a nonstationary phase argument in the integral of $K_{\tilde{a}}$ shows that for any fixed $\epsilon > 0$, the component for $|x-y| > \epsilon$ is $O_{\mathcal{S}'\to\mathcal{S}}(h^\infty)$. Hence, by Taylor expanding $\Phi(y)$ and $\Phi(x)$ at the point $\frac{x+y}{2}$,

$$\Phi(x) - \Phi(y) = (\partial\Phi)\Big(\frac{x+y}{2}\Big)(x-y) + O(\langle x\rangle^{-1}|x-y|^2) = G(x, y)(x-y),$$

with $G(x, y)$ invertible for all $|x-y| \leq \epsilon$ (ϵ small) since $G(x, y) = \partial\Phi(x) + O(\langle x\rangle^{-1}|x-y|) = \partial\Phi(x) + O(\epsilon)$, and $\partial\Phi(x)^{-1}$ is uniformly bounded. Denoting $\xi = G(x, y)^T\xi'$,

$$\begin{aligned}
&\int K_{\tilde a}(x',y')u(y')dy' \\
&= \frac{\delta(x_0-y_0)}{(2h\pi)^{d-1}}\int \tilde a(x_0,\Phi(x),(G(x,y)^T)^{-1}\xi)e^{\frac{i}{h}\langle x-y,\xi\rangle}u(x_0,\Phi(y))\,|\det G(x,y)|^{-1}d\xi|\det\partial\Phi(y)|dy \\
&= \frac{\delta(x_0-y_0)}{(2h\pi)^{d-1}}\int \tilde a(x_0,\Phi(x),(G(x,y)^T)^{-1}\xi)e^{\frac{i}{h}\langle x-y,\xi\rangle}u(x_0,\Phi(y))\,|\det G(x,y)|^{-1}d\xi|\det(\partial\Phi)(y)|dy \\
&= \frac{\delta(x_0-y_0)}{(2h\pi)^{d-1}}\int \Big(\tilde a\Big(x_0,\Phi(x),\Big(\partial\Phi\Big(\frac{x+y}{2}\Big)^T\Big)^{-1}\xi\Big)+O_{S^{s,l}_{k,\delta}}(\langle x\rangle^{-1}|x-y|)\Big)e^{\frac{i}{h}\langle x-y,\xi\rangle}u(x_0,\Phi(y))\times \\
&\qquad\qquad\times\Big|\det G(x,y)\Big|^{-1}d\xi|\det(\partial\Phi)(y)|dy, \quad \text{(A.9)}
\end{aligned}$$

The $O_{S^{s,l}_{k,\delta}}$ error follows from the Taylor expansion of

$$b(x,y,\xi):=\tilde a(x_0,\Phi(x),(G(x,y)^T)^{-1}\xi)|\det G(x,y)|^{-1}|\det(\partial\Phi)(y)|$$

at $y=x$. In fact,

$$b(x,y,\xi)=b(x,x,\xi)+\sum_j(y_j-x_j)r_j(x,y,\xi),\quad r_j\in S^{s,l-1}_{k,\delta}.$$

Hence, the integration by part using $(x_j-y_j)e^{\frac{i}{h}\langle x-y,\xi\rangle}=hD_{\xi_j}e^{\frac{i}{h}\langle x-y,\xi\rangle}$ gives a remainder in $hS^{s,l-1+\delta}_{k-1,\delta}$. Now we complete the proof of the claim by using that for $|x-y|$ small, again by a Taylor expansion

$$\Big|\det G(x,y)\Big|^{-1}|\det\partial\Phi(y)|=1+O(\langle x\rangle^{-1}|x-y|),$$

and since

$$(x-y)^\alpha e^{\frac{i}{h}\langle x-y,\xi\rangle}=(-hD_\xi)^\alpha e^{\frac{i}{h}\langle x-y,\xi\rangle}.$$

3. We conclude the proof, that is we treat the case $a\in S^{s,l}_{k,\delta}$, by a density argument, see for instance the proof of [Hin25, Theorem 5.2] for details. □

A.2. **The calculus on manifolds.** Lemma A.4 allows us to quantize symbols inside charts and then we can define a global calculus by a finite sum of such quantization in individual charts. See further in [DZ19b, Hin25].

Definition A.5 (A second-microlocal calculus on Ω_+)**.** Consider a finite family of charts covering (Φ_j,U_j) over $\partial\Omega$, associated with a partition of unity $\phi_j\in S^0(\partial\Omega)$ and $\operatorname{supp}\phi_j\subset U_j$ that $\sum_j\phi_j=1$. Let $\tilde\phi_j\in S^0(\partial\Omega)$ with $\operatorname{supp}\phi_j\cap\operatorname{supp}(1-\tilde\phi_j)=\emptyset$, $\operatorname{supp}\tilde\phi_j\subset U_j$. Let $a(x_0,x,\xi)\in S^{s,l}_{k,\delta}((T^*\partial\Omega)_+)$. We define its global left quantization by

$$\mathrm{Op}_h(a)=\sum_j\tilde\phi_j\Phi_j^*\,\mathrm{Op}_h(\Phi_j^{-*}(\phi_j(x,\xi)a(x_0,x,\xi)))\Phi_j^{-*}\tilde\phi_j, \tag{A.10}$$

where we confuse Φ with the symplectomorphism on $T^*\partial\Omega$ induced by Φ on $\partial\Omega$. We define the residual class by $h^\infty\Psi^{-\infty,-\infty}_{-\infty,\delta}(\Omega_+) = h^\infty\Psi^{-\infty,-\infty}(\Omega_+)$, consisting of all operators with kernel in $h^\infty\delta(x_0-y_0)C_b^\infty([0,\infty)_{x_0};\mathcal{S}(\partial\Omega_x\times\partial\Omega_y)$. We define $\Psi^{s,l}_{k,\delta}(\Omega_+) = \mathrm{Op}_h(S^{s,l}_{k,\delta}((T^*\partial\Omega)_+)) + O(h^\infty\Psi^{-\infty,-\infty}(\Omega_+))$.

As the corollary of Lemma A.10, we have the following definition:

Definition A.6. We define the principal symbol map by

$$\sigma_h : \Psi^{s,l}_{k,\delta}(\Omega_+) \to S^{s,l}_{k,\delta}((T^*\partial\Omega)_+)/hS^{s,l-(1-\delta)}_{k-1,\delta}((T^*\partial\Omega)_+), \tag{A.11}$$

such that $\sigma_h(\mathrm{Op}_h(a)) = a$ for each $a \in S^{s,l}_{k,\delta}((T^*\partial\Omega)_+)$. Let $b \in S^{s',l'}_{k',\delta}((T^*\partial\Omega)_+)$. We have:

$$\sigma_h(\mathrm{Op}_h(a)\,\mathrm{Op}_h(b)) = ab, \tag{A.12}$$

$$\sigma_h([\mathrm{Op}_h(a),\mathrm{Op}_h(b)]) = \{a,b\} = \langle\partial_\xi a,\partial_x b\rangle - \langle\partial_x a,\partial_\xi b\rangle. \tag{A.13}$$

Moreover, we can as well define the calculus on $\partial\Omega$ by restricting to $x_0 = 0$. In fact the calculus introduced in this section is uniform in the x_0 which is, in fact, treated as a parameter.

Lemma A.7 (Off-chart information is residual)**.** *Given $\chi,\tilde\chi \in S^0(\Omega_+)$ with $\operatorname{supp}\chi\cap\operatorname{supp}(1-\tilde\chi) = \emptyset$ and $\operatorname{supp}\chi \subset \{(x_0,x) : x_0 \le C\langle x\rangle\}$ for some $C > 0$. Then for any $A \in \Psi^{s,l}_{k,\delta}(\Omega_+)$, we have*

$$(1-\tilde\chi)A\chi \in h^\infty\langle x_0\rangle^{-\infty}\Psi^{-\infty,-\infty}(\Omega_+). \tag{A.14}$$

Proof. Disjoint $1-\tilde\chi$, χ give $(1-\tilde\chi)A\chi \in h^\infty\Psi^{-\infty,-\infty}$. Conclude with $x_0 \le C\langle x\rangle$ on $\operatorname{supp}\chi$. □

Now we want to show that the calculus is well defined, that is $\Psi^{s_1,l_1}_{k_1,\delta}\circ\Psi^{s_2,l_2}_{k_2,\delta} \subseteq \Psi^{s_1+s_2,l_1+l_2}_{k_1+k_2,\delta}$. We start with a result about the boundedness of $e^{i\langle D_\xi,D_y\rangle}$.

Lemma A.8. *Let $s_j,l_j,k_j \in \mathbb{R}$ for $j \in \{1,2\}$, and $M_0 > 0$. Then, if $X := (x,\xi) \in \partial\Omega$, $Y := (y,\eta) \in \mathbb{R}^{2(d-1)}$, and $\chi \in C_c^\infty(\mathbb{R}^{d-1})$, $\chi = 1$ near 0, $\operatorname{supp}\chi \subseteq B(0,1)$,*

$$\chi\Big(\frac{y-x}{\langle y\rangle+\langle x\rangle}\Big)e^{ih\langle D_\xi,D_y\rangle} : S^{s_1,l_1}_{k_1,\delta}((T^*\partial\Omega_X)_+)\otimes S^{s_2,l_2}_{k_2,\delta}((T^*\partial\Omega_Y)_+) \to S^{s_1,l_1}_{k_1,\delta}((T^*\partial\Omega_X)_+)\otimes S^{s_2,l_2}_{k_2,\delta}((T^*\partial\Omega_Y)_+)$$

is continuous (uniformly in h and for all $M \in [0,M_0]$) and for all $N \in \mathbb{N}$

$$\chi\Big(\frac{y-x}{\langle y\rangle+\langle x\rangle}\Big)\Big(e^{ih\langle D_\xi,D_y\rangle}a(\bar x,\xi,\bar y,\eta) - \sum_{k=0}^{N-1}\frac{(ih)^k}{k!}\langle D_\xi,D_y\rangle^k a(\bar x,\xi,\bar y,\eta)\Big) \tag{A.15}$$
$$\in h^N S^{s_1,l_1-(1-\delta)N}_{k_1-N,\delta}((T^*\partial\Omega_X)_+)\otimes S^{s_2,l_2}_{k_2,\delta}((T^*\partial\Omega_Y)_+).$$

Proof. 1. By [Zwo12, Theorem 4.8 (ii)]

$$\chi\Big(\frac{y-x}{\langle y\rangle+\langle x\rangle}\Big)e^{ih\langle D_\xi,D_y\rangle}a(\bar x,\xi,\bar y,\eta) = \frac{1}{(2h\pi)^{d-1}}\int_{\mathbb{R}^{2d-2}}\chi\Big(\frac{y-x}{\langle y\rangle+\langle x\rangle}\Big)e^{-\frac{i}{h}\langle\tilde\xi,\tilde y\rangle}a(\bar x,\xi+\tilde\xi,y_0,y+\tilde y,\eta)\,d\tilde y d\tilde\xi, \tag{A.16}$$

for $a \in \mathscr{S}$. We can give meaning to this integral as a tempered distribution when $a \in S^{s_1,l_1}_{k_1,\delta}\otimes S^{s_2,l_2}_{k_2,\delta}$ by a stationary phase approach. (See, for instace, the proof of [Zwo12, Theorem 3.18].)

2. Now we prove (A.15). First of all, for $\gamma \in (0,1)$,

$$\begin{aligned} e^{ih\langle D_\xi, D_y\rangle} a(\bar{x},\xi,\bar{y},\eta) =& \frac{1}{(2h\pi)^{d-1}} \int e^{-\frac{i}{h}\langle \tilde{\xi},\tilde{y}\rangle} a(\bar{x},\xi+\tilde{\xi},y_0,y+\tilde{y},\eta)\tilde{\chi}\left(\frac{\tilde{y}}{\langle y\rangle}, \frac{\tilde{\xi}}{\langle \xi\rangle^\gamma}\right) d\tilde{y}d\tilde{\xi} \\ &+ \frac{1}{(2h\pi)^{d-1}} \int e^{-\frac{i}{h}\langle \tilde{\xi},\tilde{y}\rangle} a(\bar{x},\xi+\tilde{\xi},y_0,y+\tilde{y},\eta)(1-\tilde{\chi})\left(\frac{\tilde{y}}{\langle y\rangle}, \frac{\tilde{\xi}}{\langle \xi\rangle^\gamma}\right) d\tilde{y}d\tilde{\xi} \\ =:& I_1(\bar{x},\xi,\bar{y},\eta) + I_2(\bar{x},\xi,\bar{y},\eta). \end{aligned}$$

2a. We start by considering I_1. Set $Z := \frac{\tilde{y}}{\langle y\rangle}$, $\Xi := \frac{\tilde{\xi}}{\langle \xi\rangle^\gamma}$, and define $\lambda := \frac{\langle y\rangle\langle \xi\rangle^\gamma}{h}$. Then,

$$I_1(\bar{x},\xi,\bar{y},\eta) = \frac{\langle y\rangle^{d-1}\langle \xi\rangle^{\gamma(d-1)}}{(2h\pi)^{d-1}} \int e^{-i\lambda\langle \Xi, Z\rangle} A(Z,\Xi)\, dZ\, d\Xi,$$

where $A(Z,\Xi) := a(\bar{x},\xi+\langle\xi\rangle^\gamma\Xi, y_0, y+\langle y\rangle Z,\eta)\tilde{\chi}(Z,\Xi)$. Now, A is compactly supported in (Z,Ξ) uniformly in $(\bar{x},\xi,\bar{y},\eta)$. Thus, stationary phase may be applied uniformly with large parameter $\lambda = \frac{\langle y\rangle\langle\xi\rangle^\gamma}{h}$. Namely, (see [Zwo12, (3.5.11)])

$$I_1(\bar{x},\xi,\bar{y},\eta) = \sum_{k=0}^{N-1} \frac{(ih)^k}{k!}\langle D_\xi, D_y\rangle^k a(\bar{x},\xi,\bar{y},\eta) + h^N R_N(\bar{x},\xi,\bar{y},\eta).$$

with

$$R_N(\bar{x},\xi,\bar{y},\eta) = (\langle y\rangle\langle\xi\rangle^\gamma)^{-N} \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!} \Big[e^{i\frac{h}{\langle y\rangle\langle\xi\rangle^\gamma} t\langle D_\Xi, D_Z\rangle} (i\langle D_\Xi, D_Z\rangle)^N A \Big]\Big|_{Z=\Xi=0} dt. \quad \text{(A.17)}$$

Hence, (see the last line of the proof of [Zwo12, Theorem 3.13])

$$|R_N| \le C_N(\langle y\rangle\langle\xi\rangle^\gamma)^{-N} \sum_{|\alpha|\le 2d-1} \sup_{Z,\Xi\in \operatorname{supp}\tilde{\chi}} |\partial^\alpha_{Z,\Xi}\langle D_\Xi, D_Z\rangle^N A(Z,\Xi)|.$$

This is estimated by the symbol seminorms of a. Hence, for all N, we showed that $R_N \in S^{s_1,l_1+\delta(N+4d-1)}_{k_1-N,\delta} \otimes S^{s_2,l_2-N}_{k_2,\delta}$ and, thus,

$$\begin{aligned} &\chi\Big(\frac{y-x}{\langle y\rangle+\langle x\rangle}\Big)\Big(I_1(\bar{x},\xi,\bar{y},\eta) - \Big(\sum_{k=0}^{N-1}\frac{(ih)^k}{k!}\langle D_{\tilde\xi}, D_{\tilde y}\rangle^k a(\bar{x},\xi+\tilde\xi,y_0,y+\tilde y,\eta)|_{\tilde\xi=\tilde y=0}\Big)\Big) \\ &\qquad \in h^N S^{s_1,l_1+(\delta-1)N+\delta(4d-1)}_{k_1-N,\delta} \otimes S^{s_2,l_2}_{k_2,\delta} \end{aligned}$$

because of the support of χ.

