Between combinatorics and analysis

(with a little help from statistical physics)

Alan Sokal

UCL Inaugural Lecture

10 March 2010

Coloringss of graphs

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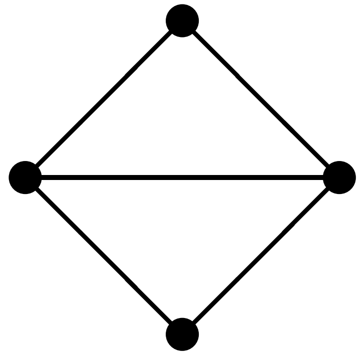

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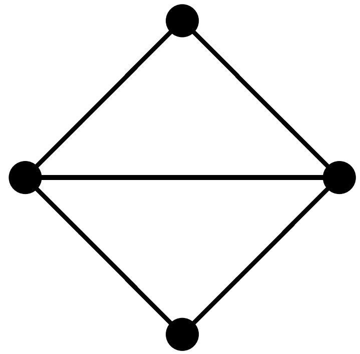
- Let G be a (finite) graph e.g. 

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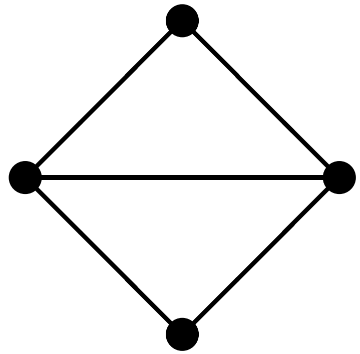
- Let G be a (finite) graph e.g. 

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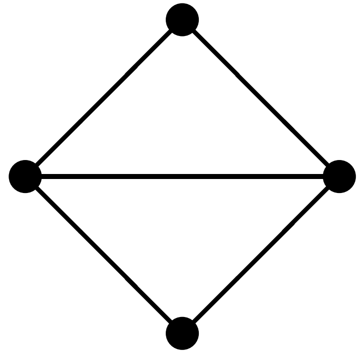
- Let G be a (finite) graph e.g. 

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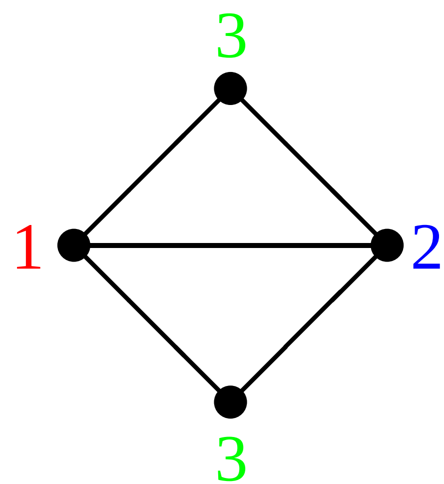
e.g. 

Some applications of graph coloring

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Sudoku	boxes of 9×9 grid	numbers $1, \ldots, 9$	same row, same column, same 3×3 square

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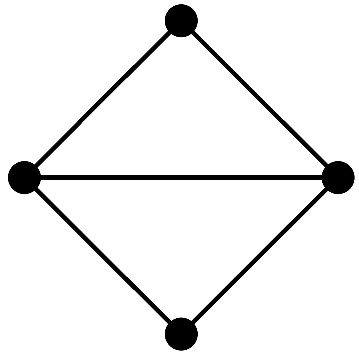

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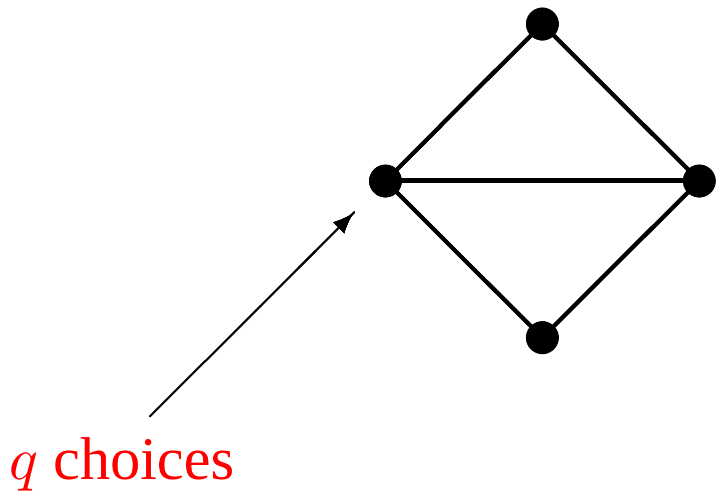

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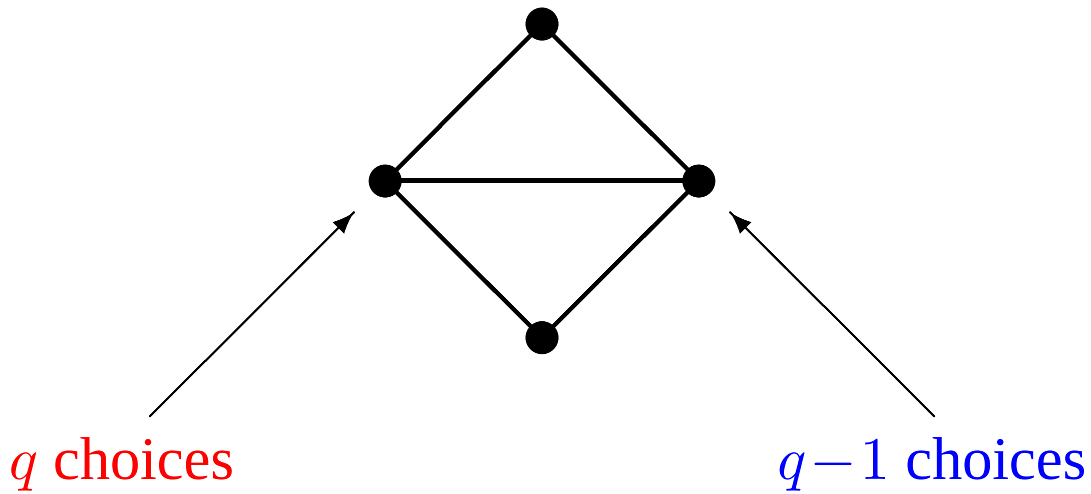

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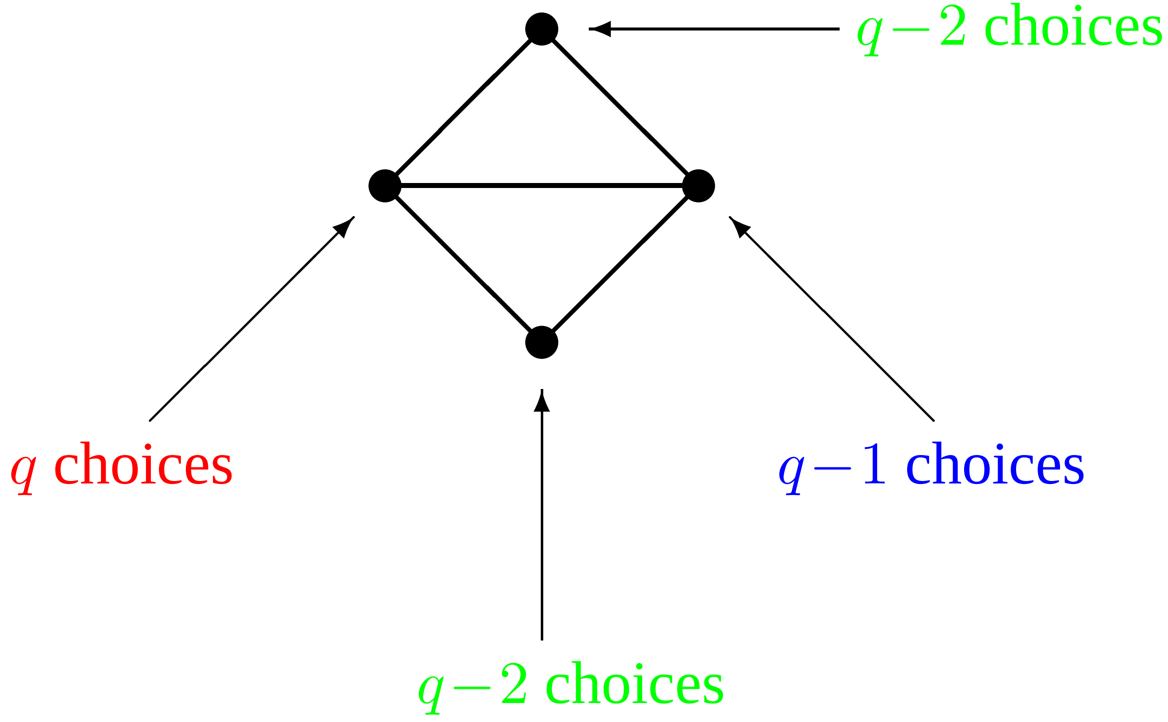

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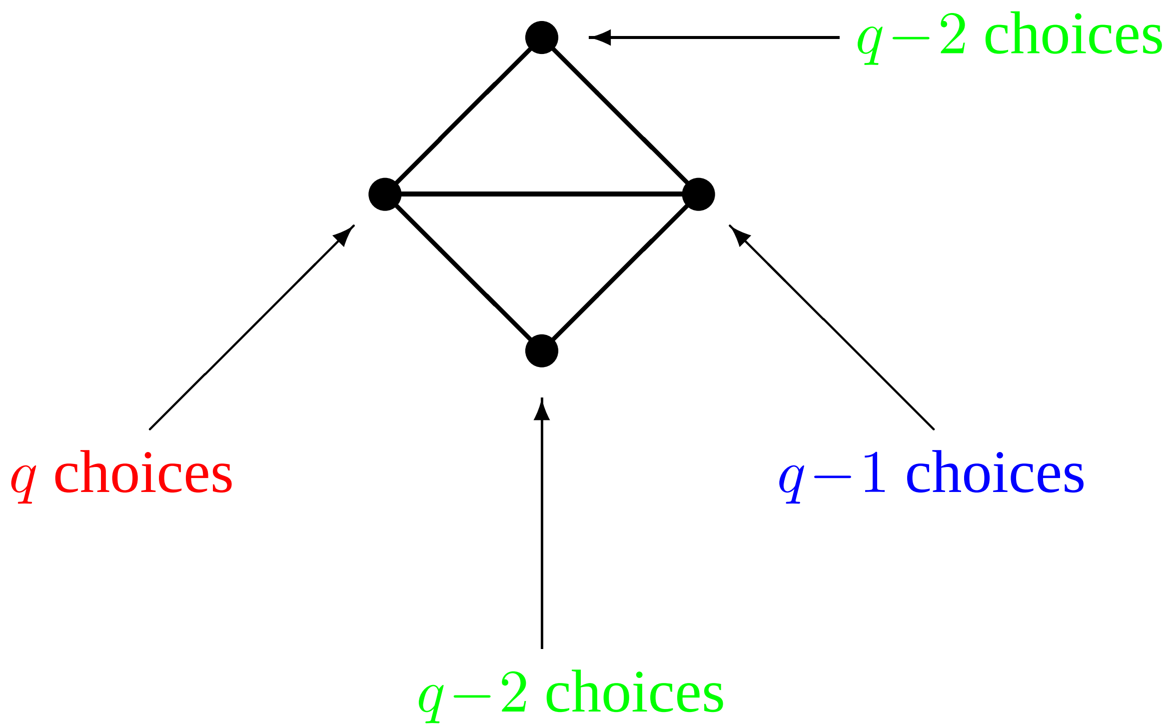

