MATHEMATICS 3103 (Functional Analysis)
YEAR 2012–2013, TERM 2

PROBLEM SET #3

This problem set is due at the *beginning* of class on Monday 18 February. I urge you to start on it *early*, as it is a bit long, and some of the problems are not entirely trivial. Only Problems 2, 4(a,b) and 6 will be formally assessed, but I strongly urge you not to neglect the others!

Topics: Normed linear spaces. Completeness of $\mathcal{C}(X)$. Definition and completeness of ℓ^p $(1 < p < \infty)$. Elementary properties of normed linear spaces. Subspaces and quotient spaces. Continuous (= bounded) linear mappings. The space $\mathcal{B}(X, Y)$ of continuous linear mappings. Examples of bounded and unbounded linear operators. Special properties of finite-dimensional spaces.

Readings:

- Handout #3: Introduction to normed linear spaces.

1. **An alternate characterization of norms, and an alternate approach to ℓ^p.** In Lemma 3.9(b) we saw that the closed unit ball of a normed linear space is convex. Here we want to prove the converse of this result. We will then use this fact to give an alternate proof (not needing Hölder's inequality) that the ℓ^p norm is indeed a norm.

   (a) Let X be a real or complex vector space, and let $N\colon X \to \mathbb{R}$ be a function with the following properties:

      (i) $N(x) \geq 0$ for all $x \in X$ (**nonnegativity**);
      (ii) $N(x) = 0$ if and only if $x = 0$ (**nondegeneracy**);
      (iii) $N(\lambda x) = |\lambda|\, N(x)$ for all $x \in X$ and all $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) (**homogeneity**);
      (iv) The set $B = \{x \in X\colon\, N(x) \leq 1\}$ is convex.

   Prove that N is a norm on X.

   Note that (i)–(iii) are just the standard properties (i)–(iii) of a norm. So what you are proving here is that, in the presence of these properties, the convexity of the "unit ball" is *equivalent* to the triangle inequality.

   (b) For any sequence $x = (x_1, x_2, \ldots)$ of real (or complex) numbers and any real number $p \in [1, \infty)$, define as usual

$$\|x\|_p \;=\; \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \,,$$

and define ℓ^p to be the set of sequences for which $\|x\|_p < \infty$. First give a direct elementary proof that ℓ^p is a vector space. [*Hint:* Bound $|x_i + y_i|^p$ above in terms of $|x_i|^p$ and $|y_i|^p$.] Then prove that the set $B = \{x\colon \|x\|_p \le 1\}$ is convex. [*Hint:* Use the convexity of the function $t \mapsto t^p$ on $[0, \infty)$.] Conclude from part (a) that $\|\cdot\|_p$ is a norm on ℓ^p.

Why does this proof fail if $0 < p < 1$?

2. **Inequivalent norms on an infinite-dimensional space.** The linear space $\mathcal{C}[0, 1]$ of continuous real-valued functions on the interval $[0, 1]$ can be equipped with many different norms, and here I would like to consider two of them:

- the sup norm $\|\cdot\|_\infty$ (Example 8 of Handout #1); and
- the L^1 norm $\|\cdot\|_1$ (Example 9 of Handout #1).

(a) Is the identity mapping of $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$ into $(\mathcal{C}[0, 1], \|\cdot\|_1)$ bounded? Prove your assertion; and if the map is bounded, find its operator norm.

(b) Same question for the identity mapping of $(\mathcal{C}[0, 1], \|\cdot\|_1)$ into $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$.

This phenomenon is linked to the fact that $(\mathcal{C}[0, 1], \|\cdot\|_1)$ is *incomplete*. If both of the spaces were complete, this could not happen, as we shall see when we study the Open Mapping Theorem.

3. **Topological isomorphism versus isometric isomorphism.** Recall that if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear spaces, then a bijective linear map $T\colon X \to Y$ is a **topological isomorphism** in case there exist numbers $m > 0$ and $M < \infty$ such that

$$m\|x\|_X \;\le\; \|Tx\|_Y \;\le\; M\|x\|_X \qquad \text{for all } x \in X\ .$$

It is called an **isometric isomorphism** if we can take $m = M = 1$, i.e. if

$$\|Tx\|_Y \;=\; \|x\|_X \qquad \text{for all } x \in X\ .$$

This is obviously a much stronger property.