Now, we fix $K \ge \frac{4d-1}{1-\delta}$ so that $S^{s_1,l_1+(\delta-1)K+\delta(4d-1)}_{k_1-N,\delta} \subseteq S^{s_1,l_1}_{k_1-N,\delta}$ and, hence, for all N

$$\begin{aligned} &\chi\Big(\frac{y-x}{\langle y\rangle+\langle x\rangle}\Big)\Big(I_1(\bar{x},\xi,\bar{y},\eta) - \Big(\sum_{k=0}^{N+K-1}\frac{(ih)^k}{k!}\langle D_\xi, D_y\rangle^k a(\bar{x},\xi,y_0,y,\eta)\Big)\Big) \\ &\qquad \in h^N S^{s_1,l_1+(\delta-1)N}_{k_1-N,\delta} \otimes S^{s_2,l_2}_{k_2,\delta}. \end{aligned}$$

Moreover, $\chi\Big(\frac{y-x}{\langle y\rangle+\langle x\rangle}\Big)\langle D_\xi, D_y\rangle^k a(\bar{x},\xi,y_0,y,\eta) \in S^{s_1,l_1+(\delta-1)k}_{k_1-k,\delta}\otimes S^{s_2,l_2}_{k_2,\delta} \subseteq S^{s_1,l_1+(\delta-1)N}_{k_1-N,\delta}\otimes S^{s_2,l_2}_{k_2,\delta}$ for $k\geq N$ which implies

$$\chi\Big(\frac{y-x}{\langle y\rangle+\langle x\rangle}\Big)\bigg(I_1(\bar{x},\xi,\bar{y},\eta)-\bigg(\sum_{k=0}^{N-1}\frac{(ih)^k}{k!}\langle D_{\tilde{\xi}},D_{\tilde{y}}\rangle^k a(\bar{x},\xi,\bar{y},\eta)\bigg)\bigg)$$
$$\in h^N S^{s_1,l_1+(\delta-1)N}_{k_1-N,\delta}\otimes S^{s_2,l_2}_{k_2,\delta}.$$

2b. Now we focus on I_2. By changing coordinates $(\tilde{y},\tilde{\xi})\mapsto(\langle y\rangle\tilde{y},\langle\xi\rangle^\gamma\tilde{\xi})$

$$\begin{aligned}I_2=&\langle y\rangle^{d-1}\langle\xi\rangle^{\gamma(d-1)}\int e^{-\frac{i}{h}\langle y\rangle\langle\xi\rangle^\gamma\langle\tilde{\xi},\tilde{y}\rangle}a(\bar{x},\xi+\langle\xi\rangle^\gamma\tilde{\xi},y_0,y+\langle y\rangle\tilde{y},\eta)(1-\tilde{\chi})(\tilde{y},\tilde{\xi})\,d\tilde{y}d\tilde{\xi}\\ =&\langle y\rangle^{d-1}\langle\xi\rangle^{\gamma(d-1)}\int e^{-\frac{i}{h}\langle y\rangle\langle\xi\rangle^\gamma\langle\tilde{\xi},\tilde{y}\rangle}a(\bar{x},\xi+\langle\xi\rangle^\gamma\tilde{\xi},y_0,y+\langle y\rangle\tilde{y},\eta)(\psi_1+\psi_2)(\tilde{y},\tilde{\xi})\,d\tilde{y}d\tilde{\xi}\end{aligned}\tag{A.18}$$

with $1-\tilde{\chi}=\psi_1+\psi_2$, $\operatorname{supp}\psi_1\subset\{|\tilde{y}|\geq c\}$, $\operatorname{supp}\psi_2\subset\{|\tilde{y}|\leq 2c,\,|\tilde{\xi}|\geq c\}$ for some $c>0$ small. We estimate

$$I_2'=\langle y\rangle^{d-1}\langle\xi\rangle^{\gamma(d-1)}\int_{\mathbb{R}^{2(d-1)}}e^{-\frac{i}{h}\langle y\rangle\langle\xi\rangle^\gamma\langle\tilde{\xi},\tilde{y}\rangle}a(\bar{x},\xi+\langle\xi\rangle^\gamma\tilde{\xi},y_0,y+\langle y\rangle\tilde{y},\eta)\psi_1(\tilde{y},\tilde{\xi})\,d\tilde{y}d\tilde{\xi},$$

by a nonstationaryphase approach using the operator $L_{\tilde{y}}=-h\frac{\langle\tilde{y},D_{\tilde{\xi}}\rangle}{|\tilde{y}|^2\langle y\rangle\langle\xi\rangle^\gamma}$ and $L_{\tilde{\xi}}=-h\frac{\langle\tilde{\xi},D_{\tilde{y}}\rangle}{|\tilde{\xi}|^2\langle y\rangle\langle\xi\rangle^\gamma}$. In fact,

$$\begin{aligned}D_{\tilde{\xi}}a(\bar{x},\xi+\langle\xi\rangle^\gamma\tilde{\xi},y_0,y+\langle y\rangle\tilde{y},\eta)&=\langle\xi\rangle^\gamma(D_\xi a)(\bar{x},\xi+\langle\xi\rangle^\gamma\tilde{\xi},y_0,y+\langle y\rangle\tilde{y},\eta)\\ &\in\langle\xi\rangle^\gamma\langle\langle x\rangle^\delta\sqrt{|\xi+\langle\xi\rangle^\gamma\tilde{\xi}|^2+M^2}\rangle^{k_1-1}S^{K,K+\delta}_{0,\delta}\times S^{s_2,l_2}_{k_2,\delta}\end{aligned}\tag{A.19}$$

for some $K>0$. Hence, $L_{\tilde{y}}a\in h\langle y\rangle^{-1}\langle\tilde{y}\rangle^{-1}\langle\langle x\rangle^\delta\sqrt{|\xi+\langle\xi\rangle^\gamma\tilde{\xi}|^2+M^2}\rangle^{k_1-1}S^{K,K+\delta}_{0,\delta}\times S^{s_2,l_2}_{k_2,\delta}$ and, iterating, $L^N_{\tilde{y}}a\in h^N\langle y\rangle^{-N}\langle\tilde{y}\rangle^{-N}\langle\langle x\rangle^\delta\sqrt{|\xi+\langle\xi\rangle^\gamma\tilde{\xi}|^2+M^2}\rangle^{k_1-N}S^{K,K+\delta N}_{0,\delta}\times S^{s_2,l_2}_{k_2,\delta}$ with the same K. Now, for large N, $L^N_{\tilde{y}}a\in h^N\langle y\rangle^{-N}\langle\tilde{y}\rangle^{-N}\langle\xi\rangle^{-J_N}\langle\tilde{\xi}\rangle^{-J_N}S^{K,K+\delta N}_{0,\delta}\times S^{s_2,l_2}_{k_2,\delta}$ when by a smooth partition we restrict to $\langle\xi+\langle\xi\rangle^\gamma\tilde{\xi}\rangle>\epsilon(\langle\xi\rangle+\langle\xi\rangle^\gamma|\tilde{\xi}|)$, ϵ small, with $J_N\to\infty$ as $N\to\infty$. For $\langle\xi+\langle\xi\rangle^\gamma\tilde{\xi}\rangle\leq\epsilon(\langle\xi\rangle+\langle\xi\rangle^\gamma|\tilde{\xi}|)$ we iteratively integrate by part using the operator $L_{\tilde{\xi}}=-h\frac{\langle\tilde{\xi},D_{\tilde{y}}\rangle}{|\tilde{\xi}|^2\langle y\rangle\langle\xi\rangle^\gamma}$. Namely, we get $L^N_{\tilde{\xi}}L^N_{\tilde{y}}a\in h^{2N}\langle y\rangle^{-N}\langle\tilde{y}\rangle^{-N}\langle\xi\rangle^{-J_N}\langle\tilde{\xi}\rangle^{-J_N}S^{K,K+\delta N}_{0,\delta}\times S^{s_2,l_2}_{k_2,\delta}$.

Now we estimate

$$I_2''=\langle y\rangle^{d-1}\langle\xi\rangle^{\gamma(d-1)}\int_{\mathbb{R}^{2(d-1)}}e^{-\frac{i}{h}\langle y\rangle\langle\xi\rangle^\gamma\langle\tilde{\xi},\tilde{y}\rangle}a(\bar{x},\xi+\langle\xi\rangle^\gamma\tilde{\xi},y_0,y+\langle y\rangle\tilde{y},\eta)\psi_2(\tilde{y},\tilde{\xi})\,d\tilde{y}d\tilde{\xi},$$

and this can be done as for I_2' using the operator $L_{\tilde{\xi}}=-h\frac{\langle\tilde{\xi},D_{\tilde{y}}\rangle}{|\tilde{\xi}|^2\langle y\rangle\langle\xi\rangle^\gamma}$.

Hence, we have that for every $J>0$, choosing the number of integrations N sufficiently large

$$\chi\Big(\frac{y-x}{\langle y\rangle+\langle x\rangle}\Big)I_2\in h^J S^{-2J,-2J}_{K_0,\delta}\otimes S^{s_2,l_2}_{k_2,\delta}\subseteq h^J S^{-J,-J}_{K_0-J,\delta}\otimes S^{s_2,l_2}_{k_2,\delta},$$

with K_0 uniform in J as claimed. □

Now we show a preliminary result of boundedness for the quantization of a symbol in our class.

Lemma A.9. *Let $a \in S^{s,l}_{k,\delta}((T^*\mathbb{R}^{d-1})_+)$ for $s,l,k \in \mathbb{R}$. Then, there is $N \in \mathbb{N}$ such that $\operatorname{Op}_h(a) : H^{2N,2N}((\partial\Omega)_+) \to H^{0,0}((\partial\Omega)_+)$ and $\operatorname{Op}_h(a) : H^{0,0}((\partial\Omega)_+) \to H^{-2N,-2N}((\partial\Omega)_+)$ are bounded uniformly in M and h.*

Proof. 1. We want to prove that $\operatorname{Op}_h(a) : H^{2N,2N} \to H^{0,0}$ is bounded for some N. Let $u \in H^{2N,2N}$ and $v := \langle x\rangle^{2N}\langle hD_x\rangle^{2N}u$. We follow a Schur test argument,

$$\operatorname{Op}_h(a)u(x) = \int k(x,y)v(y)\,dy, \quad k(x,y) = (2\pi h)^{-(d-1)}\int e^{\frac{i}{h}\langle x-y,\xi\rangle}a(x,\xi)\langle\xi\rangle^{-2N}\langle y\rangle^{-2N}\,d\xi.$$

By a stationary phase argument, for some $J > 0$ and for $K > J + (1-\delta)^{-1}(|s| + |l| + |k| + d)$, $N > |s| + |l| + |k| + K + d$

$$\begin{aligned}|k(x,y)| \leq& Ch^{-(d-1)}\langle h^{-1}(x-y)\rangle^{-2K}\langle y\rangle^{-2N}\int_{\mathbb{R}^{d-1}}|(1-\Delta_\xi)^K(a(x,\xi)\langle\xi\rangle^{-2N})|\,d\xi\\ \leq& Ch^{-(d-1)}\langle h^{-1}(x-y)\rangle^{-2K}\langle y\rangle^{-2N}\langle x\rangle^J \leq Ch^{-(d-1)}\langle h^{-1}(x-y)\rangle^{J-2K},\end{aligned}$$

where in the last estimate we used that by Peetre's estimate $\langle y\rangle^{-2N}\langle x\rangle^J \leq \langle x-y\rangle^J$. Now by the change of variables $x \mapsto hx + y$ and $y \mapsto hy + x$,

$$\sup_x \int |K(x,y)|\,dy + \sup_y \int |K(x,y)|\,dx \leq C,$$

uniformly in h. Hence, $\|\operatorname{Op}_h(a)u\| \leq C\|v\| = C\|\langle x\rangle^{2N}\langle hD_x\rangle^{2N}u\|$, as claimed.

2. Now, we prove the boundedness of $\operatorname{Op}_h(a) : H^{0,0} \to H^{-2N,-2N}$ by duality. Namely, the kernel of $\operatorname{Op}_h(a)^*$ is $(2\pi h)^{-(d-1)}\int e^{\frac{i}{h}\langle x-y,\xi\rangle}\overline{a(y,\xi)}\,d\xi$ and we conclude similarly to Step 1. □

From these lemma a composition law for symbols in the class we are studying follows

Lemma A.10 (Composition of symbols in $S^{s_1,l_1}_{k_1,\delta}$)**.** *Let $a \in S^{s_1,l_1}_{k_1,\delta}((T^*\partial\Omega)_+)$ and $b \in S^{s_2,l_2}_{k_2,\delta}((T^*\partial\Omega)_+)$. Then, there exists $c \in S^{s_1+s_2,l_1+l_2}_{k_1+k_2,\delta}((T^*\partial\Omega)_+)$ such that $\operatorname{Op}_h(a)\operatorname{Op}_h(b) = \operatorname{Op}_h(c)$ with*

$$c - \sum_{k=0}^{N-1}\frac{(ih)^k}{k!}\langle D_\xi, D_y\rangle^k a(x_0,x,\xi)b(x_0,y,\eta)|_{y=x,\,\eta=\xi} \in h^N S^{s_1+s_2,l_1+l_2-(1-\delta)N}_{k_1+k_2-N,\delta}((T^*\partial\Omega)_+) \quad \text{(A.20)}$$

Proof. 1. We first consider a and b uniformly in $C^\infty_b([0,\infty)_{x_0};\mathcal{S}(\partial\Omega))$. By the composition formula [Zwo12, (4.3.14)]

$$c(x,\xi) := e^{ih\langle D_\xi,D_y\rangle}a(x_0,x,\xi)b(x_0,y,\eta)|_{y=x,\,\eta=\xi}. \quad \text{(A.21)}$$

By Lemma A.8,

$$\chi\Big(\frac{y-x}{\langle x\rangle + \langle y\rangle}\Big)e^{ih\langle D_\xi,D_y\rangle}a(x_0,x,\xi)b(y_0,y,\eta) \in S^{s_1,l_1}_{k_1,\delta} \otimes S^{s_2,l_2}_{k_2,\delta}.$$

Restricting to $y_0 = x_0$, $y = x$ and $\eta = \xi$, we have $c \in S^{s_1+s_2,l_1+l_2}_{k_1+k_2,\delta}$.

