MATHEMATICS 3103 (Functional Analysis)
YEAR 2012–2013, TERM 2

PROBLEM SET #6

This problem set is due at the *beginning* of class on Monday 18 March. Only Problem 1 will be formally assessed; but Problems 2 and 3 present some extensions/applications of the Hahn–Banach theorem that I think you will find interesting.

Topics: The Hahn–Banach theorem and its corollaries. Separability of X and X^*. Duals of subspaces and quotient spaces. Reflexivity of X and X^*.

Readings:

- Handout #6: The Hahn–Banach theorem and duality of Banach spaces.

1. Let X be an infinite-dimensional normed linear space. Is X^* necessarily infinite-dimensional? Prove your assertion.

2. **Generalization of the Hahn–Banach theorem.** Let X be a real vector space. A **sublinear functional** on X is a function $p\colon X \to \mathbb{R}$ satisfying

   (a) $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$; and

   (b) $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha \ge 0$.

   I stress that (b) is required to hold only for $\alpha \ge 0$, and that p is not required to be nonnegative.[1] Note that it follows from (a) and (b) that $p(0) = 0$ and $-p(-x) \le p(x)$ for all $x \in X$ (why?). And it follows from this that if ℓ is a linear functional satisfying $\ell(x) \le p(x)$ for all x, then in fact we have

   $$-p(-x) \;\le\; -\ell(-x) \;=\; \ell(x) \;\le\; p(x)$$

   for all x.

   I would like you to prove the following generalization of the Hahn–Banach theorem:

[1] If we were to require that $p(\alpha x) = |\alpha| p(x)$ for arbitrary $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$ in the complex case), then p would be called a **seminorm** (and in this case it is not hard to see that $p \ge 0$). The seminorms are a *subclass* of sublinear functionals. And the norms are a subclass of seminorms.

Generalized Hahn–Banach theorem. Let X be a real vector space, and let p be a sublinear functional on X. Now let $M \subseteq X$ be a linear subspace, and let ℓ be a linear functional on M that satisfies

$$\ell(x) \;\le\; p(x) \qquad \text{for all } x \in M\,.$$

Then there exists a linear functional $\widetilde{\ell}$ on X that extends ℓ (i.e. $\widetilde{\ell} \upharpoonright M = \ell$) and satisfies

$$\widetilde{\ell}(x) \;\le\; p(x) \qquad \text{for all } x \in X\,.$$

Do you see how this implies the Hahn–Banach theorem proved in class, by taking $p(x) = c\|x\|$ with $c = \|\ell\|_{M^*} \ge 0$?

Hint for the proof: Imitate the proof of the Hahn–Banach theorem given in the lecture notes, using p in place of the norm.

Remark. The "generalized Hahn–Banach theorem" quoted here is often referred to as the Hahn–Banach theorem *tout court*, and the version proven in the lecture notes is considered to be a special case. This point of view is quite sensible, since as you will see in Problem 3, the version using general sublinear functionals is much more powerful than the version using only norms.

3. **Separation theorems for convex sets.** In this problem you will apply the "generalized Hahn–Banach theorem" of Problem 2 to prove some important theorems saying, roughly speaking, that two disjoint convex sets can be separated by a closed hyperplane.

   (a) Let X be a real vector space, and let $K \subseteq X$ be a subset satisfying $\bigcup\limits_{\alpha>0} \alpha K = X$. [Note that in particular this implies that $0 \in K$ (why?).] Then define the **Minkowski functional** (or **gauge**) associated to K by

   $$p_K(x) \;=\; \inf\{\alpha > 0\colon\, x \in \alpha K\}\,.$$

   We have $0 \le p_K(x) < \infty$ for all x (why?). Geometrically, $p_K(x)$ is the smallest factor by which we have to rescale K so that it contains x.

   Prove that:

   (i) If K is convex, then p_K is a sublinear functional.

   (ii) If X a *normed* linear space and $K \ni 0$ is an *open* convex subset of X, then $K = \{x\colon\, p_K(x) < 1\}$ and there exists a constant $c < \infty$ such that $0 \le p_K(x) \le c\|x\|$.

   (b) Let X be a real normed linear space and let A, B be disjoint convex subsets of X. Prove that:

   (i) If A is open, then there exists $\ell \in X^*$ and $\alpha \in \mathbb{R}$ such that

   $$\ell(a) \;<\; \alpha \;\le\; \ell(b) \qquad \text{for all } a \in A \text{ and } b \in B\,.$$

(ii) If $d(A, B) > 0$ [this holds in particular if one of the sets is compact and the other is closed], then there exists $\ell \in X^*$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\ell(a) \;\le\; \alpha \;<\; \beta \;\le\; \ell(b) \qquad \text{for all } a \in A \text{ and } b \in B\,.$$

[*Hint:* To prove (i), apply the result of Problem 2 to the Minkowski functional p_K where K is any translate of $A - B$ that contains 0. To prove (ii), apply the result of part (i) to $A' = A + B(0, \epsilon)$ for a suitable $\epsilon > 0$.]