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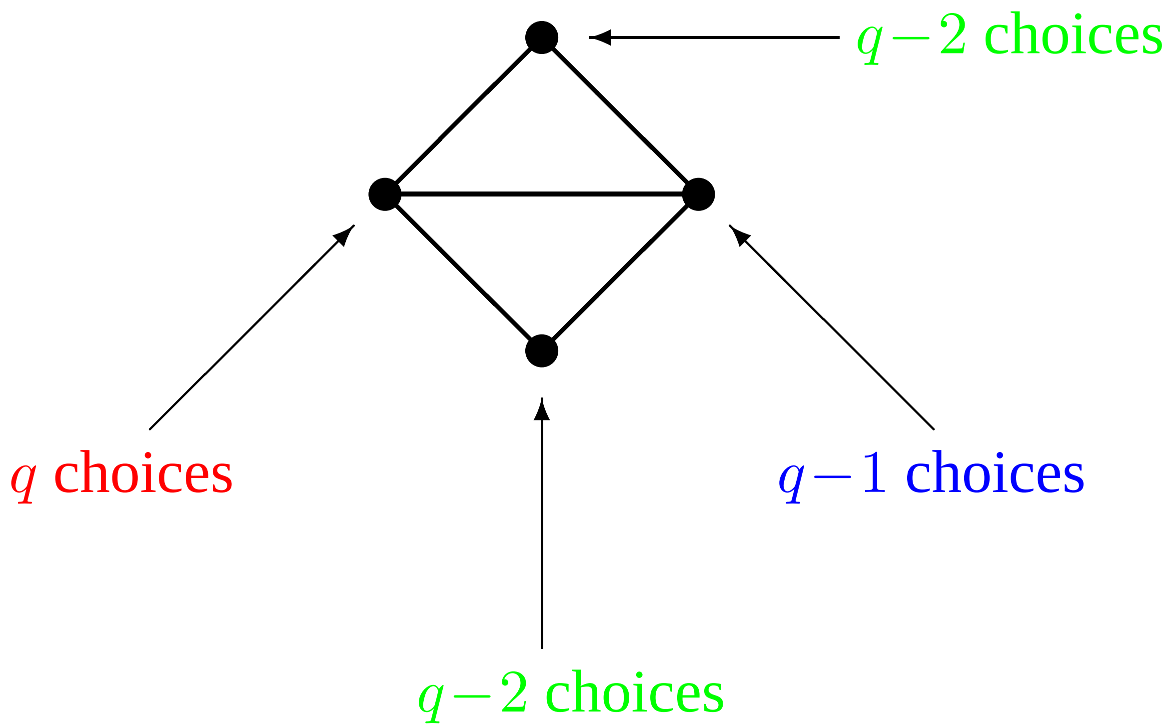

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- Note that here $P_G(q)$ is a polynomial in q

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This is a general fact (which I will prove later):

Theorem (Birkhoff 1912): For every graph G, $P_G(q)$ is the restriction to positive integers q of a polynomial in q (called the chromatic polynomial of G).

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In particular, we can ask about the real or complex roots of P_G.

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Or in simpler language:

Every planar graph can be (properly) colored with four colors.

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But the real or complex roots of P_G are still of interest …

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$$W_{ij} = \begin{cases} 1 & \text{if } i \text{ is colored differently from } j \\ 1 + v_{ij} & \text{if } i \text{ is colored the same as } j \end{cases}$$

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- Note that if we take $v_{ij} = -1$ for all edges ij, then the weight becomes

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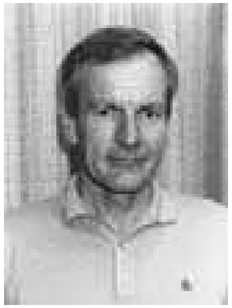
- Note in particular that $Z_G^{\mathrm{Potts}}(q, -1) = P_G(q)$

The Potts model in statistical physics

- Potts (1952) introduced this as a model in statistical physics:

  Each atom in a crystal lattice can be in any one of q states.



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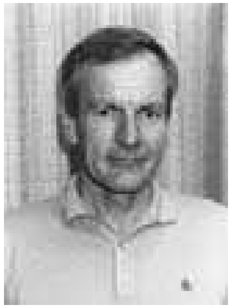

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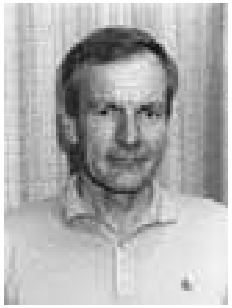

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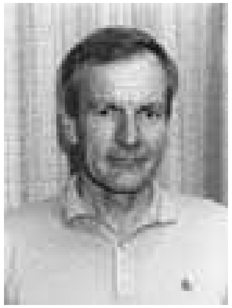

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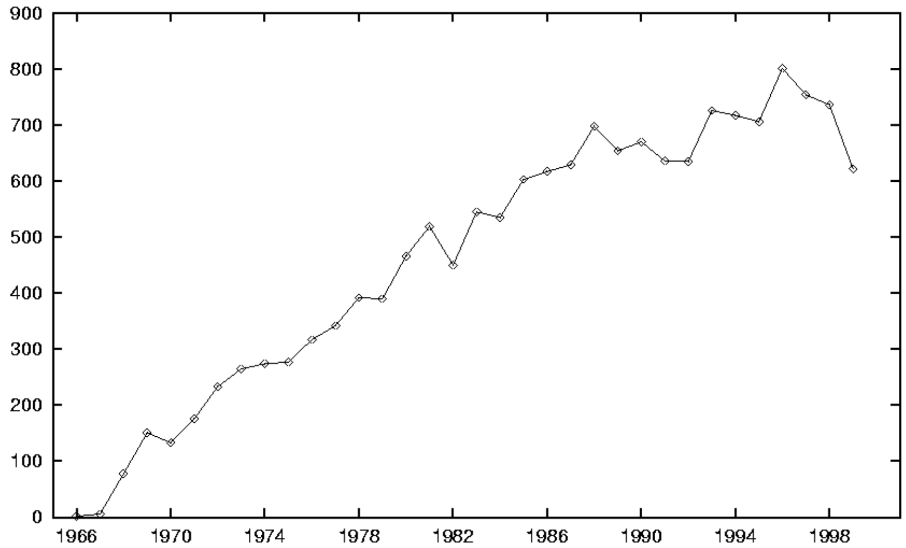

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Theorem (Fortuin + Kasteleyn 1969): For every graph G, $Z_G^{\mathrm{Potts}}(q, \mathbf{v})$ is the restriction to positive integers q of a polynomial in q (and $\mathbf{v}$):

$$Z_G^{\mathrm{Potts}}(q, \mathbf{v}) \;=\; \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e$$

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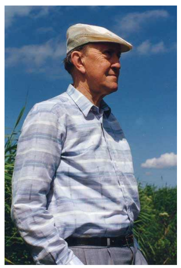

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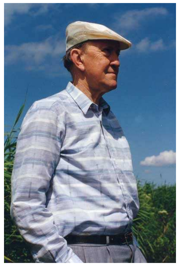

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Corollary (Birkhoff 1912): $P_G(q) = Z_G^{\mathrm{Potts}}(q, -1)$ is a polynomial.

Proof of the Fortuin–Kasteleyn representation

PROOF. Write

$$Z_G^{\mathrm{Potts}}(q, \mathbf{v}) \;=\; \sum_{\sigma\colon\, V \to \{1,2,\ldots,q\}} \; \prod_{e=ij \in E} \Big[1 + v_e \delta(\sigma_i, \sigma_j) \Big]$$

where $\delta(\sigma_i, \sigma_j)$ is the Kronecker delta $\delta(a,b) \;=\; \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$

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- It includes the chromatic polynomial as a special case: $P_G(q) = Z_G(q, -1)$

- Note that $Z_G(q, \mathbf{v})$ is multiaffine in $\mathbf{v}$, i.e. of degree 1 in each v_e separately.

Complex roots of the chromatic polynomial — why?

- Since $P_G(q)$ and $Z_G(q, \mathbf{v})$ are polynomials in q and $\mathbf{v}$, we can study their real or complex roots . . .

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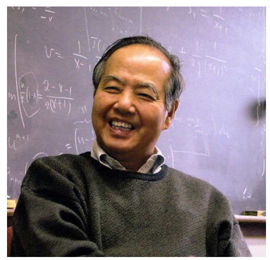

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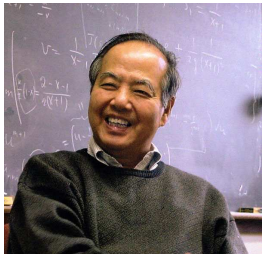

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and is connected with the physics of phase transitions.

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- Most things in Nature vary smoothly.
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- These abrupt changes are called phase transitions.

The physics of phase transitions

- Most things in Nature vary smoothly.
- But some changes are abrupt, e.g. water boiling or freezing.
- These abrupt changes are called phase transitions.
- Mathematically, a phase transition occurs whenever some physical quantity (e.g. density) varies nonanalytically as a function of some control parameter (e.g. temperature).