2. We have an asymptotic expansion for c. In fact, by Lemma A.8

$$\chi\Big(\frac{y-x}{\langle x\rangle+\langle y\rangle}\Big)\Big(e^{ih\langle D_\xi,D_y\rangle}a(x_0,x,\xi)b(y_0,y,\eta)-\sum_{j=0}^{N-1}\frac{(ih)^j}{j!}\langle D_\xi,D_y\rangle^j a(x_0,x,\xi)b(y_0,y,\eta)\Big)\in h^N S^{s_1,l_1-(1-\delta)N}_{k_1-N,\delta}\otimes S^{s_2,l_2}_{k_2,\delta}$$

which implies

$$c-\sum_{j=0}^{N-1}\frac{(ih)^j}{j!}\langle D_\xi,D_y\rangle^j a(x_0,x,\xi)b(y_0,y,\eta)|_{y_0=x_0,\,y=x,\,\eta=\xi}\in h^N S^{s_1+s_2,l_1+l_2-(1-\delta)N}_{k_1+k_2-N,\delta}.$$

3. Now, we treat the case $a,b\notin\mathcal{S}$ by a density argument. Let $\chi\in C_c^\infty(T^*\partial\Omega)$, $\chi=\chi(|x|,\xi)$, $\chi|_{|x|+|\xi|_g<1}=1$. Then, let $a_m(x_0,x,\xi):=a(x_0,x,\xi)\chi(|x|/m,\xi/m)$ and $b_m(y_0,y,\eta):=b(y_0,y,\eta)\chi(|y|/m,\eta/m)$. $a_m,b_m\in\mathcal{S}$ uniformly for all x_0, y_0. $a_m\to a$ in $S^{s_1+0,l_1+0}_{k_1+0,\delta}$ and $b_m\to b$ in $S^{s_2+0,l_2+0}_{k_2+0,\delta}$. Hence, by Step 1, $\operatorname{Op}_h(c_m)=\operatorname{Op}_h(a_m)\operatorname{Op}_h(b_m)$ with

$$c_m(x,\xi):=e^{ih\langle D_\xi,D_y\rangle}a_m(x_0,x,\xi)b_m(x_0,y,\eta)|_{y=x,\,\eta=\xi}\in S^{s_1+s_2,l_1+l_2}_{k_1+k_2,\delta}.$$

Then, by Lemma A.8, $c_m\to c$ in $S^{s_1+s_2+0,l_1+l_2+0}_{k_1+k_2+0,\delta}$ with

$$\begin{aligned}c(x,\xi)&:=e^{ih\langle D_\xi,D_y\rangle}a(x_0,x,\xi)b(x_0,y,\eta)|_{y=x,\,\eta=\xi}=\\&\sum_{k=0}^{N-1}\frac{(ih)^k}{k!}\langle D_\xi,D_y\rangle^k a(x_0,x,\xi)b(x_0,y,\eta)|_{y=x,\,\eta=\xi}+h^N S^{s_1+s_2,l_1+l_2-(1-\delta)N}_{k_1+k_2-N,\delta}((T^*\partial\Omega)_+).\end{aligned}\tag{A.22}$$

Hence, $c\in S^{s_1+s_2,l_1+l_2}_{k_1+k_2,\delta}((T^*\partial\Omega)_+)$.

4. It remains to show that $\operatorname{Op}_h(c)=\operatorname{Op}_h(a)\operatorname{Op}_h(b)$. By Lemma A.9,

$$\langle(\operatorname{Op}_h(a_m)\operatorname{Op}_h(b_m)-\operatorname{Op}_h(a)\operatorname{Op}_h(b))u,v\rangle\to 0,$$

and

$$\langle(\operatorname{Op}_h(c_m)-\operatorname{Op}_h(c))u,v\rangle\to 0.$$

Passing to the limit, $\operatorname{Op}_h(c)=\operatorname{Op}_h(a)\operatorname{Op}_h(b)$ as $\mathcal{S}\to\mathcal{S}'$ bounded operators, as claimed. □

We show that $\Psi^{0,0}_{0,\delta}(\Omega_+)$ are bounded on L^2.

Lemma A.11. *Let $a\in S^{0,0}_{0,\delta}((T^*\partial\Omega)_+)$. Then $\operatorname{Op}_h(a)$ is bounded on $C_b^\infty([0,\infty)_{x_0};L^2(\partial\Omega))$, uniformly in $h\in(0,h_0]$ and $M\in[0,M_0]$.*

Proof. 1. $\operatorname{Op}(a):L^2\to L^2$ is bounded if $a\in S^{-N,-N}_{-N,\delta}$ for some large N by Lemma A.9 and Lemma A.10.

2. Now, for all $\epsilon>0$, $\operatorname{Op}_h(a):L^2\to L^2$ is bounded if $a\in S^{-\epsilon,-\epsilon}_{-\epsilon,\delta}$. In fact, $\operatorname{Op}_h(a):L^2\to L^2$ is bounded if $\operatorname{Op}_h(a)^*\operatorname{Op}_h(a):L^2\to L^2$ is bounded. Hence, if $a\in S^{-N/2,-N/2}_{-N/2,\delta}$ then $\operatorname{Op}_h(a)$ is bounded onto L^2 since $\operatorname{Op}_h(a)^*\operatorname{Op}_h(a)\in\Psi^{-N,-N}_{-N,\delta}$ by Lemma A.10. By repeating this procedure

$\operatorname{Op}_h(a) : L^2 \to L^2$ is bounded when $a \in S^{-\epsilon,-\epsilon}_{-\epsilon,\delta}$.
3. Finally, let $a \in S^{0,0}_{0,\delta}$. Let $c := \sqrt{C - |a|^2} \in S^{0,0}_{0,\delta}$ with $C > \sup |a|^2 + 1$,

$$0 \le \langle \operatorname{Op}_h(c)^* \operatorname{Op}_h(c)u, u\rangle = \langle \operatorname{Op}_h(|c|^2 + r)u, u\rangle = \langle \operatorname{Op}_h(C - |a|^2 + r)u, u\rangle$$
$$= C\|u\|^2 - \|\operatorname{Op}_h(a)u\|^2 + \langle \operatorname{Op}_h(\tilde{r})u, u\rangle \le C\|u\|^2 - \|\operatorname{Op}_h(a)u\|^2 + \tilde{C}\|u\|^2, \quad \text{(A.23)}$$

where $r, \tilde{r} \in S^{0,\delta-1}_{-1,\delta} \subset S^{\frac{\delta-1}{2},\frac{\delta-1}{2}}_{\frac{\delta-1}{2},\delta}$ and the last inequality follows from Step 2. Hence, $\|\operatorname{Op}_h(a)u\| \le C\|u\|$, as claimed. □

We use the properties of the calculus to show that we can define for all $s, l, k \in \mathbb{R}$ an invertible operator in $S^{s,l}_{k,\delta}$.

Lemma A.12. *Let $N > 0$. Let $s, l \in \mathbb{R}$ and $k \ge 0$. We define $\Gamma^{s,l}_{k,\delta} \in \Psi^{s,l}_{k,\delta}(\Omega_+)$ by*

$$\Gamma^{s,l}_{k,\delta} = (1 - h^2\Delta_x)^{\frac{s}{4}}\langle x\rangle^{\frac{l}{2}} \operatorname{Op}_h(\langle\langle x\rangle^{\delta}\sqrt{|\xi|^2 + M^2}\rangle^{\frac{k}{2}})^* \operatorname{Op}_h(\langle\langle x\rangle^{\delta}\sqrt{|\xi|^2 + M^2}\rangle^{\frac{k}{2}})\langle x\rangle^{\frac{l}{2}}(1 - h^2\Delta_x)^{\frac{s}{4}} \quad \text{(A.24)}$$

Then there exists $h_0 > 0$ such that for $h \in (0, h_0]$, $k \le N$, $\Gamma^{s,l}_{k,\delta} : \mathcal{S}'((\partial\Omega)_+) \to \mathcal{S}'((\partial\Omega)_+)$ is invertible with $(\Gamma^{s,l}_{k,\delta})^{-1} \in \Psi^{-s,-l}_{-k,\delta}$. For $k < 0$, we then define $\Gamma^{s,l}_{k,\delta} = (\Gamma^{-s,-l}_{-k,\delta})^{-1} \in \Psi^{s,l}_{k,\delta}((\partial\Omega)_+)$.

Proof. It suffices to show the invertibility of $A = \operatorname{Op}_h(\langle\langle x\rangle^{\delta}\sqrt{|\xi|^2 + M^2}\rangle^{\frac{k}{2}})$. Construct its principal parametrix $B_0 = \operatorname{Op}_h(\langle\langle x\rangle^{\delta}\sqrt{|\xi|^2 + M^2}\rangle^{-\frac{k}{2}})$ and we have $hR = AB_0 - \mathrm{I} \in \Psi^{0,\delta-1}_{-1,\delta}$. Now for small h, $(\mathrm{I} + hR)$ is invertible and we construct $B = B_0 \sum_{j=0}^{\infty}(-hR)^j \in \Psi^{0,0}_{-\frac{k}{2},\delta}$. This gives $AB = \mathrm{I}$ and a right inverse as desired. The existence of a left inverse is similar. □

Now, we construct the microlocal parametrices in this calculus.

Lemma A.13. *Let $a \in S^{s,l}_{k,\delta}$, $b \in S^{s',l'}_{k',\delta}$. Assume that a is elliptic near* $\operatorname{supp} b$*, that is there exists $C > 0$ such that*

$$|a(\bar{x}, \xi)| \ge C\langle\xi\rangle^s\langle x\rangle^l\langle\langle x\rangle^{\delta}\sqrt{|\xi|^2 + M^2}\rangle^k \text{ near } \operatorname{supp} b,$$

Then, there exists $E \in \Psi^{s'-s,l'-l}_{k'-k,\delta}$ such that

$$\operatorname{Op}_h(b) = E\operatorname{Op}_h(a) + h^{\infty}\Psi^{-\infty,-\infty}_{-\infty,\delta} = \operatorname{Op}_h(a)E + h^{\infty}\Psi^{-\infty,-\infty}_{-\infty,\delta}.$$

Moreover, the parametrices depend continuously on a and b in the corresponding symbol seminorms.

Proof. 1. We proceed by induction. Let $\chi \in S^{0,0}_{0,\delta}$, a elliptic near $\operatorname{supp}\chi$, $\chi = 1$ near $\operatorname{supp} b$. Then, $e_0 := \frac{\chi}{a} \in S^{-s,-l}_{-k,\delta}$, $\operatorname{supp} e_0 \subset \operatorname{supp}\chi$. Then, by Lemma A.10, $\operatorname{Op}_h(\chi) = \operatorname{Op}_h(e_0)\operatorname{Op}_h(a) + h\operatorname{Op}_h(r_1) + h^{\infty}\Psi^{-\infty,-\infty}$ with $r_1 \in S^{0,\delta-1}_{-1,\delta}$, $\operatorname{supp} r_1 \subset \operatorname{supp}\chi$.

2. Now, assume there exist $e_0, \dots, e_{N-1}$ such that

$$\operatorname{Op}_h(\chi) = \Big(\sum_{j=0}^{N-1} h^j \operatorname{Op}_h(e_j)\Big)\operatorname{Op}_h(a) + h^N \operatorname{Op}_h(r_N) + h^{\infty}\Psi^{-\infty,-\infty}$$

with $r_N \in S^{0,-N(1-\delta)}_{-N,\delta}$, $\operatorname{supp} r_N \subset \operatorname{supp}\chi$. Then, with $e_N = \frac{r_N}{a} \in S^{-s,-l-N(1-\delta)}_{-k-N,\delta}$

$$\operatorname{Op}_h(\chi) = \operatorname{Op}_h\Big(\sum_{j=0}^{N} h^j e_j\Big)\operatorname{Op}_h(a) + h^{N+1}\operatorname{Op}_h(r_{N+1}) + h^\infty \Psi^{-\infty,-\infty},$$

with $r_{N+1} \in S^{0,(N+1)(\delta-1)}_{-(N+1),\delta}$, $\operatorname{supp} r_{N+1} \subset \operatorname{supp}\chi$, concluding the induction.

3. Now, let $e \sim \sum_{j\geq 0} h^j e_j$. Then, $E := \operatorname{Op}_h(b)\operatorname{Op}_h(e) \in \Psi^{s'-s,l'-l}_{k'-k,\delta}$, as claimed.

4. For the construction of the symbol of E_R we can proceed similarly. The continuity statement follows from the recursive construction and the fixed ellipticity lower bound of a on $\operatorname{supp}\chi$. □

Appendix B. The scattering geometry and Hardy constants

B.1. **The scattering geometry.** We show that the boundary of any asymptotically conic domain is a scattering manifold.