Recall that c_0 is the space of real sequences that converge to zero; and let c be the space of real sequences that converge to some finite limit. It is easy to see that c is the vector space spanned by c_0 together with the single sequence $(1, 1, 1, \ldots)$. We equip both c_0 and c with the sup norm.

Now define a map $T\colon c \to c_0$ by

$$\begin{aligned} (Tx)_1 \;&=\; \lim_{n\to\infty} x_n \\ (Tx)_{i+1} \;&=\; x_i - \lim_{n\to\infty} x_n \quad \text{for } i \ge 1 \end{aligned}$$

(a) Prove that T is a bounded linear map from c to c_0, of norm $\|T\| \le 2$.

(b) Prove that T is a bijection from c to c_0, and find an explicit formula for T^{-1}.

(c) Prove that T^{-1} is a bounded linear map from c_0 to c, of norm $\|T^{-1}\| \le 2$. Conclude that c and c_0 are topologically isomorphic.

On the other hand, let us prove that c and c_0 are *not* isometrically isomorphic, i.e. there does *not* exist any isometric isomorphism of c onto c_0. To do this, let us make a definition: If C is a convex subset of a real vector space V, a point $x \in C$ is called an **extreme point** of C if there do not exist *distinct* points $y, z \in C$ and a number $0 < \lambda < 1$ such that $x = \lambda y + (1-\lambda) z$.

(d) Prove that the closed unit ball of c_0 has no extreme points.

(e) Prove that the point $(1, 1, 1, \ldots)$ is an extreme point of the closed unit ball of c.

(f) Conclude from (d) and (e) that there cannot exist an isometric isomorphism of c onto c_0.

4. **Operator norms of linear mappings defined by matrices.** Let $A = (a_{ij})$ be an $m \times n$ matrix of real numbers, and define the linear map $A\colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$(Ax)_i \;=\; \sum_{j=1}^{n} a_{ij} x_j \qquad \text{for } i = 1, 2, \ldots, m\,.$$

(a) Suppose that we equip both the domain space $\mathbb{R}^n$ and the range space $\mathbb{R}^m$ with the sup norm. Prove that the operator norm of the linear mapping A is given by

$$\|A\|_{(\mathbb{R}^n, \|\cdot\|_\infty) \to (\mathbb{R}^m, \|\cdot\|_\infty)} \;=\; \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|\,.$$

[*Hint:* Prove separately the two inequalities represented by the = sign.]

(b) Suppose that we equip both the domain space $\mathbb{R}^n$ and the range space $\mathbb{R}^m$ with the ℓ^1 norm. Prove that the operator norm of the linear mapping A is given by

$$\|A\|_{(\mathbb{R}^n, \|\cdot\|_1) \to (\mathbb{R}^m, \|\cdot\|_1)} \;=\; \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|\,.$$

These formulae for the operator norm of a linear mapping are important in numerical linear algebra.

Now let $A = (a_{ij})_{i,j=1}^{\infty}$ be an infinite matrix of real numbers, and let us try to define a linear map on infinite sequences by

$$(Ax)_i \;=\; \sum_{j=1}^{\infty} a_{ij} x_j \qquad \text{for } i = 1, 2, \ldots\,. \tag{$*$}$$

Of course we have to worry in general about convergence; but there is manifestly no problem when x belongs to c_{00} (the space of infinite sequences having at most finitely many nonzero entries), so this formula certainly defines a linear map A from c_{00} to $\mathbb{R}^{\mathbb{N}}$ (the space of *all* infinite sequences of real numbers). We can then define, for any $1 \le p, q \le \infty$, the quantity

$$\|A\|_{p\to q} \;=\; \sup_{x\in c_{00}\setminus\{0\}} \frac{\|Ax\|_q}{\|x\|_p} \;,$$

which of course might take the value $+\infty$ (in particular we recall that $\|Ax\|_q = +\infty$ if $Ax \notin \ell^q$).

(c) Show that

$$\|A\|_{\infty\to\infty} \;=\; \sup_i \sum_{j=1}^{\infty} |a_{ij}| \;.$$

Show further that if $\|A\|_{\infty\to\infty} < \infty$, then the sum $(*)$ is absolutely convergent whenever $x \in \ell^\infty$ and defines a sequence $Ax \in \ell^\infty$, and that the resulting mapping $A\colon \ell^\infty \to \ell^\infty$ is bounded and has operator norm $\|A\|_{\ell^\infty\to\ell^\infty} = \|A\|_{\infty\to\infty}$.