(“nonanalytic” = in sense of complex analysis)

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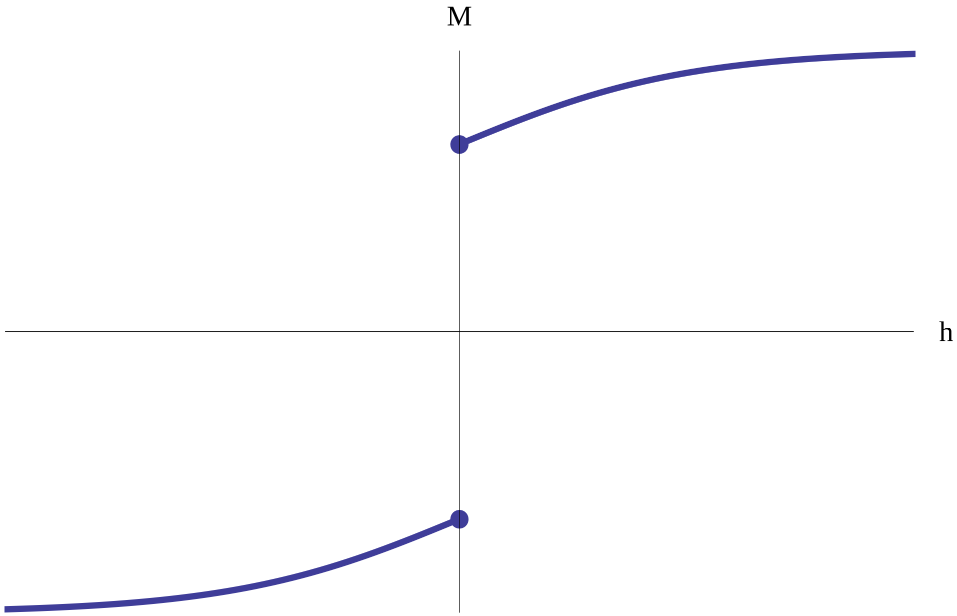

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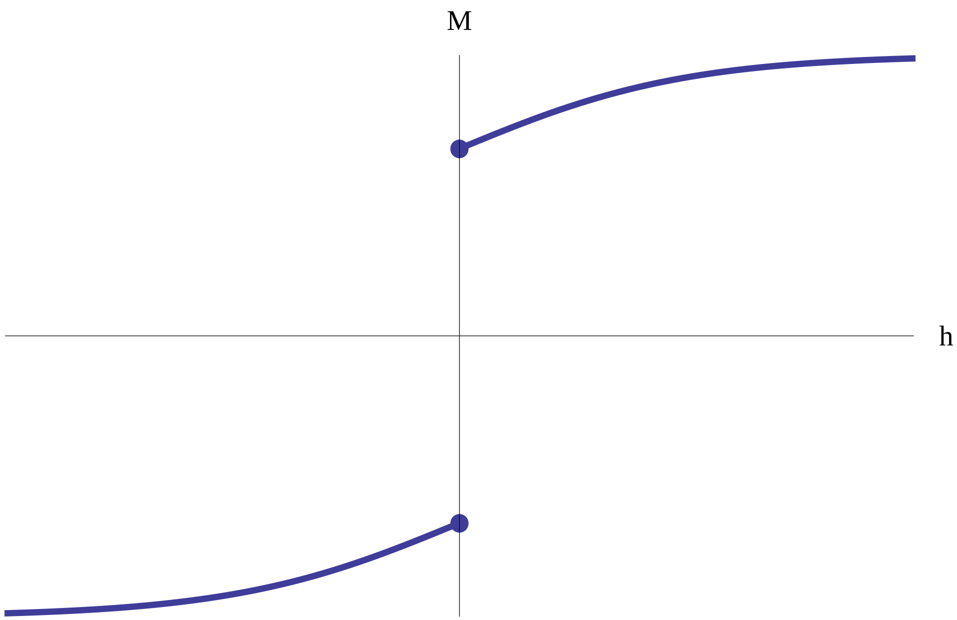

The discontinuity at $h = 0$ is a phase transition.

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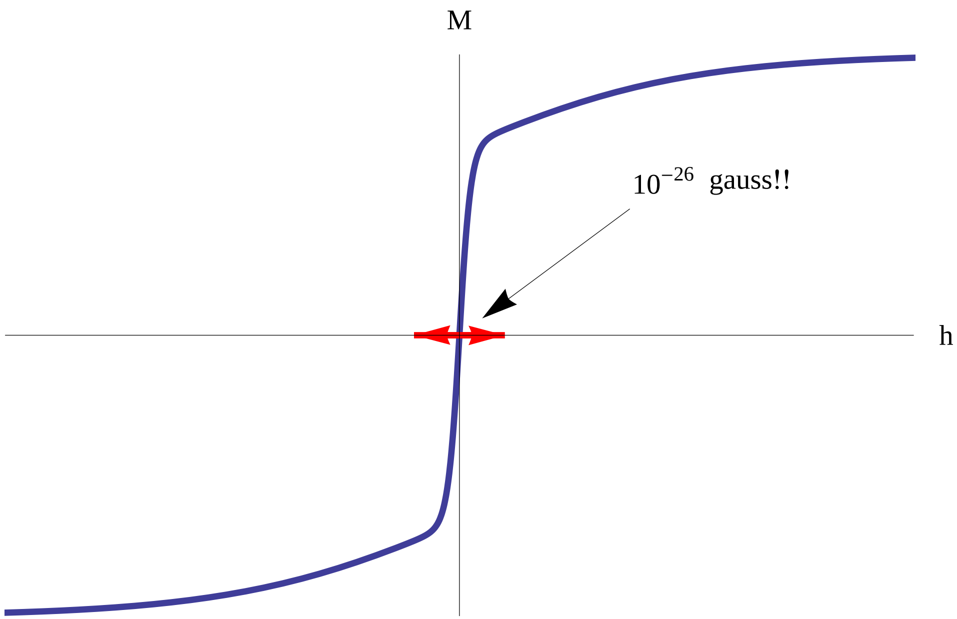
MORAL: Phase transitions never occur in a physical system with finitely many degrees of freedom.

The mathematics of phase transitions

For a real (finite!) sample of iron one actually gets



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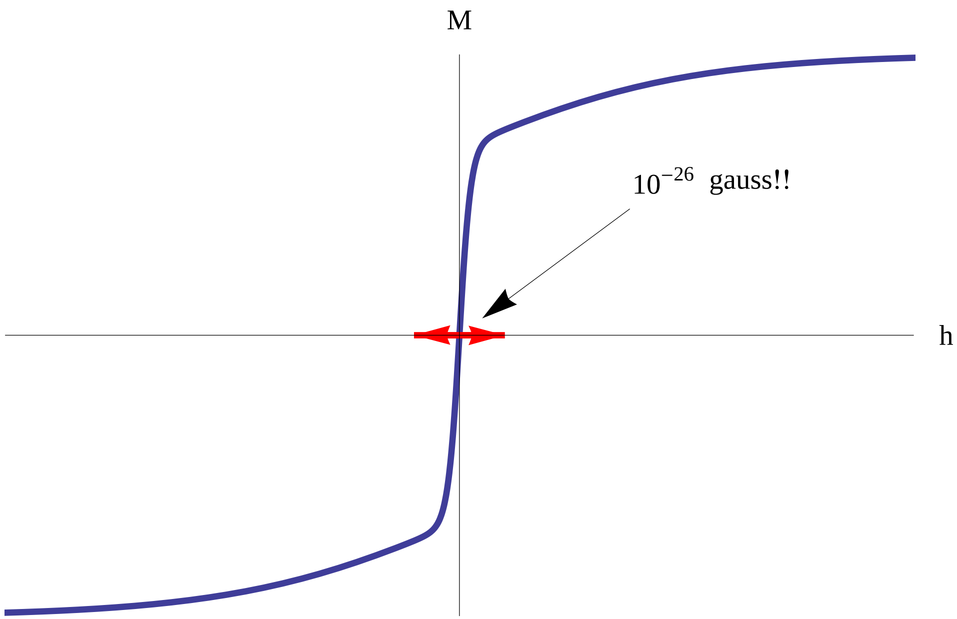
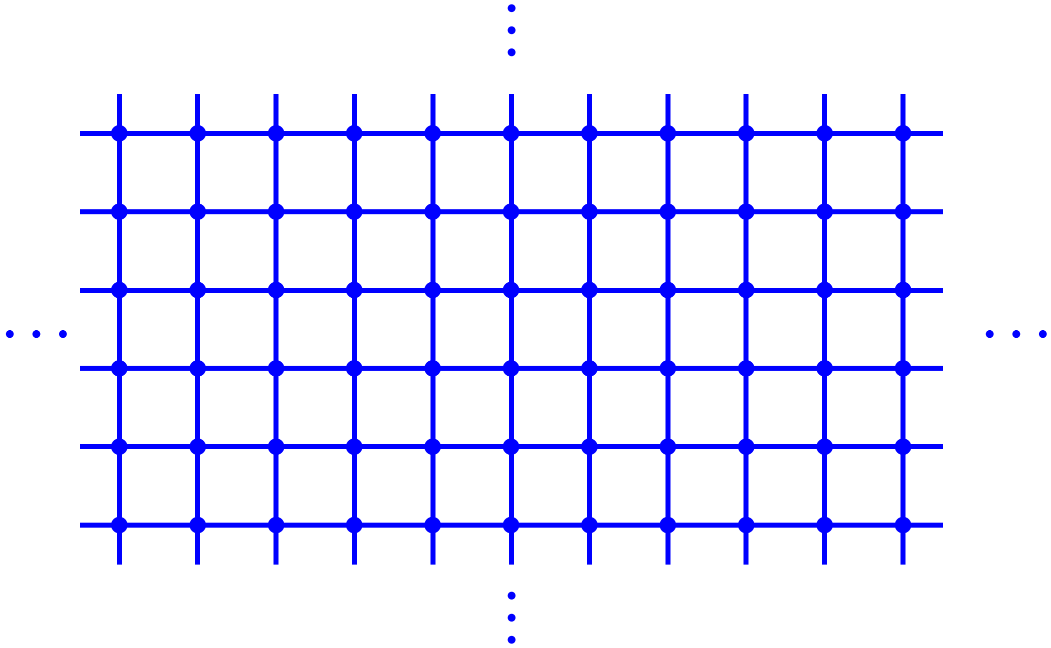


But for all practical purposes that *is* a phase transition!

The mathematics of phase transitions

So it makes sense to first study phase transitions in an idealized system where the discontinuity is a true discontinuity:

namely, the Ising or Potts model on an infinite graph, such as the square lattice $\mathbb{Z}^2$:



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It then turns out that $\lim_{n\to\infty} Z_{G_n}(q, \mathbf{v})$ does not exist,

but $f(q, \mathbf{v}) = \lim_{n\to\infty} \dfrac{1}{|G_n|} \log Z_{G_n}(q, \mathbf{v})$ does.

(Physicists call f the free energy per unit volume.)