Lemma B.1. *The boundary of an AC smooth domain $\Omega \subset \mathbb{R}^d$ is a (Lipschitz) scattering manifold.*

Proof. The case $d = 2$ is trivial and we assume $d > 2$. We just need to write what is the metric induced by $\mathbb{R}^d$ on $\partial\Omega$. In fact, $\psi = \psi(\rho,\omega) \in C^\infty([0,1]\times\partial Y; \mathbb{S}^{d-1})$ that $\psi(0,\omega) = \omega$, $\rho^{-1}\psi(\rho,\omega)$ parametrizes $\partial\Omega$. Then in the local coordinates $\omega = (\omega_1,\dots,\omega_{d-2})$, we have

$$g = \rho^{-4}d\rho^2 + \rho^{-2}\Big(|\partial_\rho\psi|^2 d\rho^2 + 2\sum_{j=1}^{d-2}\langle\partial_\rho\psi,\partial_{\omega_j}\psi\rangle d\rho d\omega^j + \sum_{i,j=1}^{d-2}\langle\partial_{\omega_i}\psi,\partial_{\omega_j}\psi\rangle d\omega^i d\omega^j\Big). \tag{B.1}$$

Note the smoothness of ψ at $\rho = 0$ implies

$$g(0,\omega) = \rho^{-4}d\rho^2 + g_\infty(\omega),\ g_\infty(\omega) = \sum_{i,j=1}^{d-2}\langle\partial_{\omega_i}\psi(0,\omega),\partial_{\omega_j}\psi(0,\omega)\rangle d\omega^i d\omega^j \tag{B.2}$$

where $g_\infty \in \operatorname{Sym}^2(T^*\partial Y))$ is positive-definite. Then [JSB99, Proposition 2.1] implies $\partial\Omega$ is a scattering manifold. □

B.2. **Properties of the Hardy constant.**

Lemma B.2. *Let Ω be an AC Lipschitz domain. Then there is some $r_0 \geq 0$ that for each $\epsilon > 0$, there exists $\gamma \in (0,\frac{1}{2})$ that $\mathfrak{h}^0_{U^{r_0}_\gamma,\partial\Omega} \geq \frac{1}{2} - \epsilon$. If $\partial\Omega$ is only singular at up to one point, one can take $r_0 = 0$.*

Proof. There exists $r_0 \geq 0$ such that $\partial\Omega \cap \{|z| > r_0\}$ is smooth. There then exists $\gamma \in (0,\frac{1}{2})$ that one can adopt the Fermi normal coordinates identifying $U^{r_0}_\gamma$ with a subset of $\{(x_0,x) : x \in \partial\Omega \cap \{|z| > r_0\}, x_0 \leq 2\gamma|x|\}$ equipped with metric $g_+(x_0,x) = dx_0^2 + g(x) + O(\gamma\operatorname{Sym}^2(T^*\partial\Omega))$. This implies

$$\sqrt{g_+(x_0,x)} = (1+O(\gamma))\sqrt{g(x)},\ \operatorname{dist}(z,\partial\Omega) = x_0,\ |\partial_{x_0}u|^2 \leq |\nabla_{g_+}u|^2 \tag{B.3}$$

on $U_\gamma^{r_0}$. Let $u \in \dot{H}^1(U_\gamma^{r_0})$, extended trivially by 0 onto the rest of $\{(x_0,x) : x \in \partial\Omega \cap \{|z| > r_0\}, x_0 \le 2\gamma|x|\}$. Now the one-dimensional Hardy inequality in x_0 implies

$$\int_{x_0=0}^{\infty} \Big|\frac{u(x_0,x)}{x_0}\Big|^2 dx_0 \le 4\int_{x_0=0}^{\infty} \Big|\partial_{x_0} u(x_0,x)\Big|^2 dx_0 \tag{B.4}$$

for all u with $u(0,x)=0$. We have

$$\begin{aligned}\Big\|\frac{u}{\operatorname{dist}(z,\partial\Omega)}\Big\|^2 &= (1+O(\gamma))\int_{\partial\Omega}\int_{x_0=0}^{\infty}\Big|\frac{u(x_0,x)}{x_0}\Big|^2\sqrt{g(x)}dx_0dx \\ &\le 4(1+O(\gamma))\int_{\partial\Omega}\int_{x_0=0}^{\infty}\Big|\nabla_{g_+}u(x_0,x)\Big|^2\sqrt{g(x)}dx_0dx = 4(1+O(\gamma))\|\nabla_{g_+}u\|^2. \end{aligned}\tag{B.5}$$

Make γ small to conclude the proof for generic AC Lipschitz domains. If $\partial\Omega$ is singular at up to one point, one can set that point $z=0$ and the above works with $r_0=0$. □

Similarly one can show:

Lemma B.3. *Let $\mathcal{P}$ be a curvilinear polygon, and let $\mathcal{V}=\{x_j\}_{j=1}^J$ contain all vertices of $\mathcal{P}$. Then for each $\epsilon>0$, there exists $\gamma\in(0,\frac12)$ that $\mathfrak{h}^0_{U_\gamma^0,\partial\Omega}\ge\frac12-\epsilon$.*

Lemma B.4. *Suppose $\Omega\subset\mathbb{R}^2$ such that there are $\gamma_\pm\in C^\infty(\mathbb{R};\mathbb{R}^2)$ and $\alpha\in(0,2\pi)$ such that $\gamma_\pm(0)=0$, $|\gamma_\pm'(s)|=1$, and*

$$\partial\Omega\cap B(0,1) = \big(\gamma_+([0,\infty))\cup\gamma_-([0,\infty)\big)\cap B(0,1). \tag{B.6}$$

Let $\alpha\in(0,2\pi)$ be the angle from $\gamma_+'(0)$ to $\gamma_-'(0)$. Then there is a neighborhood U of 0 and $F:\overline{U\cap\mathcal{W}_\alpha}\to\overline{\Omega}$, $F\in C^{1,1}(\overline{\mathcal{W}_\alpha})\cap C^\infty(\overline{\mathcal{W}_\alpha}\setminus\{0\})$ such that

$$F(s,0)=\gamma_+(s),\qquad F(s\cos\alpha, s\sin\alpha)=\gamma_-(s),$$

and

$$dF(s,0)\begin{pmatrix}0\\-1\end{pmatrix}=n_+(s),\qquad dF(s\cos\alpha,s\sin\alpha)\begin{pmatrix}-\sin\alpha\\\cos\alpha\end{pmatrix}=n_-(s),$$

where $n_\pm$ are the outward unit normals to Ω at $\gamma_\pm(s)$.

Proof. 1. Without loss of generality, we may assume that $\gamma_+'(0)=(1,0)$ and $\gamma_-'(0)=(\cos\alpha,\sin\alpha)$. We now use polar coordinates $\Theta_\pm:[0,\epsilon)\to\mathbb{R}$ (for some $\epsilon>0$) with $\Theta_-(0)=\alpha$, $\Theta_+(0)=0$ such that

$$\gamma_\pm = |\gamma_\pm|(\cos\Theta_\pm,\sin\Theta_\pm) \text{ on } [0,\epsilon).$$

2. Let $q_\pm(r)=\frac{|\gamma_\pm(r)|}{r}$ for $r>0$, $q_\pm(0)=1$ by smooth extension. We construct $\tilde{R}(r,\theta)$ and $\Theta(r,\theta)\in C^\infty$ such that

$$\begin{gathered}\Theta(r,\alpha)=\Theta_-(r),\qquad \Theta(r,0)=\Theta_+(r),\qquad \tilde{R}(r,\alpha)=q_-(r),\qquad \tilde{R}(r,0)=q_+(r),\\ \Theta(0,\theta)=\theta,\qquad \tilde{R}(0,\theta)=1.\end{gathered}\tag{B.7}$$

Let $\tilde{R}$, $\Theta\in C^\infty([0,\epsilon)_r\times[0,\alpha]_\theta)$ that $\tilde{R}(r,0)=q_+(r)$, $\tilde{R}(r,\alpha)=q_-(r)$, $\Theta(r,0)=\Theta_+(r)$, $\Theta(r,\alpha)=\Theta_-(r)$, $\Theta(0,\theta)=\theta$ and

$$\partial_\theta\tilde{R}(r,0)=-\langle n_+(r),(\cos(\Theta_+(r)),\sin(\Theta_+(r)))\rangle,\quad \partial_\theta\Theta(r,0)=-\frac{\langle n_+(r),(-\sin(\Theta_+(r),\cos(\Theta_+(r)))\rangle}{q_+(r)},$$

$$\partial_\theta \tilde{R}(r,\alpha) = \langle n_-(r), (\cos(\Theta_-(r)), \sin(\Theta_-(r)))\rangle, \quad \partial_\theta \Theta(r,\alpha) = \frac{\langle n_-(r), (-\sin(\Theta_-(r)), \cos(\Theta_-(r)))\rangle}{q_-(r)},$$

Then,

$$F(r\cos\theta, r\sin\theta) := r\tilde{R}(r,\theta)\big(\cos\Theta(r,\theta), \sin\Theta(r,\theta)\big),$$

satisfies the required properties.

3. It remains to show that that $F \in C^{1,1}$. By Taylor expanding $\tilde{R}$ and Θ at $r = 0$,

$$F(r\cos\theta, r\sin\theta) = r(\cos(\theta), \sin(\theta)) + r^2\tilde{F}(r,\theta),$$

with $\tilde{F} \in C^\infty$ bounded with all bounded derivatives. Since $DF = I + O(r)$ and $D^2F = O(1)$, $F \in C^{1,1}$. □

Remark B.5. In fact, when $\alpha \neq \pi$, it is possible to find $F \in C^\infty(\overline{W_\alpha})$ mapping $\overline{W_\alpha}$ to $\overline{\Omega}$. However, when $\alpha = \pi$ there is an obvious obstruction to this if $\partial\Omega \notin C^\infty$ since $\partial\mathcal{W}_\pi \in C^\infty$.

Lemma B.6. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be two uniformly Lipschitz domains, with $\{z = 0\} \subset \partial\Omega_1 \cap \partial\Omega_2$. Let $F : \Omega_1 \to \Omega_2$ be a bi-Lipschitz map such that $F(0) = 0$. Then, for every $r_0 \geq 0$, we have*

$$\sqrt{\|\det DF\|_{L^\infty}\|\det DF^{-1}\|_{L^\infty}}\|DF\|_{L^\infty}\||F^{-1}(z)|/|z|\|_{L^\infty}\mathfrak{r}_{\Omega_2}^{r_0} \geq \mathfrak{r}_{\Omega_1}^{r_0\||F^{-1}(z)|/|z|\|_{L^\infty}}, \tag{B.8}$$

$$\sqrt{\|\det DF^{-1}\|_{L^\infty}\|\det DF\|_{L^\infty}}\|DF^{-1}\|_{L^\infty}\||F(z)|/|z|\|_{L^\infty}\mathfrak{r}_{\Omega_1}^{r_0} \geq \mathfrak{r}_{\Omega_2}^{r_0\||F(z)|/|z|\|_{L^\infty}}. \tag{B.9}$$

Proof. Let $u \in \dot{H}^1(\Omega_2)$ and $\tilde{r}_0 = \||F^{-1}(z)|/|z|\|_{L^\infty} r_0$. Then,

$$\begin{aligned}
\Big\|\frac{u(z)}{|z| + r_0}\Big\|^2_{L^2(\Omega_2)} &= \int_{\Omega_1} \frac{|u(F(\tilde{z}))|^2}{(|F(\tilde{z})| + r_0)^2}|\det DF(\tilde{z})|\, d\tilde{z} \\
&\leq \|\det DF\|_{L^\infty(\Omega_1)}\||F^{-1}(z)|/|z|\|^2_{L^\infty}\int_{\Omega_1}\frac{|u(F(\tilde{z}))|^2}{(|\tilde{z}| + \tilde{r}_0)^2}\, d\tilde{z} \\
&\leq (\mathfrak{r}_{\Omega_1}^{\tilde{r}_0})^{-2}\|\det DF\|_{L^\infty(\Omega_1)}\||F^{-1}(z)|/|z|\|^2_{L^\infty}\int_{\Omega_1}|\nabla(u\circ F)(\tilde{z})|^2\, d\tilde{z} \\
&\leq (\mathfrak{r}_{\Omega_1}^{\tilde{r}_0})^{-2}\|\det DF\|_{L^\infty(\Omega_1)}\||F^{-1}(z)|/|z|\|^2_{L^\infty}\int_{\Omega_1}|(\nabla u)(F(\tilde{z}))|^2|DF(\tilde{z})|^2\, d\tilde{z} \\
&\leq (\mathfrak{r}_{\Omega_1}^{\tilde{r}_0})^{-2}\|\det DF\|_{L^\infty(\Omega_1)}\|DF\|^2_{L^\infty(\Omega_1)}\||F^{-1}(z)|/|z|\|^2_{L^\infty}\int_{\Omega_2}|(\nabla u)(z)|^2|\det DF^{-1}(z)|\, dz \\
&\leq (\mathfrak{r}_{\Omega_1}^{\tilde{r}_0})^{-2}\|\det DF\|_{L^\infty(\Omega_1)}\|DF\|^2_{L^\infty(\Omega_1)}\|\det DF^{-1}\|_{L^\infty(\Omega_2)}\||F^{-1}(z)|/|z|\|^2_{L^\infty}\|\nabla u\|^2_{L^2(\Omega_2)}.
\end{aligned} \tag{B.10}$$

Hence, the lemma follows after reversing the roles of Ω_1 and Ω_2. □

We determine the Hardy constant on exact cones in any dimensions.

Lemma B.7. *Let $Y \subset \mathbb{S}^{d-1}$ be open with smooth boundary. Let the exact cone with smooth cross section Y, $W_Y = [0,\infty)_r \times Y \subset \mathbb{R}^d$ be equipped with the metric $dr^2 + r^2 g_{\mathbb{S}^{d-1}}$. We then have*

$$\mathfrak{r}^0_{W_Y} = \sqrt{\lambda_1(-\Delta_Y) + \frac{(d-2)^2}{4}}, \tag{B.11}$$

where $-\Delta_Y$ is the Dirichlet Laplace–Beltrami operator on Y with the round metric and λ_1 denotes the first (positive) eigenvalue. In particular, if $Y = (0,\alpha)$, then $\mathfrak{r}^0_{W_Y} = \frac{\pi}{\alpha}$.

Proof. 1. We show $\mathfrak{r}^0_{W_Y} \geq \sqrt{\lambda_1(-\Delta_Y) + \frac{(d-2)^2}{4}}$. Let (r, y) be the coordinates on W_Y. Let $u \in \dot{H}^1(W_Y)$. Consider $-\Delta_{W_Y} = \partial_r^* \partial_r - r^{-2}\Delta_Y$. This implies

$$\|\nabla u\|^2_{L^2(W_Y)} = \langle -\Delta_{W_Y} u, u\rangle_{L^2(W_Y)} = \|\partial_r u\|^2_{L^2(W_Y)} + \langle -r^{-2}\Delta_Y u, u\rangle_{L^2(W_Y)}. \tag{B.12}$$

We claim

$$\|\nabla u\|^2_{L^2(W_Y)} \geq \Big(\lambda_1(-\Delta_Y) + \frac{(d-2)^2}{4}\Big)\|r^{-1}u\|^2_{L^2(W_Y)}. \tag{B.13}$$

The second term of (B.12) can be bounded from below by

$$\begin{aligned}\langle -r^{-2}\Delta_Y u, u\rangle_{L^2(W_Y)} = \int_0^\infty r^{-2}\langle -\Delta_Y u, u\rangle_{L^2(Y)}\, r^{d-1} dr &\geq \lambda_1(-\Delta_Y)\int_0^\infty r^{-2}\|u\|^2_{L^2(Y)}\, r^{d-1} dr \\ &= \lambda_1(-\Delta_Y)\|r^{-1}u\|^2_{L^2(W_Y)}. \end{aligned}\tag{B.14}$$

The case $d = 2$ immediately follows from the non-negativity of the first term of (B.12). For $d > 2$, we first analyze the first term of (B.12). We note $\partial_r^* = -r^{-(d-1)}\partial_r r^{d-1} = -\partial_r - \frac{d-1}{r}$ gives a Rellich-type identity:

$$\langle \partial_r u, r^{-1}u\rangle_{L^2(W_Y)} = \langle u, \partial_r^*(r^{-1}u)\rangle_{L^2(W_Y)} = -\langle r^{-1}u, \partial_r u\rangle_{L^2(W_Y)} - (d-2)\|r^{-1}u\|^2_{L^2(W_Y)}. \tag{B.15}$$

This implies

$$\|r^{-1}u\|^2_{L^2(W_Y)} = -\frac{2}{d-2}\Re\langle \partial_r u, r^{-1}u\rangle_{L^2(W_Y)} \leq \frac{2}{d-2}\|\partial_r u\|_{L^2(W_Y)}\|r^{-1}u\|_{L^2(W_Y)} \tag{B.16}$$

and a Hardy inequality on W_Y:

$$\|r^{-1}u\|_{L^2(W_Y)} \leq \frac{2}{d-2}\|\partial_r u\|_{L^2(W_Y)}. \tag{B.17}$$

The claim now follows from (B.12).

2. We now show that $\mathfrak{r}^0_{W_Y} \leq \sqrt{\lambda_1(-\Delta_Y) + \frac{(d-2)^2}{4}}$. Let $\chi \in C_c^\infty((\frac{1}{2}, 4); [0, 1])$ with $\chi \equiv 1$ on $[1, 2]$ and let $\phi \in \dot{H}^1(Y) \cap H^2(Y)$ satisfy

$$(-\Delta_Y - \lambda_1(-\Delta_Y))\phi(y) = 0,\ \|\phi\|_{L^2(Y)} = 1. \tag{B.18}$$

For $\epsilon > 0$, define

$$u_\epsilon(r, y) = \beta_\epsilon^{-1}\chi(\epsilon \log r) r^{-\frac{d-2}{2}}\phi(y),\ \beta_\epsilon = \Big(\int_0^\infty r^{-1}\chi^2(\epsilon \log r)\, dr\Big)^{\frac{1}{2}} \geq \epsilon^{-\frac{1}{2}}, \tag{B.19}$$

normalized so that $\|r^{-1}u_\epsilon\|_{L^2(W_Y)} \equiv 1$. We compute

$$\partial_r u_\epsilon = \beta_\epsilon^{-1}\Big(\epsilon\chi'(\epsilon \log r) - \frac{d-2}{2}\chi(\epsilon \log r)\Big) r^{-\frac{d}{2}}\phi(y) = -\frac{d-2}{2} r^{-1}u_\epsilon + O_{L^2(W_Y)}(\epsilon). \tag{B.20}$$

Now we have

$$\|\nabla u_\epsilon\|^2_{L^2(W_Y)} = \lambda_1(-\Delta_Y)\|r^{-1}u_\epsilon\|^2_{L^2(W_Y)} + \|\partial_r u_\epsilon\|^2_{L^2(W_Y)} = \lambda_1(-\Delta_Y) + \frac{(d-2)^2}{4} + O(\epsilon). \tag{B.21}$$

Pushing $\epsilon \to 0$ saturates the claimed bound. □

Appendix C. DtN quasimodes for 2D AC smooth domains

Let Ω be a smooth AC Lipschitz domain and (x_0, x) denote Fermi normal coordinates in a neighborhood,

$$U = \{(x_0, x) \,:\, 0 \leq x_0 < c\langle x\rangle\}$$

of $\partial\Omega$ with x the arc length coordinates on $\partial\Omega$. Recall that, in these coordinates,

$$g_+(x_0, x) = dx_0^2 + [R(x_0, x)]^2 dx^2, \qquad R(x_0, x) := 1 + \kappa(x)x_0,$$

where $\kappa(x)$ denotes the curvature of $\partial\Omega$ and

$$|\partial_x^\alpha \kappa(x)| \leq C_\alpha \langle x\rangle^{-1-|\alpha|}$$

Then, in these coordinates,

$$-h^2\Delta + h^2M^2 = \big(h^2D_{x_0}^2 - iR^{-1}\partial_{x_0}Rh^2D_{x_0} + R^{-2}h^2D_x^2 + iR^{-1}\partial_x Rh^2D_x + h^2M^2\big),$$

and, conjugating away the volume form, we have

$$R^{1/2}(-h^2\Delta + h^2M^2)R^{-1/2} = h^2D_{x_0}^2 + (R^{-1/2}hD_xR^{-1/2})^2 + h^2M^2 - \frac{h^2\kappa^2}{4R^2}.$$

Lemma C.1. *Let $\chi, \chi_+ \in C_c^\infty(\mathbb{R})$ with $0 \notin \operatorname{supp}\chi_+$, $\operatorname{supp}\chi \cap \operatorname{supp}(1-\chi_+) = \emptyset$. Then there are b_0, b_1 such that, with*

$$p := i\sqrt{\xi^2R^{-2} + h^2M^2}\chi_+(\xi) + i\frac{h}{2R}\kappa(x) + h^2(hM^2b_0(x,\xi) + x_0b_1(x_0,x,\xi))\chi_+(\xi),$$

we have

$$R^{1/2}(-h^2\Delta + h^2M^2)R^{-1/2}\chi(hD_x) - (hD_{x_0} - p^w)(hD_{x_0} + p^w)\chi(hD_x) \in h^\infty\Psi^{-\infty,-\infty}(U).$$

Moreover,

$$\begin{aligned} b_0 &\sim \sum_j h^j b_{0,j}, \qquad b_{0,j} \in S^{\mathrm{comp},-1-j} \\ b_1 &\sim \sum_j h^j b_{1,j}, \qquad \partial_{x_0}^\ell b_{1,j} \in S^{\mathrm{comp},-1-j-\ell}. \end{aligned} \tag{C.1}$$

Proof. We proceed as in Section 3. Observe that, we require

$$\begin{aligned} &(h^2D_{x_0}^2 + (R^{-1/2}hD_xR^{-1/2})^2 + h^2M^2 - \frac{h^2\kappa^2}{4R^2})\chi(hD_x) \\ &= ((hD_{x_0})^2 - (ih\partial_{x_0}p)^w - (p^w)^2)\chi(hD_x) + h^\infty\Psi^{-\infty,-\infty}. \end{aligned}$$

Thus, our goal is to solve

$$((p^w)^2 + (ih\partial_{x_0}p)^w)\chi(hD_x) = (-(R^{-1/2}hD_xR^{-1/2})^2 - h^2M^2 + \frac{h^2\kappa^2}{4R^2})\chi(hD_x) + O(h^\infty)_{\Psi^{-\infty,-\infty}}.$$

Since $\operatorname{supp}(1-\chi_+) \cap \operatorname{supp}\chi = \emptyset$, we may ignore χ_+ in all of the below calculations at the cost of an $h^\infty\Psi^{-\infty,-\infty}$ error. We also do not track decay in ξ since every symbol will be compactly supported.

Recall that

$$(b_1^w b_2^w) = b_3^w, \qquad b_3 =: b_1 \sharp b_2 \sim \sum_j \frac{h^j i^j}{j!} (A(D))^j b_1(y,\eta) b_2(x,\xi)\Big|_{\substack{y=x\\ \eta=\xi}}, \qquad A(D) := \frac{1}{2}\sigma(D_y, D_\eta, D_x, D_\xi).$$

In particular, if $b_1 \sim \sum_j h^j b_{1,j}$ and $b_2 \sim \sum_j h^j b_{2,j}$

$$b_1 \sharp b_2 \sim \sum_j h^j \sum_{\ell=0}^{j} \sum_{m=0}^{\ell} \frac{i^m}{m!} (A(D))^m b_{1,j-\ell}(y,\eta) b_{2,\ell-m}(x,\xi)\Big|_{\substack{y=x\\ \eta=\xi}},$$

Therefore, if b_1 and b_2 satisfy (C.1), then so do $b_1 \sharp b_2$ and $h\partial_{x_0} b_1$. (Indeed, each power of h in the asymptotic series comes exactly one (x,y)- derivative). Hence, with $p_0 = i\sqrt{\xi^2 R^{-2} + h^2 M^2} + i\frac{h}{2R}\kappa$,

$$(p^w)^2 + (ih\partial_{x_0} p)^w = (p_0 \sharp p_0 + h^2(p_0 \sharp b + b \sharp p_0) + h^4 b \sharp b)^w + (ih\partial_{x_0} p_0 + ih^3 \partial_{x_0} b)^w.$$

Now,

$$\begin{aligned} p_0 \sharp p_0 \sim & -(\xi^2 R^{-2} + h^2 M^2) - \sqrt{\xi^2 R^{-2} + h^2 M^2} h R^{-1} \kappa - \tfrac{h^2\kappa^2}{4R^2} \\ & - \sum_{j\geq 1} \frac{h^{2j}(-1)^j}{(2j)! 2^{2j}} (D_\eta D_y - D_x D_\xi)^{2j} (\sqrt{\xi^2 R(x)^{-2} + h^2 M^2} + \tfrac{1}{2} h R(x)^{-1} \kappa(x)) \\ & \qquad \times (\sqrt{\eta^2 R(y)^{-2} + h^2 M^2} + \tfrac{1}{2} h R(y)^{-1} \kappa(y))|_{y=x,\eta=\xi} \\ \sim & -(\xi^2 R^{-2} + h^2 M^2) - \sqrt{\xi^2 R^{-2} + h^2 M^2} h R^{-1} \kappa - \tfrac{h^2\kappa^2}{4R^2} + h^2 \sum_j x_0 c_{1,j}(x_0, x, \xi) h^j + h^3 M^2 \sum_j c_{0,j}(x,\xi) h^j, \end{aligned} \tag{C.2}$$

with

$$|\partial_x^m c_{i,j}(x,\xi)| \leq C_m \langle x \rangle^{-2-j-m},$$

and

$$ih\partial_{x_0} p_0 = \frac{h\xi^2 (R^{-3}\kappa)}{\sqrt{\xi^2 R^{-2} + h^2 M^2}} + \frac{h^2\kappa^2}{2R^2} = h\sqrt{\xi^2 R^{-2} + h^2 M^2} R^{-1}\kappa - \frac{h^3 M^2 R^{-1}\kappa}{\sqrt{\xi^2 R^{-2} + h^2 M^2}} + \frac{h^2\kappa^2}{2R^2}.$$

So our goal is to solve

$$\begin{aligned} p_0 \sharp (hM^2 b_0 + x_0 b_1) + (hM^2 b_0 + x_0 b_1) \sharp p_0 + h^2 (hM^2 b_0 + x_0 b_1) \sharp (hM^2 b_0 + x_0 b_1) + ih\partial_{x_0}(x_0 b_1) \\ \sim -\sum_j x_0 c_{1,j} h^j - hM^2 \sum_j c_{0,j} + \frac{hM^2 R^{-1}\kappa}{\sqrt{\xi^2 R^{-2} + h^2 M^2}} \end{aligned}$$

Setting

$$\tilde{b}_0 := \frac{1}{2p_0(0,x,\xi)}\Big(-\sum_{j\geq 0} c_{0,j}h^j + \frac{R^{-1}\kappa}{\sqrt{\xi^2R^{-2}+h^2M^2}}\Big),$$
$$\tilde{b}_1 := \frac{1}{2p_0}(-\sum_{j\geq 0} c_{1,j}h^j + \frac{hM^2}{x_0}\Big(1-\frac{p_0(x_0,x,\xi)}{p_0(0,x,\xi)}\Big)\Big(\sum_{j\geq 0} c_{0,j}h^j + \frac{R^{-1}\kappa}{\sqrt{\xi^2R^{-2}+h^2M^2}}\Big),$$

we obtain that $\tilde{b}_i$ satisfy (C.1). Suppose by induction that $\tilde{b}_{K-1,i}$ satisfies (C.1) and, with $\tilde{b}_{K-1} = x_0\tilde{b}_{K-1,1} + hM^2\tilde{b}_{K-1,0}$,

$$p_0\sharp\tilde{b}_{K-1}+\tilde{b}_{K-1}\sharp p_0+h^2\tilde{b}_{K-1}\sharp\tilde{b}_{K-1}+ih\partial_{x_0}\tilde{b}_{K-1}+\sum_j x_0c_{1,j}h^j+hM^2\sum_j c_{0,j}-\frac{hM^2R^{-1}\kappa}{\sqrt{\xi^2R^{-2}+h^2M^2}}$$
$$= h^{K+1}M^2e_{K,0}+x_0h^Ke_{K,1}.$$

with $e_{K,i}\in S^{\mathrm{comp},-1-K}$. Define

$$\tilde{b}_{K,0} := \tilde{b}_{K-1,0} - \frac{h^K}{2p_0(0,x,\xi)}e_{K,0},$$
$$\tilde{b}_{K,1} := \tilde{b}_{K-1,1} - \frac{h^K}{2p_0}\Big(e_{K,1}+\frac{hM^2}{x_0}\Big(1-\frac{p_0(x_0,x,\xi)}{p_0(0,x,\xi)}\Big)e_{K,0}\Big).$$

Then, since $x_0^{-1}(p_0(0,x,\xi)-p_0(x_0,x,\xi)) = \kappa(x)e(\kappa(x)x_0,\kappa(x),\xi)$ for some smooth function e, $\tilde{b}_{K,i}$ satisfies (C.1) and

$$p_0\sharp\tilde{b}_K+\tilde{b}_K\sharp p_0+h^2\tilde{b}_K\sharp\tilde{b}_K+ih\partial_{x_0}\tilde{b}_K+\sum_j x_0c_{1,j}h^j+hM^2\sum_j c_{0,j}-\frac{hM^2R^{-1}\kappa}{\sqrt{\xi^2R^{-2}+h^2M^2}}$$
$$= h^{K+2}M^2e_{K+1,0}+x_0h^{K+1}e_{K+1,1},$$

with $e_{K+1,i}$ satisfying (C.1) and hence the proof of (C.1) is complete after using the Borel summation lemma to set $b_\ell \sim \tilde{b}_{\ell,0}+\sum_j \tilde{b}_{\ell,j}-\tilde{b}_{\ell,j-1}$.