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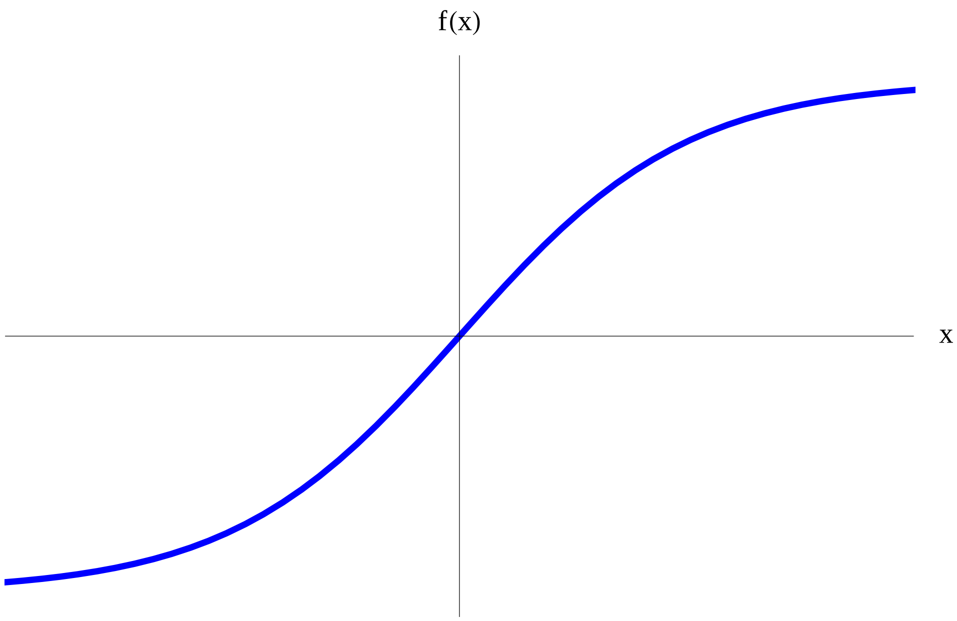

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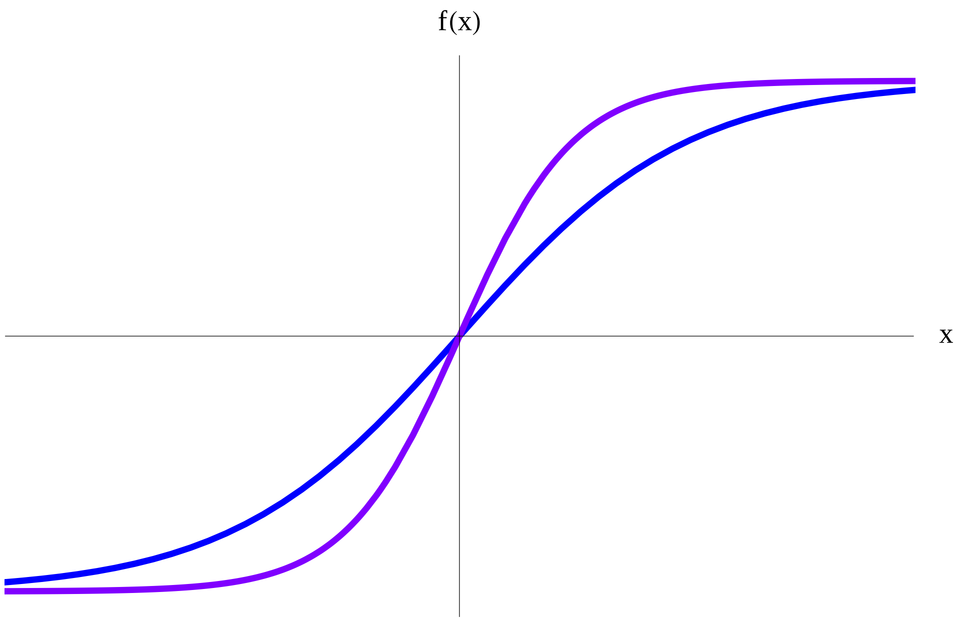

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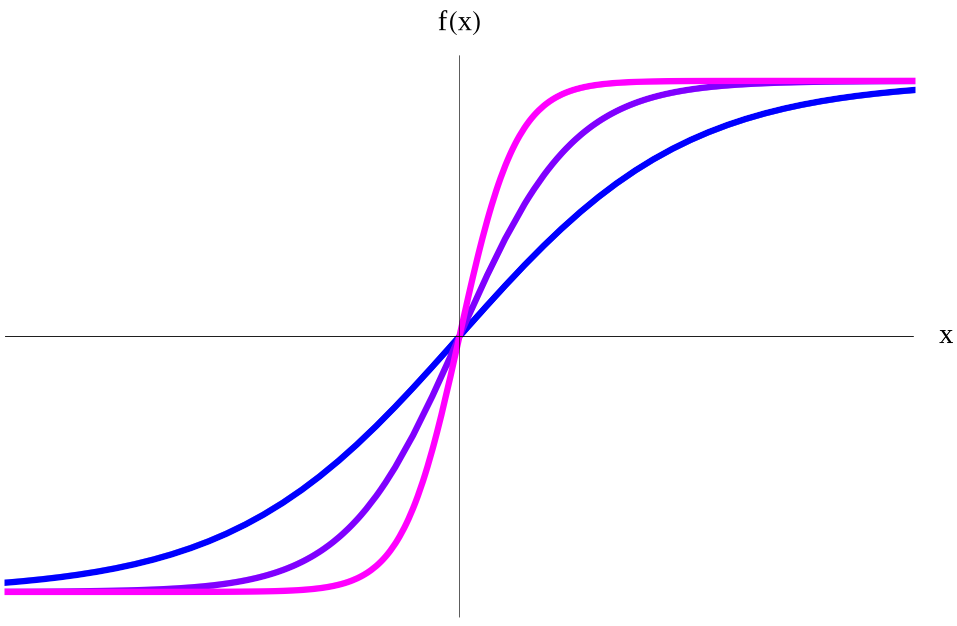

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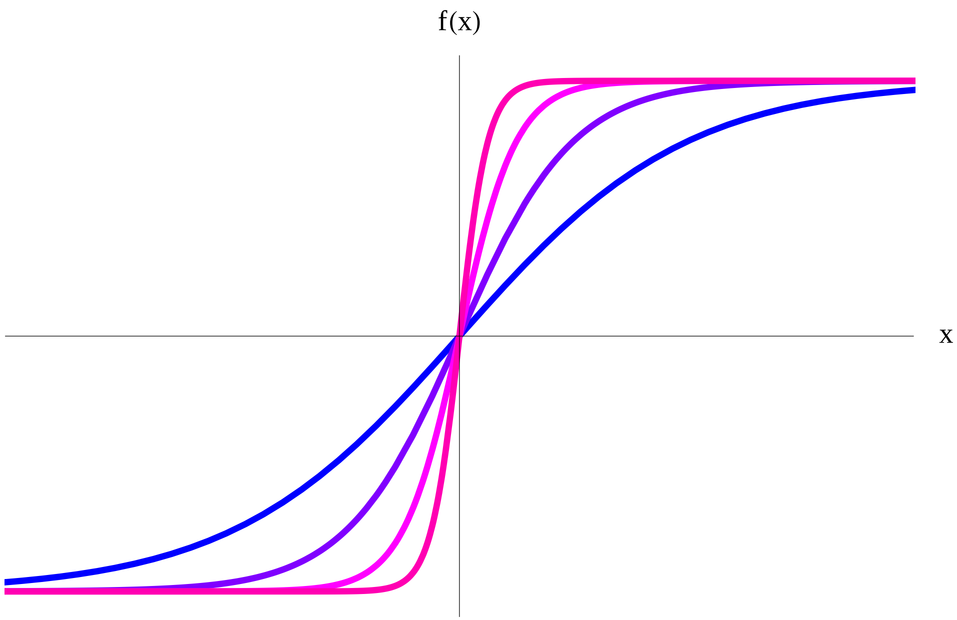

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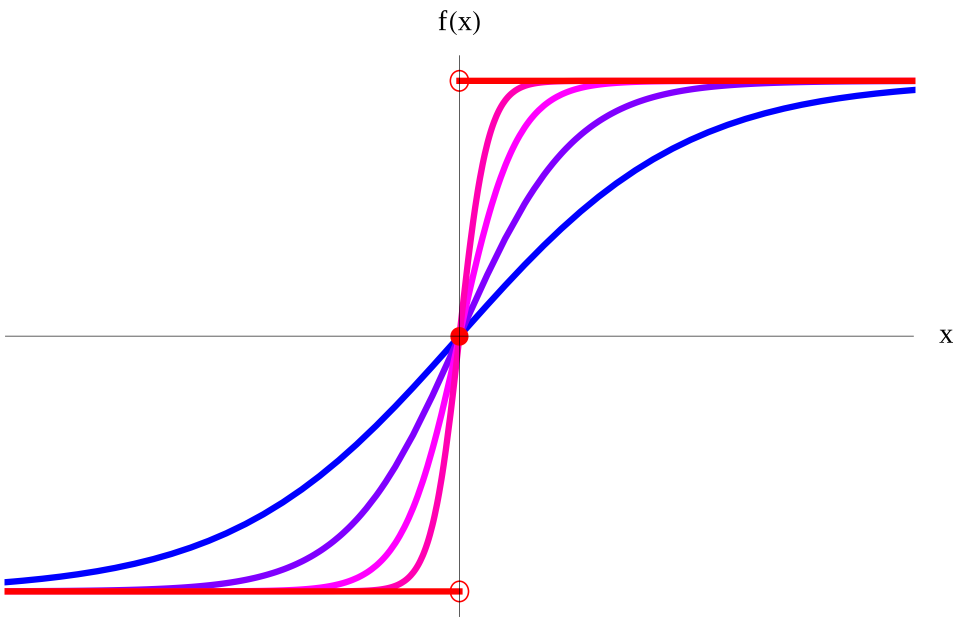

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EXAMPLE: $f_n(x) = \tanh(nx)$ has poles at $x = \pm\dfrac{\pi}{2n}\, i$.

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Apply the above to $f_n(q, \mathbf{v}) = \dfrac{1}{|G_n|} \log Z_{G_n}(q, \mathbf{v})$.

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The Yang–Lee approach to phase transitions:

- Promote one or more physical quantities (e.g. temperature) to complex variables.
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- The real limit points (as $n \to \infty$) of those complex zeros are the possible points of phase transitions.

The Lee–Yang theorem for the Ising model

Therefore ... If a domain $D \subset \mathbb{C}$ is free of zeros (uniformly in the volume G_n), then the intersection of D with the real axis is free of phase transitions.

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P.S. The Lee–Yang theorem is actually a beautiful theorem about zeros of multiaffine polynomials in several complex variables. The result quoted above is a mere corollary.

Phase transitions, summarized . . .

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But first ... some facts about the real roots ...
to motivate some conjectures about the complex roots.