□

Lemma C.2. *Let $\chi\in C_c^\infty(\mathbb{R}\setminus\{0\})$ with $0\notin\operatorname{supp}\chi$. Then, with p as in Lemma C.1*

$$h\Lambda_M^\Omega\chi(hD_x) = -i(p^w-\tfrac{ih}{2}\kappa(x))\chi(hD_x)+E_M^\Omega,$$

where

$$\|E_M^\Omega\|_{H_h^{-N,-N}\to H_h^{N,\mathfrak{r}^\Omega+\frac{1}{2}-0}} \leq C_Nh^N,$$

and for any $\psi\in C_c^\infty(\mathbb{R})$ with $0\notin\operatorname{supp}(1-\psi)$,

$$\|(1-\psi(D_x))E_M^\Omega\|_{H_h^{-N,-N}\to H_h^{N,N}} \leq C_Nh^N.$$

Proof. In this proof, we denote by $\Psi_T^{-\infty,-\infty}$ the tangential residual class and $\Psi^{-\infty,-\infty}$, the full residual class. Let $\chi_0 = \chi$ and $\chi_i \in C_c^\infty(\mathbb{R}\setminus\{0\})$, $i = 1, 2$ with $\operatorname{supp}(1-\chi_i)\cap\operatorname{supp}\chi_{i-1} = \emptyset$. Let $U(x_0)$ satisfy,

$$(hD_{x_0} - p^w - i(1-\chi_+(hD))\langle hD\rangle)U(x_0) = 0, \qquad U(0) = I. \tag{C.3}$$

We first claim that

$$(1-\chi_1(hD_x))U(x_0)\chi(hD_x) = O(e^{-cx_0/h}h^\infty)_{\Psi_T^{-\infty,-\infty}}. \tag{C.4}$$

First, notice that

$$\begin{aligned} h\partial_{x_0}\|U(x_0)f\|_{L^2}^2 &= -2\Im\langle hD_{x_0}U(x_0)f, U(x_0)f\rangle \\ &= -2\Im\langle (p^w + i(1-\chi_+(hD))\langle hD\rangle)U(x_0)f, U(x_0)f\rangle \\ &\le -c\|U(x_0)f\|_{L^2}^2. \end{aligned}$$

Hence,

$$\|U(x_0)f\|_{L^2} \le Ce^{-cx_0/h}\|f\|_{L^2}.$$

Furthermore, $a = a(x,\xi) \in S^{m,n}$, and $b \in S^{0,0}$ with $\operatorname{supp}(1-b)\cap\operatorname{supp} a = \emptyset$, real valued,

$$\begin{aligned} h\partial_{x_0}\|a^wU(x_0)f\|_{L^2}^2 &= -2\Im\langle a^whD_{x_0}U(x_0)f, a^wU(x_0)f\rangle \\ &= -2\Im\langle a^w(p^w + i(1-\chi_+(hD))\langle hD\rangle)U(x_0)f, a^wU(x_0)f\rangle \\ &= -2\Im\langle (p^w + i(1-\chi_+(hD))\langle hD\rangle)a^wU(x_0)f, a^wU(x_0)f\rangle \\ &\quad - 2\Im\langle a^w[a^w, (p^w + i(1-\chi_+(hD))\langle hD\rangle)]U(x_0)f, U(x_0)f\rangle \\ &\le -c\|a^wU(x_0)f\|_{L^2}^2 + Ch^2\|b^wU(x_0)f\|_{H_h^{m-1,n-1}}^2 + C_Nh^N\|U(x_0)f\|_{H_h^{-N,-N}}. \end{aligned}$$

We first use this estimate by letting $a \in \Psi^{m,n}$ elliptic and iterating to obtain

$$\|U(x_0)f\|_{H_h^{m,n}} \le Ce^{-cx_0/h}\|f\|_{H_h^{m,n}}.$$

Next, for any $a \in S^{m,n}$ and $b \in S^{0,0}$ with $\operatorname{supp}(1-b)\cap\operatorname{supp} a = \emptyset$, we obtain

$$\begin{aligned} \|e^{\frac{cx_0}{2h}}a^wU(x_0)f\|_{L^2}^2 &= \|a^wf\|_{L^2}^2 + \int_0^{x_0}\partial_s e^{\frac{cs}{2h}}\|a^wU(s)f\|_{L^2}ds \\ &\le \|a^wf\|_{L^2}^2 + C\int_0^{x_0}h^2e^{\frac{cs}{2h}}\|b^wU(s)f\|_{H_h^{m-1,n-1}}^2ds + Ch^N\int_0^{x_0}e^{\frac{cs}{2h}}\|U(s)f\|_{H_h^{-N,-N}}ds \\ &\le \|a^wf\|_{L^2}^2 + C\int_0^{x_0}h^2e^{\frac{cs}{2h}}\|b^wU(s)f\|_{H_h^{m-1,n-1}}^2ds + Ch^N\|f\|_{L^2}. \end{aligned}$$

Another iteration shows that

$$\|a^wU(x_0)f\|_{L^2} \le Ce^{-cx_0/h}\big(\|a^wf\|_{L^2}^2 + ch^N\|f\|_{L^2}\big),$$

which implies (C.4).

Now letting $\chi_0 \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(1-\chi_0)\cap\operatorname{supp}\chi = \emptyset$, $\operatorname{supp}(1-\chi_+)\cap\operatorname{supp}\chi_0 = \emptyset$, and $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \equiv 1$ near 0, using (C.3) to convert derivatives in x_0 into derivatives in x, we have

$$\begin{aligned} (-h^2\Delta + h^2M^2)R^{-1/2}\psi(x_0/\langle x\rangle)U\chi(hD_x) &= (-h^2\Delta + h^2M^2)R^{-1/2}\chi_+(hD_x)\psi(x_0/\langle x\rangle)U\chi(hD_x) + O(h^\infty)_{\Psi^{-\infty,-\infty}} \\ &= O(h^\infty)_{\Psi^{-\infty,-\infty}}. \end{aligned}$$

Hence, solving away the error, we have

$$h\Lambda_M^\Omega \chi(hD_x) = -h\partial_{x_0} R^{-1/2} U(x_0)|_{x_0=0}\chi(hD_x) + E_M^\Omega,$$

where,

$$\|E_M^\Omega\|_{H_h^{-N,-N}\to H_h^{N,\mathfrak{r}^\Omega+\frac{1}{2}-0}} \leq C_N h^N$$

and for any $\psi \in C_c^\infty(\mathbb{R})$ with $0 \notin \operatorname{supp}(1-\psi)$

$$\|(1-\psi(D_x))E_M^\Omega\|_{H_h^{-N,-N}\to H_h^{N,N}} \leq C_N h^N.$$

This implies the lemma. □

Lemma C.3. *Let $\psi_0 \in C_c^\infty(\mathbb{R})$ with $0 \notin \operatorname{supp}(1-\psi_0)$. Then there is a solution, u to*

$$(h\Lambda_M^\Omega - \sqrt{\omega^2+M^2})e_\pm(s;h,\omega,M) = g_1 + M^2 g_2 + \psi_0(D_x)g_3,$$

such that $e_\pm(s;h,\omega,M) = e^{\pm i\omega x/h}(1+h^2M^2e_\pm(s))$ with $e_\pm \sim \sum_j h^j e_{j,\pm}$, with $e_{j,\pm}(0)=0$ for all j and

$$\begin{aligned} |\partial_x^\ell e_{j,\pm}(x)| &\leq C_\ell \langle x\rangle^{-1-\ell-j}, \qquad 0\leq j\leq 2,\ 1\leq \ell \\ |\partial_x^\ell e_{j,\pm}(x)| &\leq C_\ell(\langle x\rangle^{-1-\ell-j} + M^2\langle x\rangle^{-1-\ell}), \qquad 3\leq j,\ 1\leq \ell. \end{aligned} \tag{C.5}$$

and for all N

$$\|g_1\|_{H_h^{N,N}} + \|g_2\|_{H_h^{N,\frac{3}{2}-0}} + \|g_2\|_{H_h^{N,\mathfrak{r}^\Omega+\frac{1}{2}-0}} \leq C_N h^N.$$

Proof. We consider the case of $u = e^{ix\sqrt{1-h^2M^2}/h}(1+h^2M^2e_+(x))$, the other being similar and drop the $\pm$ from the notation to simplify the writing.

We start by constructing $f \in S$ such that $f(0)=1$, and

$$\begin{aligned} e^{-ix\sqrt{1-h^2M^2}/h}(p^w - i\tfrac{h}{2}\kappa - 1)e^{ix\sqrt{1-h^2M^2}h} f &= O(h^\infty)_{\mathcal{S}} + O(h^\infty M^2)_{S^{-2}}, \\ f &\sim 1 + h^2M^2\sum_{j\geq 0} h^j e_j(x), \end{aligned} \tag{C.6}$$

with e_j satisfying (C.5).

Indeed, observe that, by Lemma C.2

$$\begin{aligned} &e^{-is\sqrt{1-h^2M^2}/h}(-i(p^w - i\tfrac{h}{2}\kappa)-1)e^{ix\sqrt{1-h^2M^2}h} f \\ &= (\sqrt{(hD_x)^2+h^2M^2-1} + h^3M^2 b_0^w)e^{ix\sqrt{1-h^2M^2}/h} f + O(h^\infty)_{\mathcal{S}} \\ &\sim \sum_j \frac{h^j}{j!i^j}(D_wD_\xi)^j\Big[\Big(\sqrt{(\xi+\sqrt{1-h^2M^2})^2+h^2M^2} + h^3M^2 b_0(x-\tfrac{w}{2},\xi+\sqrt{1-h^2M^2})-1\Big)f(x-w))\Big]|_{w=\xi=0}, \end{aligned}$$

and hence, differentiating in s we have

$$\begin{aligned} &\sum_j \frac{h^j}{j!i^j} D_w^\ell (D_wD_\xi)^j\Big[\Big(\sqrt{(\xi+\sqrt{1-h^2M^2})^2+h^2M^2}+h^3M^2 b_0(x-\tfrac{w}{2},\xi+\sqrt{1-h^2M^2})-1\Big)(f-1)(x-w))\Big]|_{w=\xi=0} \\ &\qquad \sim -\sum_j \frac{h^{j+3}M^2}{j!i^j} D_w^\ell (D_wD_\xi)^j b_0(x-\tfrac{w}{2},\xi+\sqrt{1-h^2M^2})|_{w=\xi=0} \end{aligned} \tag{C.7}$$

Now, reorganizing the sum, we obtain, with $e := h^{-2}M^{-2}(f-1) \sim \sum_j h^j e_j$ and $e_m(0) = 0$

$$
\begin{aligned}
&- \sqrt{1-h^2M^2}D_x^{\ell+1}e_m(x)\\
&= -\sum_{j=0}^{m-1} \frac{1}{(j+2)!i^{j+2}} D_w^\ell (D_w D_\xi)^{j+2}\Big[\Big(\sqrt{(\xi+\sqrt{1-h^2M^2})^2+h^2M^2}-1\Big)e_{m-1-j}(x-w)|_{w=\xi=0}\\
&- \sum_{j=0}^{m-3} \frac{M^2}{j!i^j} D_w^\ell (D_w D_\xi)^{j}\Big[\Big(b_0(x-\tfrac{w}{2},\xi+\sqrt{1-h^2M^2})-1\Big)e_{m-j-3}(x-w))|_{w=\xi=0}\\
&- \frac{1}{m!i^m} D_w^\ell (D_w D_\xi)^m b_0(x-\tfrac{w}{2},\xi+\sqrt{1-h^2M^2})|_{w=\xi=0}.
\end{aligned}
$$

We now proceed by induction. Observe that, since $b_0 \in S^{-2}$

$$|\partial_x^\alpha e_0| \leq C_\alpha \langle x\rangle^{-1-|\alpha|},\ |\alpha| \geq 1.$$

Now, it is easy to see that for $m \leq 2$,

$$|\partial_x^\alpha e_m| \leq C_\alpha \langle x\rangle^{-1-m-|\alpha|}, \quad |\alpha| \geq 1$$

Suppose by induction that for $0 \leq m \leq N$,

$$|\partial_x^\alpha e_m| \leq C_\alpha (\langle x\rangle^{-1-m-|\alpha|} + M^2\langle x\rangle^{-1-|\alpha|}), \quad |\alpha| \geq 1,$$

Then, using (C.7), we see that this holds for $0 \leq m \leq N+1$ and hence for all m by induction.

Using the Borel summation lemma, we have thus proved (C.6) and (C.5).

Now, to conclude the proof, observe that for $\chi \in C_c^\infty(\mathbb{R})$ with $0 \notin \operatorname{supp}\chi$ and $1 \notin \operatorname{supp}(1-\chi)$, $(1-\chi(hD_x))f = O(h^\infty)_{\mathcal{S}}$, and hence by Lemma C.2

$$
\begin{aligned}
(h\Lambda_M^\Omega - 1)f &= (-i(p^w - i\tfrac{ih\kappa}{2}) - 1)f + O(h^\infty)_{\mathcal{S}} + E_M^\Omega f\\
&= g = O(h^\infty)_{\mathcal{S}} + e^{ix\sqrt{1-M^2h^2}/h} O(h^\infty M^2)_{S^{-2}} + \psi_0(D_x)O(h^\infty)_{H_h^{N,\mathfrak{r}^\Omega+\frac12-0}}.
\end{aligned}
$$

□

Appendix D. Explicit scattering matrices on 2D straight wedges

To determine the scattering matrix on an infinite wedge, we use parts of [LPPS22a, Appendix A] which in turn relies heavily on [Pet50]. Given incoming data and $0 < \alpha < 2\pi$, we have shown that there is a unique solution $u \in Z^{\infty,\infty,-1+}_{-\frac{\pi}{\alpha}-\frac12+,-\frac12-,-\frac12-}$ to

$$(\Lambda_{\mathcal{W}_\alpha} - 1)u = 0, \tag{D.1}$$

with that incoming data. Thus, since there is a two dimensional family of incoming data, it is enough to find two linearly independent solutions to (D.1) and compute their asymptotic data.

D.1. **Solutions of the sloshing problem.** To find two linearly independent solutions of (D.1), we first construct solutions $v \in L^\infty(\mathcal{W}_{\alpha/2})$ of the two problems

$$\begin{cases} -\Delta v = 0 & \text{in } \mathcal{W}_{\alpha/2} = \{z \in \mathbb{Z} \,:\, -\alpha/2 < \arg z < 0\} \\ \partial_y v(s) = v(s) & \text{for } s > 0 \\ v(se^{-i\alpha/2}) = 0 & \text{for } s > 0. \end{cases} \tag{D.2}$$

and

$$\begin{cases} -\Delta v = 0 & \text{in } \mathcal{W}_{\alpha/2} = \{z \in \mathbb{Z} \,:\, -\alpha/2 < \arg z < 0\} \\ \partial_y v(s) = v(s) & \text{for } s > 0 \\ \partial_{\nu_{\alpha/2}} v(se^{-i\alpha/2}) = 0 & \text{for } s > 0, \end{cases} \tag{D.3}$$

where $\nu_{\alpha/2}$ is the normal vector to $\arg z = -\alpha/2$.

Remark D.1. The arguments in [LPPS22a, Appendix A] apply directly for $0 < \alpha < \pi$, but not for $\pi \leq \alpha < 2\pi$. Therefore, we recall the parts of that appendix which we require here.

Let $\mathcal{C}$ denote the logarithmic cover of $\mathbb{C} \setminus \{0\}$. Define (at first I_α is defined for $\zeta > 0$)

$$g_\alpha(\zeta) := \exp\Big(- I_\alpha(\zeta e^{-i\alpha/2}) + \log\Big(\frac{\zeta + i}{\zeta}\Big)\Big), \qquad I_\alpha(\zeta) := \frac{1}{\pi}\int_0^\infty \log(1 + w^{-2\pi/\alpha})\frac{\zeta}{w^2 + \zeta^2} dw. \tag{D.4}$$

We show below (following [Pet50]) that I_α has an analytic continuation to the logarithmic cover of

$$\mathcal{C} \setminus \big(\bigcup_{j=0}^\infty e^{-i(\pi + (2j+1)\alpha)/2}[0,1] \cup \bigcup_{j=0}^\infty e^{i(\pi + (2j+1)\alpha)/2}[0,1]\big).$$

Remark D.2. In fact, it is not hard to see that it has a meromorphic continuation to a larger Riemann surface where each point $e^{\pm i(\pi + (2j+1)\alpha)/2}$ is a branch point.

Following [LPPS22b, Pet50] we will show that $v_d := \Re f_d(z)$ and $v_n = \Re f_n(z)$ satisfy respectively (D.2) and (D.3) with

$$f_d(z) := i\int_P \frac{g_\alpha(\zeta)}{\zeta + i}\zeta^{-\mu} e^{z\zeta} d\zeta, \qquad f_n(z) := i\int_P \frac{g_\alpha(\zeta)}{\zeta + i} e^{z\zeta} d\zeta$$

where $\mu := \frac{\pi}{\alpha}$, P is a contour consisting of the union of a circle of radius 2 traversed counter clockwise and the two half lines

$$\Gamma_{out} := \{s \in (2, \infty) \,:\, se^{i(\pi + \alpha/4)}\}, \quad \Gamma_{in} := \{s \in (\infty, 2) \,:\, se^{i(-\pi + \alpha/4)}\}.$$

D.2. **Meromorphic continuation of I_α to the logarithmic cover .**

Lemma D.3. *The function $I_\alpha(\zeta)$ defined by* (D.4) *analytically continues to*

$$\mathcal{C} \setminus \big(\bigcup_{j=0}^\infty e^{-i(\pi + (2j+1)\alpha)/2}[0,1] \cup \bigcup_{j=0}^\infty e^{i(\pi + (2j+1)\alpha)/2}[0,1]\big).$$

Moreover, the function

$$\tilde{g}(\zeta) := \exp(I_\alpha(\zeta))$$

is meromorphic on $\mathcal{C}$ and its poles are given by $\{\lambda_k\}_{k=-\infty}^{\infty}$ with

$$\lambda_k = e^{-i(\pi/2-k\alpha)},\ k \le 0, \qquad \lambda_k = e^{i(\pi/2+k\alpha)},\ k \ge 1. \tag{D.5}$$

Proof. Using that $\frac{\zeta}{v^2+\zeta^2} = \frac{1}{2i}(\frac{1}{v-i\zeta} - \frac{1}{v+i\zeta})$ and changing coordinates with $v = \pm iu$, it is easy to check that,

$$I_\alpha(\zeta) = \frac{1}{2\pi i}\int_L \log(1+(-iu)^{-2\mu})\frac{1}{u-\zeta}\,du + \frac{1}{2\pi i}\int_M \log(1+(iu)^{-2\mu})\frac{1}{u-\zeta}\,du, \tag{D.6}$$

where $L := i(\infty, 0)$ and $M := -i(0, \infty)$.

We notice that (D.6) represents $\zeta \mapsto I_\alpha(\zeta)$ in the half plane $-\pi/2 < \arg\zeta < \pi/2$ since it defines an analytic function on $-\pi/2 < \arg\zeta < \pi/2$ which coincides with I_α when $\zeta > 0$.

Fix $\theta_1 < \theta_2 \in \mathbb{R}$. Then, define

$$C_R := \{u;\ \arg u \in (\theta_1, \theta_2),\ |u| = R\}$$

and

$$\Gamma^\theta_{out,R} := e^{i\theta}(R, \infty), \qquad \Gamma^\theta_{in,R} := e^{i\theta}(\infty, R).$$

Notice that for any θ_1, θ_2,

$$\lim_{R\to\infty}\int_{C_R} \log(1+(\pm iu)^{-2\mu})\frac{1}{u-\zeta}\,du = 0$$

Also, $\log(1+(-iu)^{-2\mu})$ is analytic on the logarithmic cover of

$$\mathbb{C} \setminus \Big(\{0\} \cup \bigcup_{k\in\mathbb{Z}} \{e^{i(\pi+(2k+1)\alpha)/2}\}\Big),$$

since

$$(-iu)^{-2\mu} = (-iu)^{-2\mu} = -1,$$

is equivalent to

$$-2\mu(\log(u) - i\pi/2) = -i(\pi + 2k\pi), \quad k \in \mathbb{Z}.$$

Hence, since $u \mapsto \frac{\log(1+(-iu)^{-2\mu})}{u-\zeta}$ has no singularity in the sector $\pi/2 < \arg\zeta < \pi/2 + \alpha/2$,

$$-\int_{\Gamma^{(\pi+\alpha)/2}_{out,0}} \log(1+(-iu)^{-2\mu})\frac{1}{u-\zeta}\,du = \int_L \log(1+(-iu)^{-2\mu})\frac{1}{u-\zeta}\,du.$$

Therefore, for $\arg\zeta \in (-\pi/2, \pi/2 + \alpha/2)$,

$$\zeta \mapsto \frac{i}{2\pi}\int_{\Gamma^{(\pi+\alpha)/2}_{out,0}} \log(1+(-iu)^{-2\mu})\frac{1}{u-\zeta}\,du + \frac{1}{2\pi i}\int_M \log(1+(iu)^{-2\mu})\frac{1}{u-\zeta}\,du,$$

is an analytic function which coincides with $\zeta \mapsto I_\alpha(\zeta)$ when $|\arg\zeta| < \pi/2$ and, hence, it is the holomorphic continuation of $\zeta \mapsto I_\alpha(\zeta)$ to $\arg\zeta \in (-\pi/2, \pi/2 + \alpha/2)$. In a similar way we can shift the path M into $M_1 := \Gamma^{-(\pi+\alpha)/2}_{in,0}$ in (D.6) since $u \mapsto \log(1+(iu)^{-2\mu})$ has singularities at 0 and $e^{i(-\pi+(2k+1)\alpha)/2}$.

To continue I_α further, we repeat the process of shifting, this time taking care of where the singularities of the integrand are and computing the relevant residues. For instance, if $-\pi/2 - 3\alpha/2 < \theta < -\pi/2 - \alpha/2$, and if $-\pi/2 - \alpha/2 < \arg\zeta < \theta + 2\pi$ then

$$\frac{1}{2\pi i}\int_{\Gamma_{in,0}^{-(\pi+\alpha)/2}} \log(1+(iu)^{-2\mu})\frac{du}{u-\zeta} = \frac{1}{2\pi i}\int_{\Gamma_{in,0}^{\theta}} \log(1+(iu)^{-2\mu})\frac{du}{u-\zeta} + \frac{1}{2\pi i}\int_{\Gamma} \log(1+(iu)^{-2\mu})\frac{du}{u-\zeta},$$

with Γ being the loop path around the branch cut from $e^{-i(\pi/2+\alpha)}$ to 0 (that is, $\Gamma := e^{-i(\pi/2+\alpha/2-0)}(0,1) \cup (-e^{-i(\pi/2+\alpha/2+0)})(-1,0)$) and one computes

$$\frac{1}{2\pi i}\int_{\Gamma} \log(1+(iu)^{-2\mu})\frac{du}{u-\zeta} = \log\Big(\frac{\zeta - e^{-i(\pi/2+\alpha/2)}}{\zeta}\Big).$$

Hence, for $\arg\zeta \in (-\pi/2 - 3\alpha/2, \pi/2 + \alpha/2)$, we have

$$\begin{aligned} I_\alpha(\zeta) =& \frac{i}{2\pi}\int_{\Gamma_{out,0}^{\pi/2}} \log(1+(-iu)^{-2\mu})\frac{du}{u-\zeta} \\ &+ \frac{1}{2\pi i}\int_{\Gamma_{in,0}^{-(\pi+3\alpha)/2}} \log(1+(iu)^{-2\mu})\frac{du}{u-\zeta} + \log\Big(\frac{\zeta - e^{-i(\pi/2+\alpha/2)}}{\zeta}\Big), \end{aligned}$$

Repeating this argument we obtain for $\arg\zeta \in (-3\pi/2, \pi/2)$,

$$\begin{aligned} I_\alpha(\zeta) =& \frac{i}{2\pi}\int_{\Gamma_{out,0}^{\pi/2}} \log(1+(-iu)^{-2\mu})\frac{du}{u-\zeta} \\ &+ \frac{i}{2\pi}\int_{\Gamma_{in,0}^{-3\pi/2}} \log(1+(iu)^{-2\mu})\frac{du}{u-\zeta} + \sum_{j=0}^{\lceil \pi/\alpha - \frac{3}{2}\rceil} \log\Big(\frac{\zeta - e^{-i(\pi/2+(2j+1)\alpha/2)}}{\zeta}\Big), \end{aligned} \tag{D.7}$$

To extend I_α further, we can now repeat the argument, rotating Γ_{in} and Γ_{out} in the logarithmic cover of $\mathbb{C}\setminus\{0\}$.

We will also use the following formula for $\arg\zeta \in (-\pi/2, 3\pi/2)$,

$$\begin{aligned} I_\alpha(\zeta) =& \frac{i}{2\pi}\int_{\Gamma_{out,0}^{3\pi/2}} \log(1+(-iu)^{-2\mu})\frac{du}{u-\zeta} \\ &+ \frac{i}{2\pi}\int_{\Gamma_{in,0}^{-\pi/2}} \log(1+(iu)^{-2\mu})\frac{du}{u-\zeta} + \sum_{j=0}^{\lceil \pi/\alpha - \frac{3}{2}\rceil} \log\Big(\frac{\zeta - e^{i(\pi/2+(2j+1)\alpha/2)}}{\zeta}\Big), \end{aligned} \tag{D.8}$$

The process of rotating the contour can be continued to obtain (D.5). □

It will be useful below to also record for $-\pi/2 - \alpha/2 < \arg\zeta < \pi/2 - \alpha/2$

$$\begin{aligned} I_\alpha(\zeta e^{-i\alpha/2}) =& \frac{i}{2\pi}\int_{\Gamma_{out,0}^{\pi/2-\alpha}} \log(1+(-iu)^{-2\mu})\frac{du}{u-\zeta e^{-i\alpha/2}} \\ &+ \frac{i}{2\pi}\int_{\Gamma_{in,0}^{-\pi/2-\alpha}} \log(1+(iu)^{-2\mu})\frac{du}{u-\zeta e^{-i\alpha/2}} + \log\Big(\frac{\zeta+i}{\zeta}\Big) - \log\Big(\frac{\zeta-i}{\zeta}\Big), \end{aligned} \tag{D.9}$$

Lemma D.4. *For all $M > 0$, there is $C > 0$ such that*

$$|I_\alpha(\zeta)| \le \frac{C}{|\zeta|}, \qquad -M < \arg\zeta < M, |\zeta| > 2.$$

Proof. The logarithmic terms appearing, for example in (D.7) are bounded by $C/|\zeta|$ for $|\zeta| > 1 + \frac{1}{M}$. Therefore, we only need to consider the integrals. Notice that for $|\arg\zeta - \theta| > \epsilon$, using that $2\mu > 1$,

$$\begin{aligned}
&\Big| \int_{\Gamma^\theta_{out}} \log(1 + (-iu)^{-2\mu}) \frac{du}{u-\zeta} \Big| \\
&\le C \int_1^\infty (\log|1 - t^{-1}|)) \frac{dt}{|e^{i\theta}t - \zeta|} + C \int_0^1 (1 + |\log|1 - t^{-1}|) \frac{dt}{|e^{i\theta}t - \zeta|} \le \frac{C_\epsilon}{|\zeta|}.
\end{aligned}$$

A similar argument for the integral over Γ^θ_{in} proves the claim. □

Lemma D.5. *There is $C > 0$ such that*

$$|I_\alpha(\zeta) + \mu \log\zeta| < C, \quad -3\pi/2 < \arg\zeta < 3\pi/2, \, |\zeta| < \frac{1}{2}.$$

Proof. We consider $-\frac{3\pi}{2} < \arg\zeta < \frac{\pi}{2}$. The other case being similar. Using (D.7) it is enough to study the function

$$\tilde{I}_\theta(\zeta) := \frac{i}{2\pi} \int_{\Gamma^{\pi/2}_{out,0}} \log(1 + (-iu)^{-2\mu}) \frac{du}{u-\zeta} + \frac{i}{2\pi} \int_{\Gamma^{-3\pi/2}_{in,0}} \log(1 + (iu)^{-2\mu}) \frac{du}{u-\zeta}.$$