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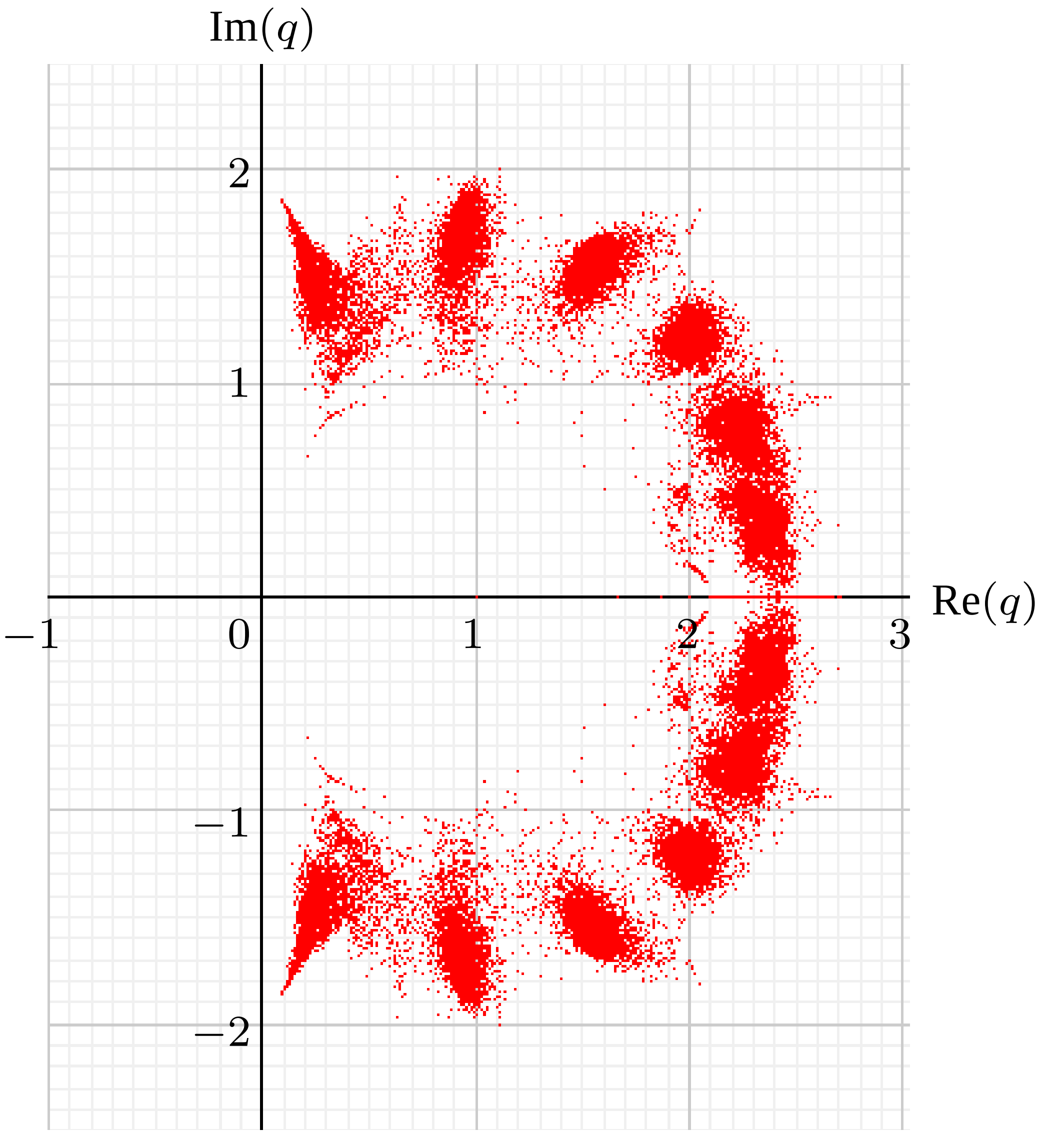
Chromatic roots of cubic graphs on 16 vertices

4060 graphs $\Longrightarrow$ $4060 \times 16 = 64960$ roots



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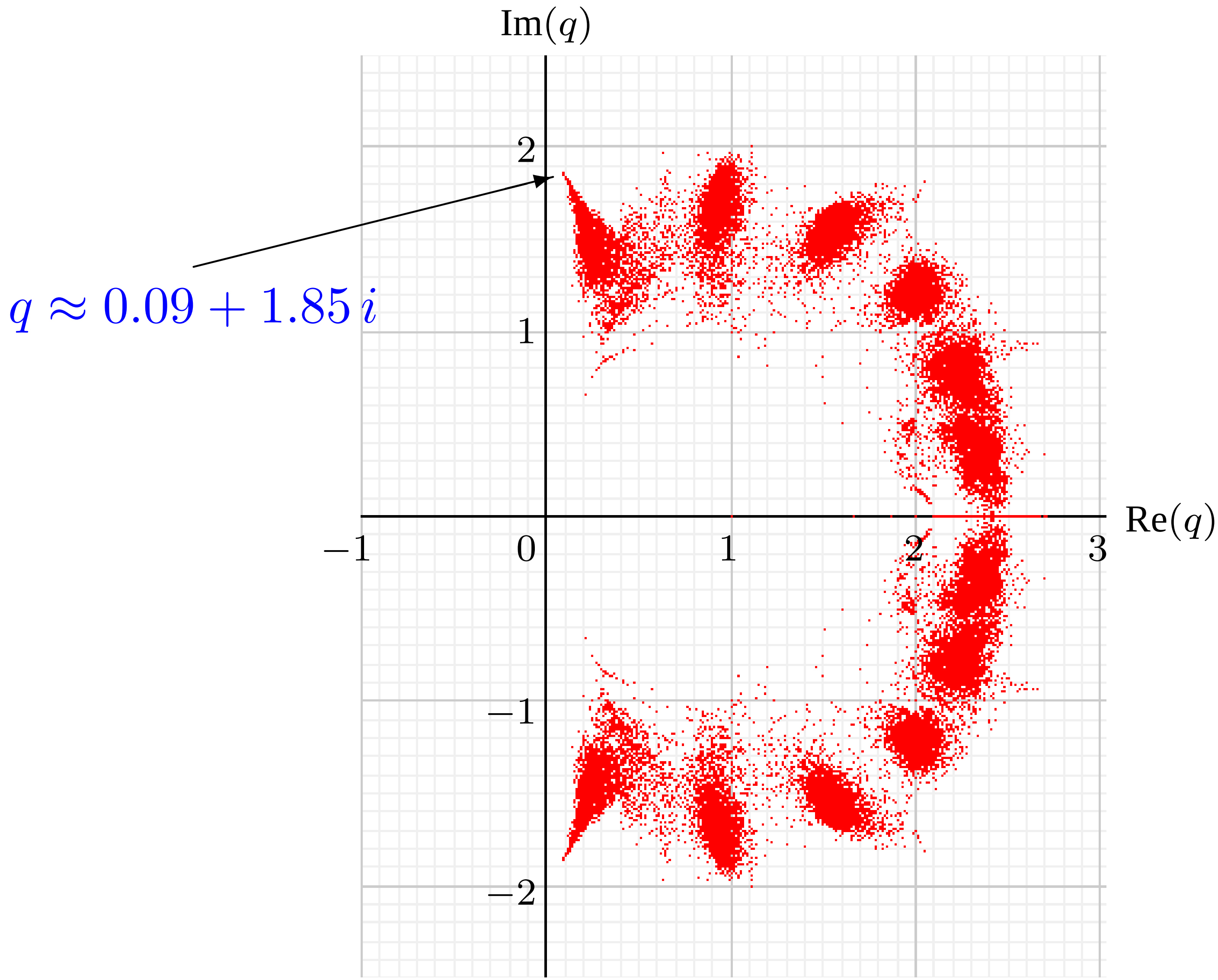

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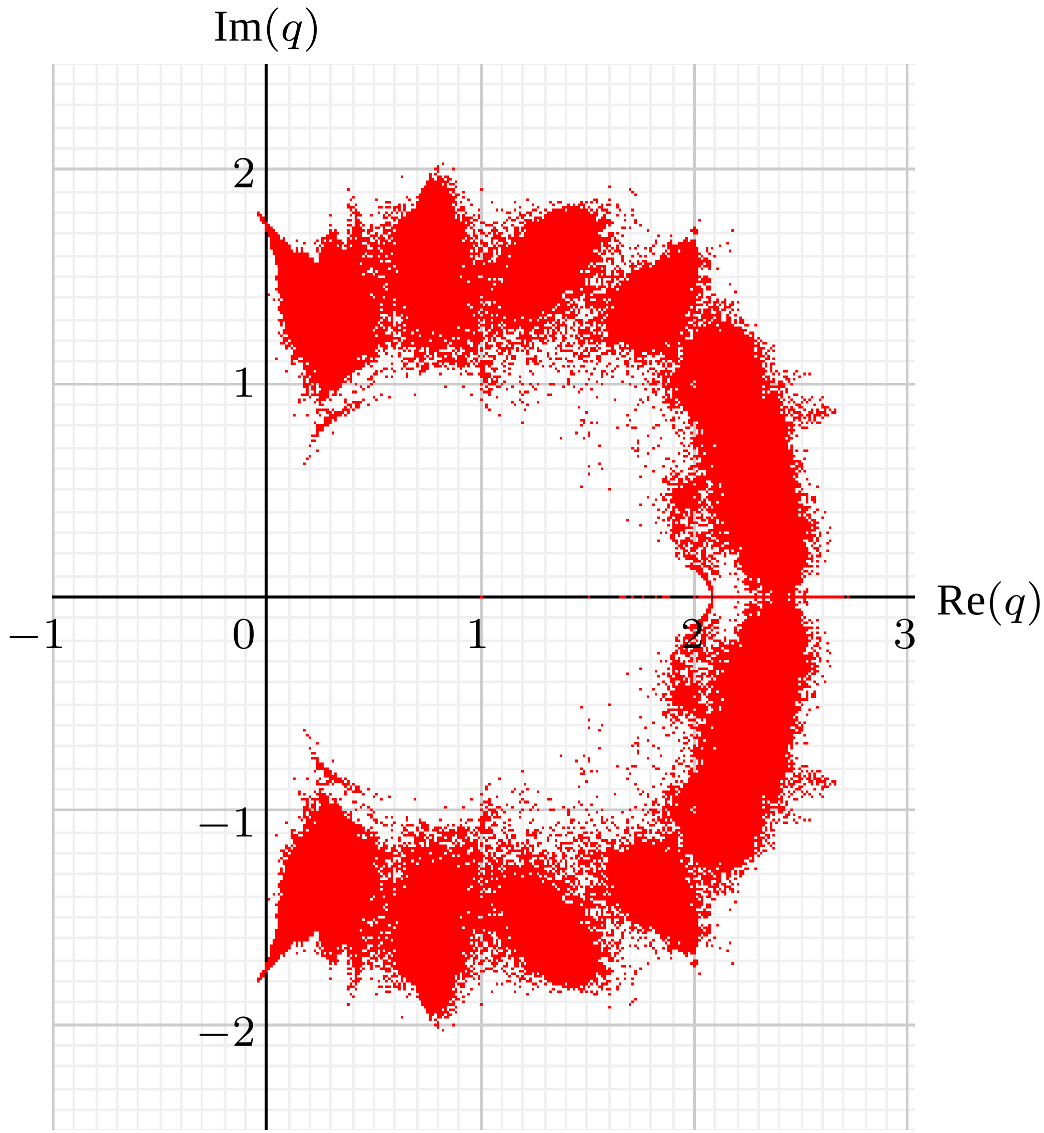

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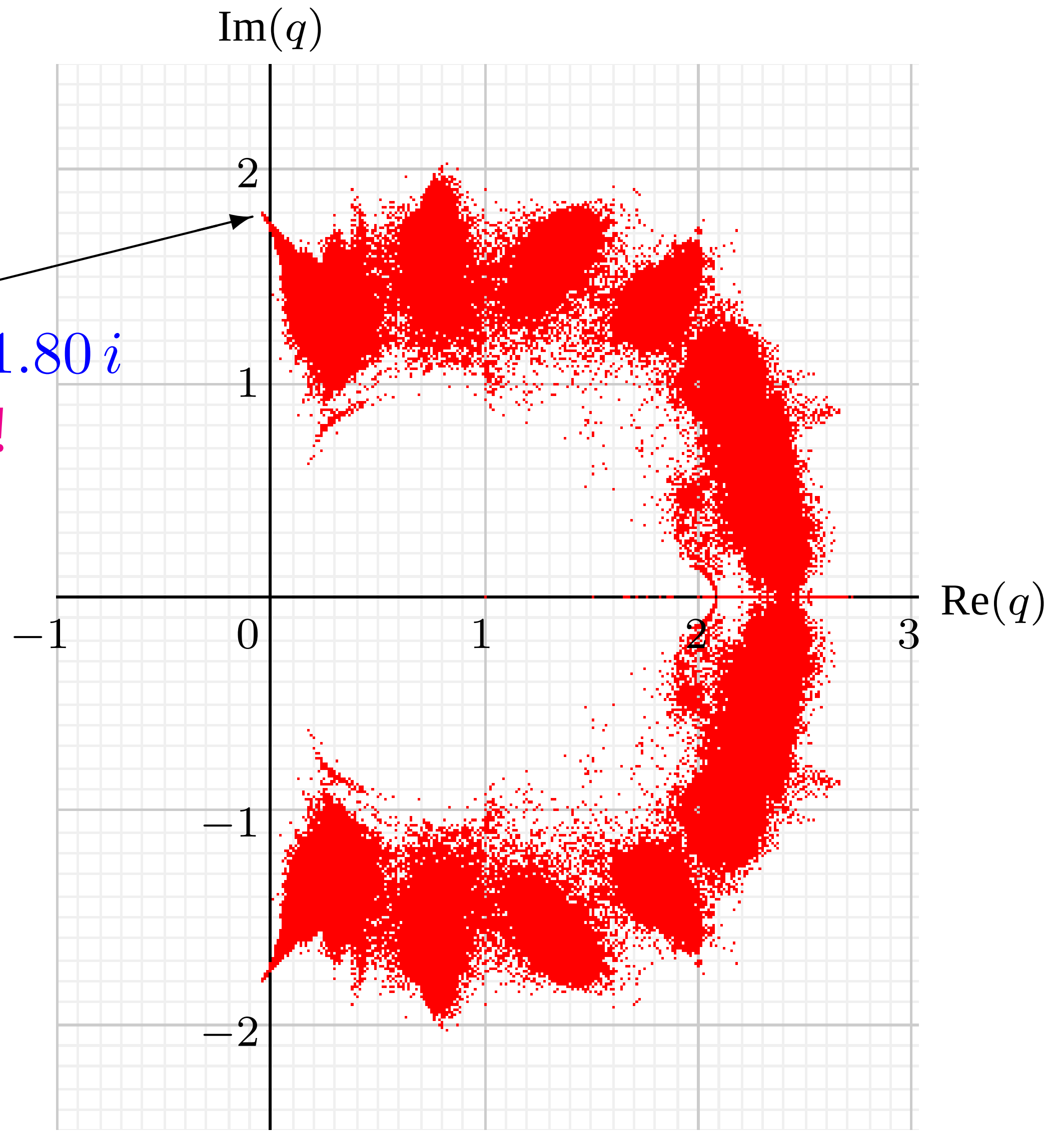

$q \approx -0.05 + 1.80\, i$

OOPS!!!!!

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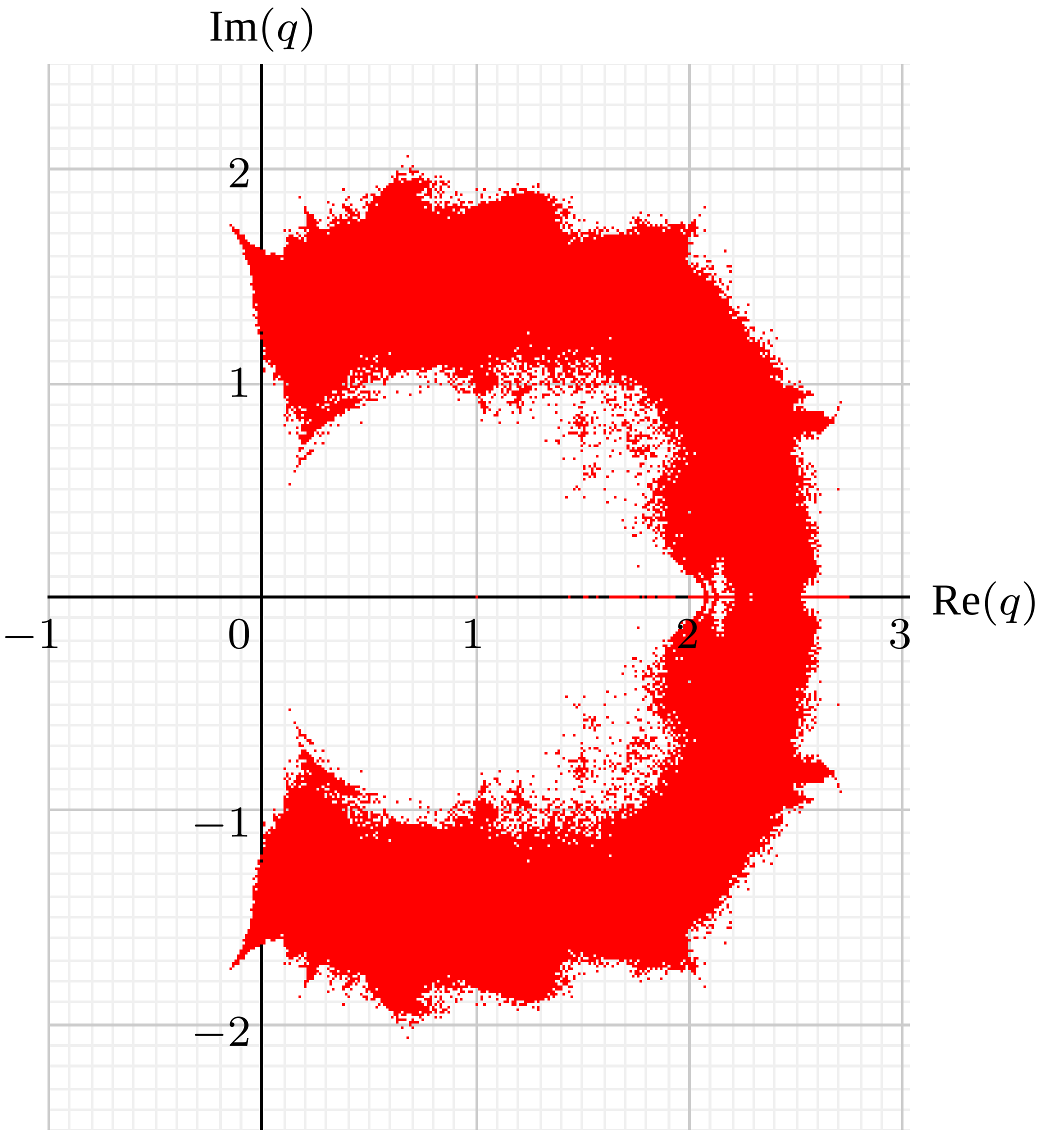