For this, we argue as in Lemma D.4 to see that the integrals along $\Gamma^{-3\pi/2}_{in,3/4}$ and $\Gamma^{\pi/2}_{out,3/4}$ are bounded for $|\zeta| < \frac{1}{2}$. Therefore, it is enough to consider the remaining integral which is given by

$$\begin{aligned}
&\frac{i}{2\pi} \int_0^{\frac{3}{4}} \Big(\frac{i}{it-\zeta} \log(1 + t^{-2\mu}) - \frac{i}{it-\zeta} \log(1 + (e^{-i\pi}t)^{-2\mu}) \Big) dt \\
&= \frac{i}{2\pi} \int_0^{\frac{3}{4}} \Big(\log(1 + t^{-2\mu}) - \log(1 + e^{2i\mu\pi}t^{-2\mu}) \Big) \frac{i}{it-\zeta} dt \\
&= \frac{i}{2\pi} \int_0^{\frac{3}{4}} \Big(\log\Big(e^{-2i\mu\pi} \frac{1 + t^{2\mu}}{1 + e^{2i\mu\pi}t^{2\mu}} \Big) \frac{i}{it-\zeta} dt \\
&= \frac{i}{2\pi} \log e^{-2i\mu\pi} \int_0^{\frac{3}{4}} \frac{i}{it-\zeta} dt + \frac{i}{2\pi} \int_0^{\frac{3}{4}} \log\Big(\frac{1 + t^{2\mu}}{1 + e^{2i\mu\pi}t^{2\mu}} \Big) \frac{i}{it-\zeta} dt \\
&= \frac{i}{2\pi} \log e^{-2i\mu\pi} \log\Big(\frac{\frac{3}{4} + i\zeta}{i\zeta} \Big) + O(1) \\
&= -\frac{i}{2\pi} \log e^{-2i\mu\pi} \log\zeta + O(1) \\
&= \big(-\mu + \lceil \tfrac{\pi}{\alpha} - \tfrac{3}{2} \rceil \big) \log\zeta + O(1).
\end{aligned}$$

Inserting this into (D.7) completes the proof. □

Lemma D.6. *For $0 < \alpha < 2\pi$, there is $r_\alpha > 0$ such that*

$$g_\alpha(-i) = r_\alpha e^{i\frac{\pi}{4}(1-\frac{\pi}{\alpha})}. \tag{D.10}$$

Proof. We compute using (D.9)

$$\begin{aligned}
&\log g_\alpha(-i)\\
&= -\frac{i}{2\pi}\int_{\Gamma^{\pi/2-\alpha}_{out,0}} \log(1+(-iu)^{-2\mu})\frac{du}{u+ie^{-i\alpha/2}} - \frac{i}{2\pi}\int_{\Gamma^{-\pi/2-\alpha}_{in,0}} \log(1+(iu)^{-2\mu})\frac{du}{u+ie^{-i\alpha/2}} + \log 2\\
&= \frac{1}{2\pi i}\int_0^\infty \log(1+t^{-2\mu})\Big(\frac{1}{t+e^{i\alpha/2}} - \frac{1}{t-e^{i\alpha/2}}\Big)dt + \log 2\\
&= \frac{i}{\pi}\int_0^\infty \log(1+t^{-2\pi/\alpha})\Big(\frac{e^{i\alpha/2}}{t^2-e^{i\alpha}}\Big)dt + \log 2\\
&= \frac{i}{\pi}\int_0^\infty \Big(\frac{2\pi}{\alpha}\log t + \log(1+t^{-2\pi/\alpha})\Big)\Big(\frac{e^{-i\alpha/2}}{t^2-e^{-i\alpha}}\Big)dt + \log 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
2\Im \log g_\alpha(-i) &= \frac{1}{\pi}\int_0^\infty \log(1+t^{-2\pi/\alpha})\Big(\frac{e^{i\alpha/2}}{t^2-e^{i\alpha}}\Big)dt - \frac{1}{\pi}\int_0^\infty \log(1+t^{-2\pi/\alpha})\Big(\frac{e^{-i\alpha/2}}{t^2-e^{-i\alpha}}\Big)dt\\
&= \frac{2}{\alpha}\int_0^\infty \log t\frac{e^{-i\alpha/2}}{t^2-e^{-i\alpha}}dt\\
&= \frac{2}{\alpha}\int_{e^{i(\pi/2-\alpha/2)}(0,\infty)} \log u\frac{e^{-i\alpha/2}}{u^2-e^{-i\alpha}}du\\
&= \frac{2}{\alpha}\int_0^\infty \big((\frac{\alpha-\pi}{2}) - i\log t\big)\frac{1}{t^2+1}dt\\
&= -\frac{\pi(\alpha-\pi)}{2\alpha}.
\end{aligned}$$

□

D.3. **Verification that v_d and v_n satisfy** (D.2) **and** (D.3)**.** .

D.3.1. *Harmonicity.* By Lemma D.4 and the fact that for ζ on the infinite part of the contour P, $\arg\zeta = \pi + \alpha/4$ and hence, since $\alpha < 2\pi$, for $-\alpha/2 \le \arg z \le 0$,

$$\Re(z\zeta) < -c_\alpha|z||\zeta|.$$

Thus, the integrand defining f_n and f_d is rapidly decaying in ζ for all $-\alpha/2 \le \arg z \le 0$, $z \neq 0$ and hence we may differentiate under the integral to see that f_d and f_n are holomorphic in $\mathcal{W}_{\alpha/2}$. In particular,

$$-\Delta v_d = -\Delta v_n = 0, \text{ in } \mathcal{W}_{\alpha/2}.$$

D.3.2. *Steklov boundary conditions.* to obtain that for $z \in (0, \infty)$,

$$(\partial_y - 1)v_d(z) = \Re(if_d'(z) - f_d(z)) = -\Re\Big(\int_P g_\alpha(\zeta)\zeta^{-\mu_\alpha}e^{z\zeta}d\zeta\Big),$$
$$(\partial_y - 1)v_n(z) = \Re(if_n'(z) - f_n(z)) = -\Re\Big(\int_P g_\alpha(\zeta)e^{z\zeta}d\zeta\Big).$$

By [Pet50, A.1]

$$I(\zeta e^{-i\alpha/2}) + \log\Big(\frac{\zeta + i}{\zeta}\Big) > 0$$

for $\zeta > 0$. Hence, $I_\alpha(\bar{\zeta}) = \overline{I_\alpha(\zeta)}$ and, using that I_α is analytic on the

$$\{|z| > 1\} \cap \mathcal{C},$$

where $\mathcal{C}$ denotes the logarithmic cover of $\mathbb{C} \setminus 0$, $I_\alpha(\zeta e^{-i\zeta})$ is bounded on $|\zeta| > 2$, and $|e^{z\zeta}| \le e^{-cz}$ for $z > 0$ and $-3\pi/2 < \arg\zeta < 3\pi/2$, may deform the contour P to

$$P_1 := \Gamma^{-\pi}_{in,2} \cup \{|z| = 2\} \cup \Gamma^{\pi}_{out,2},$$

and use that

$$g_\alpha(\bar{\zeta}) = \overline{g_\alpha(\zeta)}$$

to see that both integrals are pure imaginary and hence that

$$\partial_y v_d - v_d = \partial_y v_n - v_n = 0, \quad \text{on } z > 0.$$

D.3.3. *Neumann boundary conditions.* For the Neumann boundary condition, we change variables to $w = \zeta e^{-i\alpha/2}$ and deform the contour

$$\begin{aligned}
\partial_{\nu_{\alpha/2}} v_n(se^{-i\alpha/2}) &= \Re ie^{-i\alpha/2} f'(se^{-i\alpha/2}) \\
&= \Re\Big(-e^{-i\alpha/2}\int_P \frac{g_\alpha(\zeta)\zeta}{\zeta + i}e^{se^{-i\alpha/2}\zeta}d\zeta\Big) \\
&= \Re\Big(-e^{-i\alpha/2}\int_P \exp(-I_\alpha(\zeta e^{-i\alpha/2}))e^{se^{-i\alpha/2}\zeta}d\zeta\Big) \\
&= \Re\Big(-\int_P \exp(-I_\alpha(w))e^{sw}dw\Big) \\
&= \Re\Big(-\int_{P_1} \exp(-I_\alpha(w))e^{sw}dw\Big).
\end{aligned}$$

Using that $I_\alpha(\bar{\zeta}) = \overline{I_\alpha(\zeta)}$ shows that the integral is pure imaginary and hence that v_n satisfies the required boundary conditions.

D.3.4. *Dirichlet boundary conditions.* Finally, for the Dirichlet boundary condition, we again change variables to $w = \zeta e^{-i\alpha/2}$ and deform the contour to

$$\begin{aligned} v_d(se^{-i\alpha/2}) &= \Re f(se^{-i\alpha/2}) \\ &= \Re\Big(i\int_P \frac{g_\alpha(\zeta)}{\zeta+i}\zeta^{-\mu_\alpha}e^{se^{-i\alpha/2}\zeta}d\zeta\Big) \\ &= \Re\Big(\int_{P_1}\exp(-I_\alpha(w))(w)^{-\mu_\alpha-1}e^{sw}dw\Big) = 0 \end{aligned}$$

Using again that the integrand satisfies $I_\alpha(\bar{w}) = \overline{I_\alpha(w)}$.

D.3.5. *Boundedness and asymptotics f_d and f_n.*

Lemma D.7. *Let $q \le \mu$ and define*

$$f_q(z) := i\int_P \frac{g_\alpha(\zeta)}{\zeta+i}\zeta^{-q}e^{z\zeta}d\zeta.$$

Then

$$\|f_q\|_{L^\infty(\mathcal{W}_{\alpha/2})} < \infty,$$

and

$$\lim_{x\to\infty}|f_q(x) + 2\pi g_\alpha(-i)(-i)^{-q}e^{-ix}| = 0.$$

Proof. For $-\alpha/2 \le \arg z \le 0$, using Lemmas D.4 and D.5, we may deform the contour P to

$$P_0 := \Gamma_{in,0}^{\pi+\theta} \cup \Gamma_{out,0}^{-\pi+\theta},$$

where $|\theta - \alpha/4|$ is small enough so that $e^{z\zeta}$ is exponentially decreasing on P_0 for all $z \in \mathcal{W}_{\alpha/2}$ and $\{e^{i(\pi+\theta)}, e^{i(-\pi+\theta)}\} \notin \{\lambda_k\}_{k=-\infty,\infty}$.

Accounting for the poles of g_α, we obtain

$$\begin{aligned} f_q(z) &= i\int_{P_0}\frac{g_\alpha(\zeta)}{\zeta+i}\zeta^{-q}e^{z\zeta}d\zeta - 2\pi g_\alpha(-i)(-i)^{-q}e^{-iz} \\ &\quad - 2\pi\sum_{-\pi+\theta<\arg\lambda_k<\pi+\theta} e^{z\lambda_m}\operatorname{Res}_{\lambda_k} g_\alpha(\zeta)\zeta^{-q}/(\zeta+i) \\ &:= R_1(z) - 2\pi g_\alpha(i)(i)^{-q}e^{iz} + R_2(z) \end{aligned}$$

It is easy to check that there are finitely many k with

$$-\pi+\theta < \arg\lambda_k < \pi+\theta,$$

and moreover, for such k, $z \in \overline{\mathcal{W}_{\alpha/2}}$, $\Re z\lambda_k < -c|z|$. This implies

$$|R_2(z)| \le Ce^{-c|z|}, \qquad z \in \mathcal{W}_{\alpha/2}.$$

Using Lemma D.5, it is also easy to check that for $|q| \le \mu$,

$$|R_1(z)| \le C, \qquad z \in \overline{\mathcal{W}_{\alpha/2}}.$$

Moreover, for $x > 0$ and $\zeta \in P_0$, $x \mapsto |e^{x\zeta}|$ is decreasing and $\lim_{x\to\infty} e^{x\zeta} = 0$ for almost every $\zeta \in P_0$. In particular, the dominated convergence theorem implies that

$$\lim_{x\to\infty} |R_1(x)| = 0.$$

this completes the proof of the lemma. □

D.4. **Determining the scattering matrix.**

Proof of Lemma D.8. To determine the scattering matrix in $\mathcal{W}_\alpha = \{-\alpha < \arg z < 0\}$, we set

$$v_1(z) := \begin{cases} v_d(z) & -\alpha/2 \le \arg z \le 0 \\ -v_d(ze^{2(-\arg z+\alpha/2)}) & -\alpha \le \arg z \le \alpha/2, \end{cases}$$

$$v_2(z) := \begin{cases} v_n(z) & -\alpha/2 \le \arg z \le 0 \\ v_n(ze^{2(-\arg z+\alpha/2)}) & -\alpha \le \arg z \le \alpha/2. \end{cases}$$

Then, since v_d and v_n satisfy respectively (D.2) and (D.3), we for $i = 1, 2$

$$-\Delta v_i = 0 \text{ in } \mathcal{W}_\alpha, \qquad \partial_\nu v_i - v_i = 0 \text{ on } \partial\mathcal{W}_\alpha.$$

Moreover, since $v_i \in L^\infty$,

$$\partial_\nu v_i|_{\mathcal{W}_\alpha} = \Lambda_{\mathcal{W}_\alpha} v_i|_{\partial\mathcal{W}_\alpha}.$$

Hence, setting $u_i := v_i|_{\mathcal{W}_\alpha}$, we have

$$(\Lambda_{\mathcal{W}_\alpha} - 1)u_i = 0.$$

Therefore, to compute the scattering matrix, we only need to use Lemma D.7. Indeed,

$$\lim_{x\to\infty} |v_n(x) + 2\pi\Re(g_\alpha(-i))e^{-ix})| = 0.$$

Notice that by (D.10)

$$\Re g_\alpha(-i)e^{-ix} = \tfrac{r_\alpha}{2}(e^{i(\frac{\pi}{4}(1-\frac{\pi}{\alpha})-x)} + e^{-i(\frac{\pi}{4}(1-\frac{\pi}{\alpha})-x)})$$

Therefore,

$$\Sigma_{0,1}^{\mathcal{W}_\alpha} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -ie^{i\frac{\pi^2}{2\alpha}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A similar computation using (D.10) again shows that

$$\Re g_\alpha(-i)(-i)^{-\mu}e^{-ix} = \tfrac{r_\alpha}{2}(e^{i(\frac{\pi}{4}(1+\frac{\pi}{\alpha})-x} + e^{-i(\frac{\pi}{4}(1+\frac{\pi}{\alpha})-x)}).$$

Thus

$$\Sigma_{0,1}^{\mathcal{W}_\alpha} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -ie^{-i\frac{\pi^2}{2\alpha}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

In particular,

$$\Sigma_{0,1}^{\mathcal{W}_\alpha} = \begin{pmatrix} -i\cos(\frac{\pi^2}{2\alpha}) & \sin(\frac{\pi^2}{2\alpha}) \\ \sin(\frac{\pi^2}{2\alpha}) & -i\cos(\frac{\pi^2}{2\alpha}). \end{pmatrix}$$

as claimed. □

Lemma D.8. *The scattering matrix on the infinite wedge is given by*

$$\Sigma_{0,1}^{\mathcal{W}_\alpha} = \begin{pmatrix} -i\cos(\frac{\pi^2}{2\alpha}) & \sin(\frac{\pi^2}{2\alpha}) \\ \sin(\frac{\pi^2}{2\alpha}) & -i\cos(\frac{\pi^2}{2\alpha}). \end{pmatrix} \tag{D.11}$$

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