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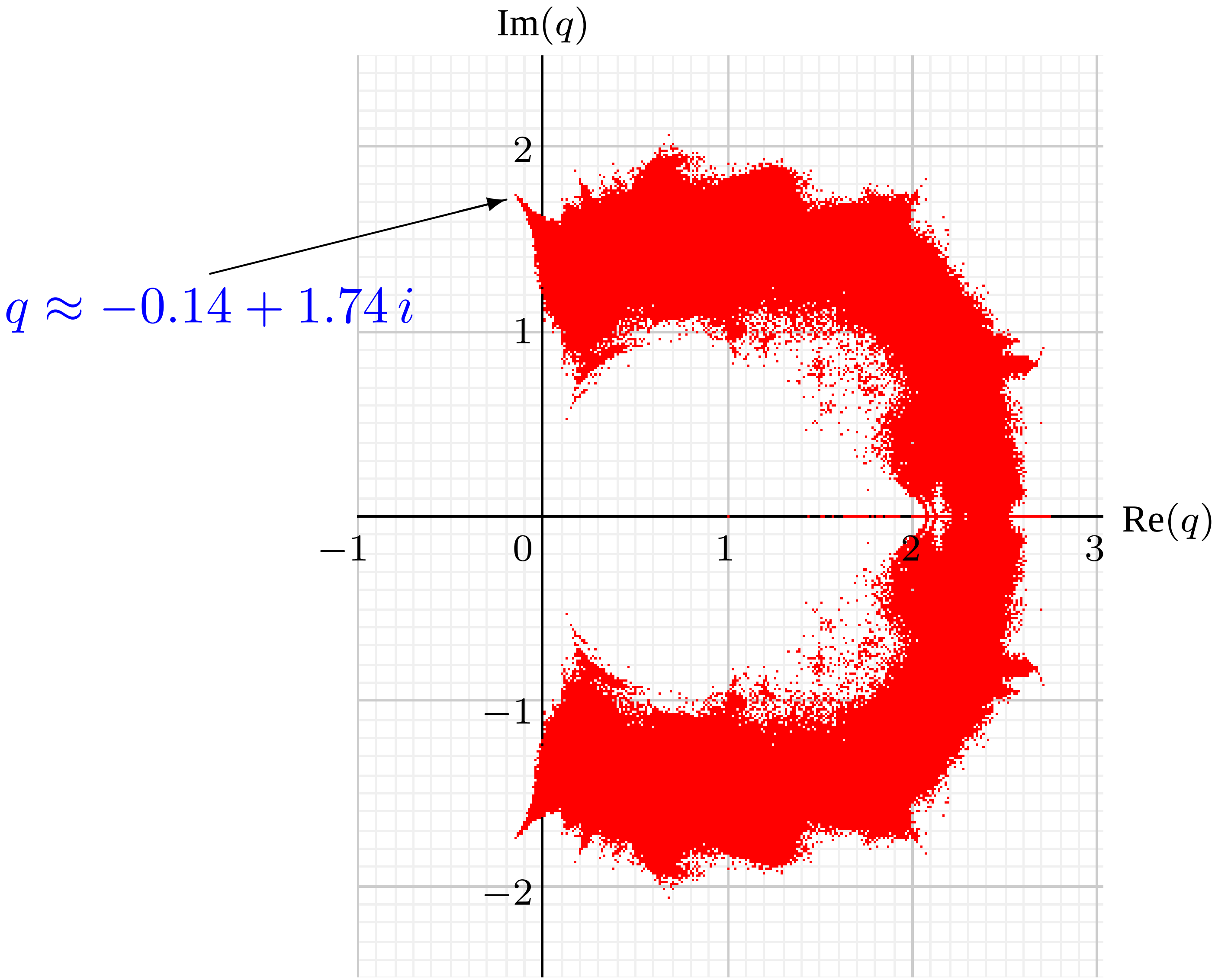

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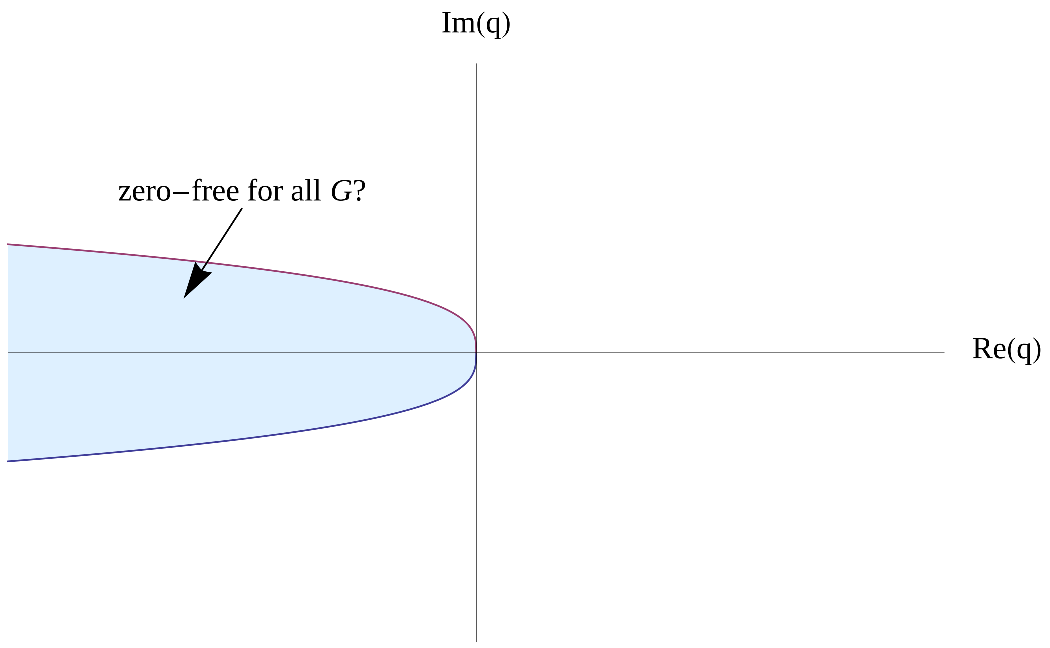

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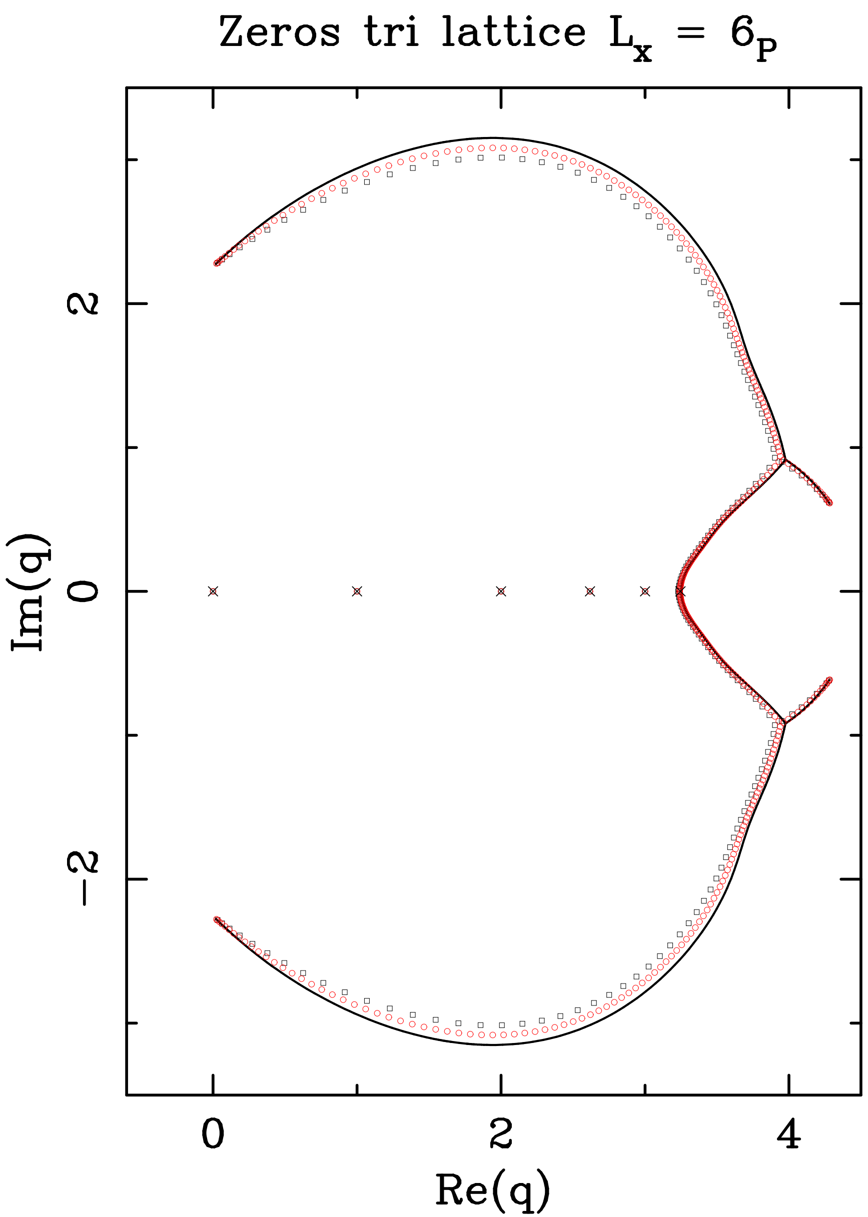

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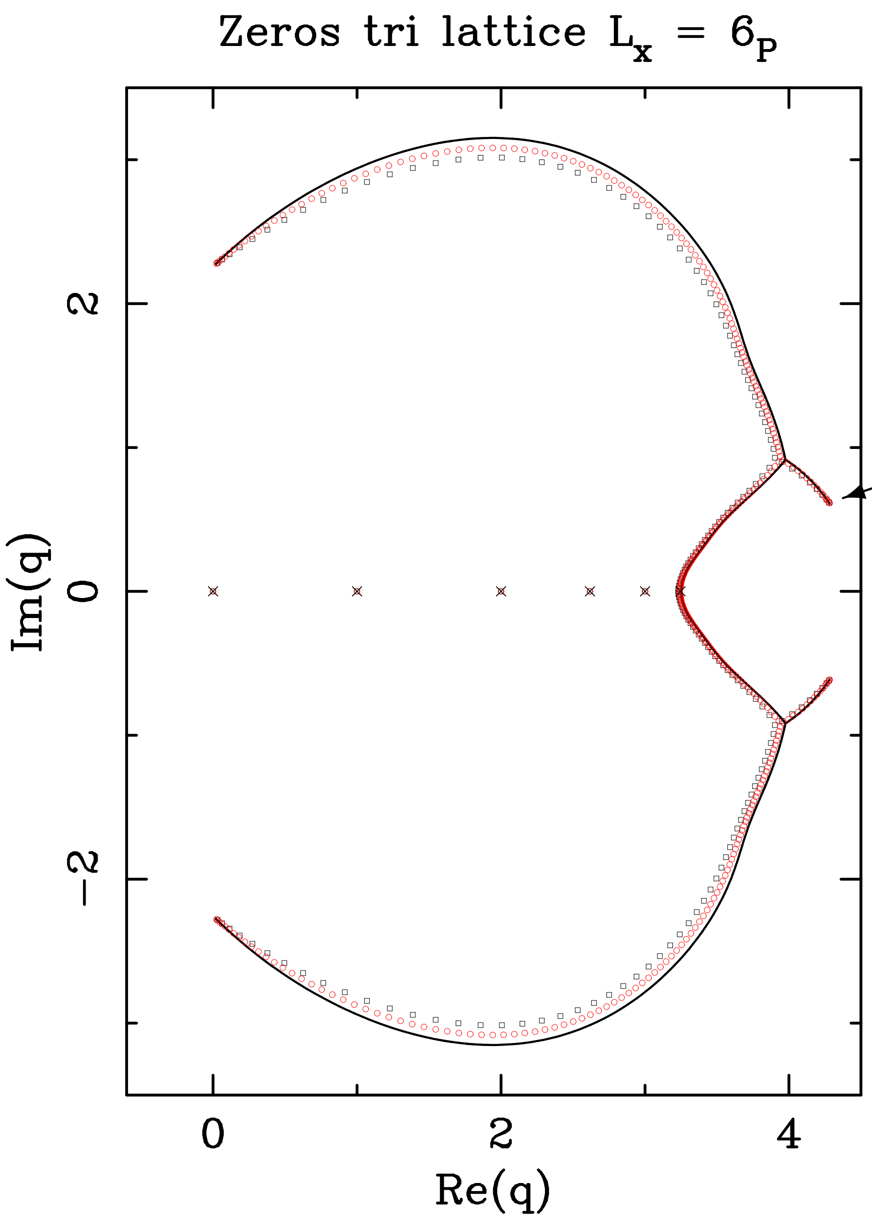

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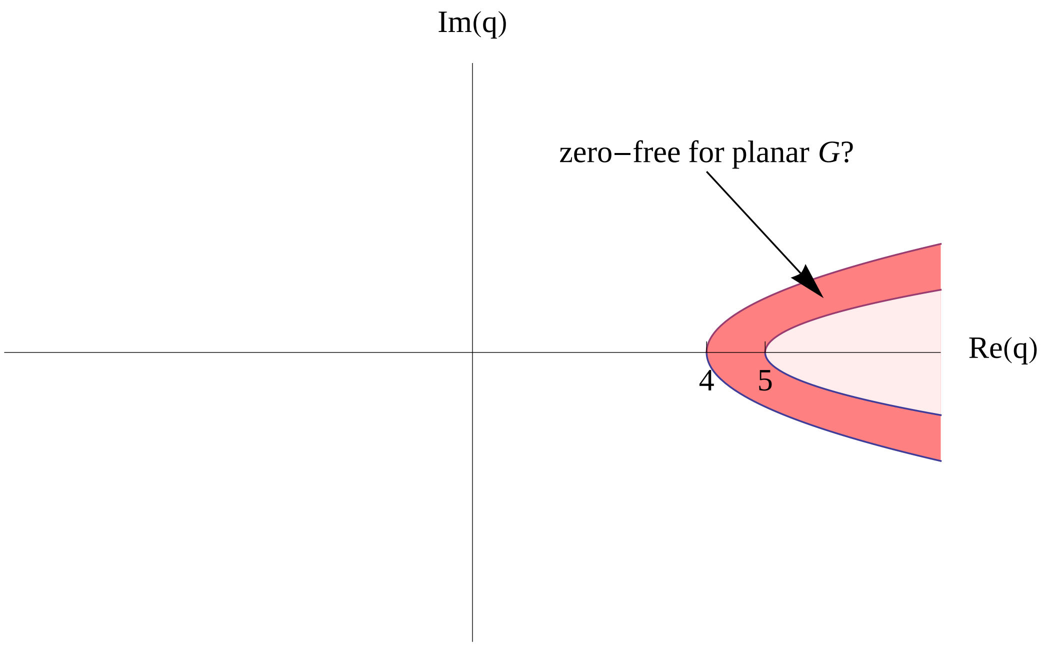

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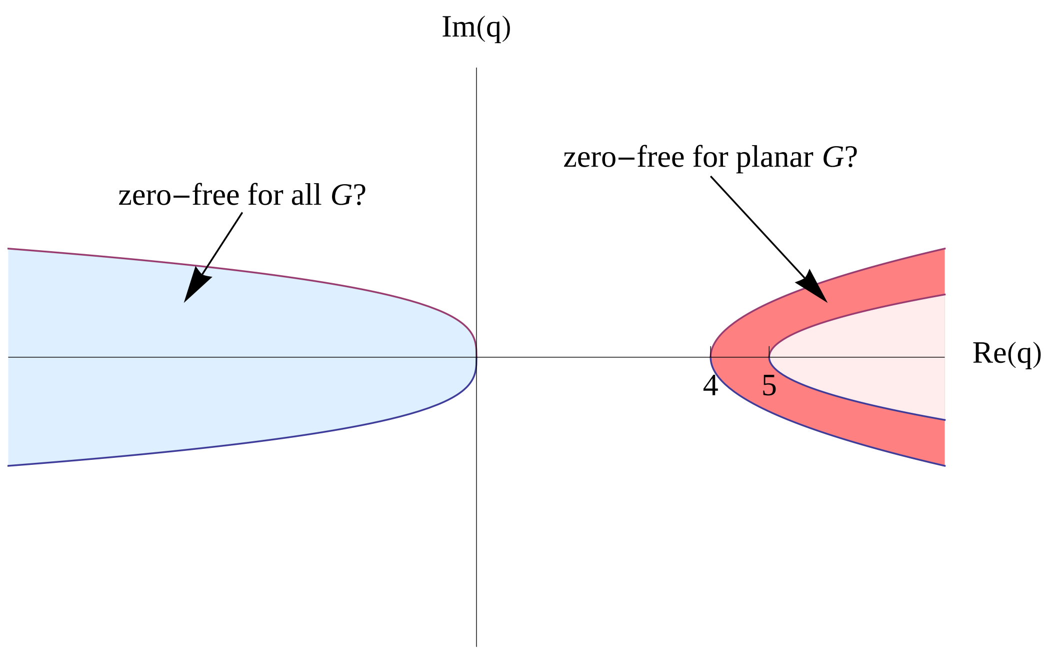

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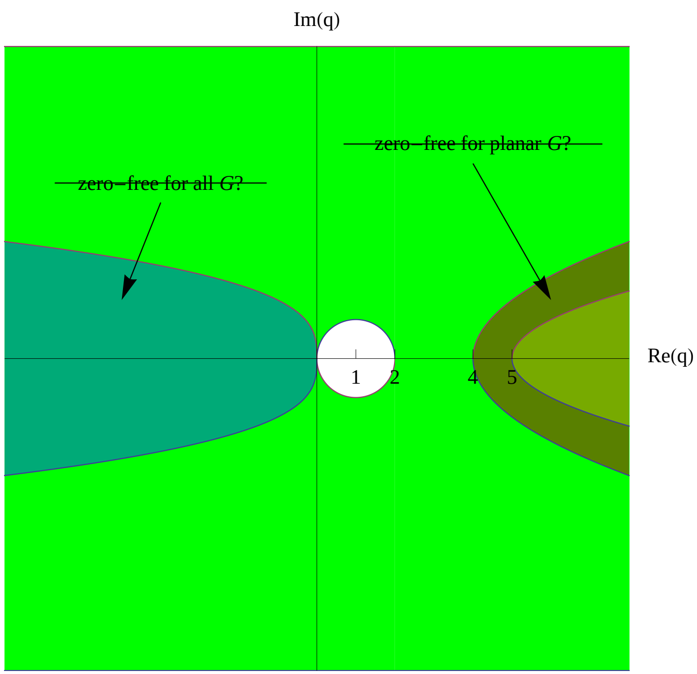

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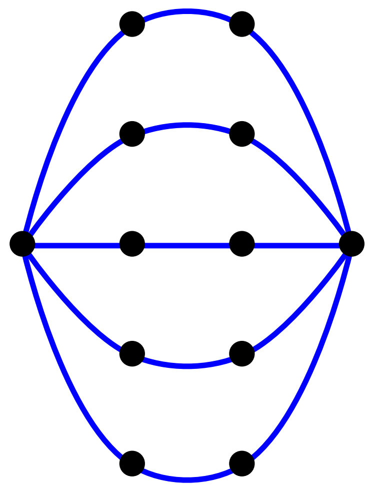
e.g. $\Theta^{(3,5)} =$ 

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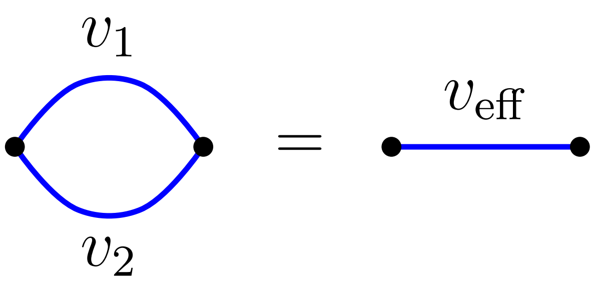
Parallel: 

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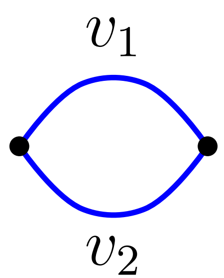
Parallel:  $=$  with $v_{\text{eff}} = v_1 + v_2 + v_1 v_2$

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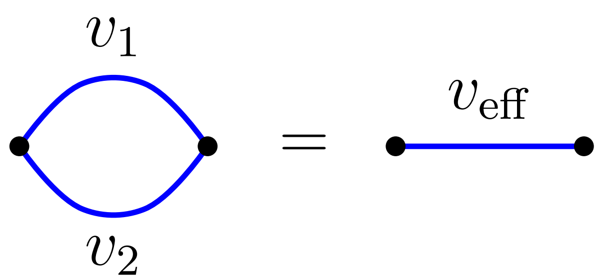

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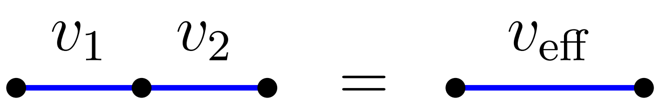
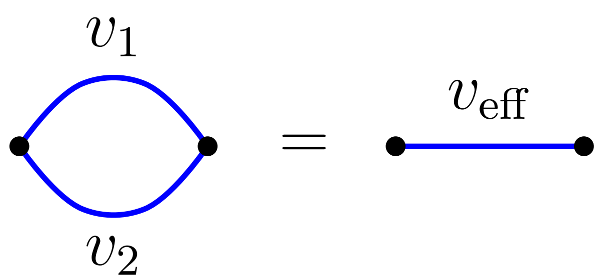
Series:



with $v_{\text{eff}} = \dfrac{v_1 v_2}{q + v_1 + v_2}$

The intuition behind the proof

The Potts model obeys series and parallel laws
(generalizing those of electrical circuits):

Parallel: 

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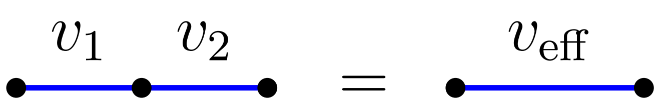
Series: 

with $v_{\text{eff}} = \dfrac{v_1 v_2}{q + v_1 + v_2}$

Hence $\dfrac{v_{\text{eff}}}{q + v_{\text{eff}}} = \dfrac{v_1}{q + v_1} \dfrac{v_2}{q + v_2}$, i.e. $\dfrac{v}{q + v}$ multiplies

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- For the chromatic polynomial ($v = -1$), $\left|\frac{v}{q+v}\right| < 1$ means $|q-1| > 1$. This is where the chromatic roots are dense.

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What next?

- So planarity does not much constrain the chromatic roots.
- But other graph-theoretic parameters can:
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- What determines where the chromatic roots of a graph go in the complex plane?
- We know very little at present.
- The study of chromatic roots is still a very young field.

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