(d) Show that

$$\|A\|_{1\to 1} \;=\; \sup_j \sum_{i=1}^{\infty} |a_{ij}| \;.$$

Show further that if $\|A\|_{1\to 1} < \infty$, then the sum $(*)$ is absolutely convergent whenever $x \in \ell^1$ and defines a sequence $Ax \in \ell^1$, and that the resulting mapping $A\colon \ell^1 \to \ell^1$ is bounded and has operator norm $\|A\|_{\ell^1\to\ell^1} = \|A\|_{1\to 1}$.

5. **A weighted right shift.** Consider the operator $W\colon \ell^2 \to \ell^2$ defined by

$$W(x_1, x_2, x_3, \ldots) \;=\; (0, x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \ldots) \;.$$

(a) Prove that W is a bounded linear operator on ℓ^2, and compute its operator norm.

(b) Compute the operator norm of W^n for each positive integer n, and show that $\lim\limits_{n\to\infty} \|W^n\|^{1/n} = 0$. (Such an operator is called **quasinilpotent**.)

(c) Does W have any eigenvalues? (Recall that $\lambda \in \mathbb{C}$ is called an **eigenvalue** of W if there exists a nonzero vector $x \in \ell^2$ such that $Wx = \lambda x$.)

6. **Fredholm integral operators.** Let $[a, b]$ be a bounded closed interval of the real line, and let $K\colon [a, b] \times [a, b] \to \mathbb{R}$ be a continuous function. For $f \in \mathcal{C}[a, b]$, define

$$(Tf)(s) \;=\; \int_a^b K(s,t)\, f(t)\, dt \;.$$

(a) Prove that Tf is continuous whenever f is.

(b) Prove that T defines a bounded linear map from $\mathcal{C}[a,b]$ to itself, whose operator norm satisfies

$$\|T\|_{\mathcal{C}[a,b]\to\mathcal{C}[a,b]} \;\le\; \sup_{s\in[a,b]} \int_a^b |K(s,t)|\, dt \;.$$

(c) Prove that the operator norm is in fact *equal* to the expression given above on the right-hand side.

7. **A property of finite-dimensional subspaces.** In a normed linear space X, let M be a closed linear subspace and let N be a finite-dimensional linear subspace. Prove that $M + N$ is closed in X.

   This generalizes the theorem that every finite-dimensional subspace of a normed linear space is closed: just consider the special case $M = \{0\}$.

8. **A criterion for separability.** In a normed linear space X, we say that a subset $A \subset X$ is **total** if the linear span of A (i.e. the set of all *finite* linear combinations of elements of A) is dense in X.

   (a) Prove that if there there exists in X a countable total set, then X is separable.

   (b) Prove, conversely, that if X is separable, then there exists in X a countable total set consisting of linearly independent vectors.

9. **Can one achieve $d(x, M) = 1$ in F. Riesz's lemma?**

   (a) Let X be the closed linear subspace of $\mathcal{C}[0,1]$ (equipped as usual with the sup norm) consisting of the functions that vanish at 0. Let M be the linear subspace of X consisting of the functions $f \in X$ for which $\int_0^1 f(t)\, dt = 0$. It is easy to see that M is closed (why?) and proper (why?). Prove that there does *not* exist $f \in X$ with $\|f\| = 1$ such that $d(f, M) \ge 1$.

   (b) Let $X = \ell^1$, and let $c = (c_1, c_2, \ldots) \in \ell^\infty$ be any sequence whose absolute value does not attain its supremum, i.e. $|c_n| < \|c\|_\infty$ for all n. (For instance, $c_n = 1 - 1/n$ is an example.) Then let M be the subspace of X defined by

   $$M \;=\; \{x \in \ell^1 \colon \sum_{n=1}^{\infty} c_n x_n = 0\} \;.$$

   It is easy to see that M is closed (why?) and proper (why?). Prove that there does *not* exist $x \in X$ with $\|x\| = 1$ such that $d(x, M) \ge 1$.

(c) Let X be any real normed linear space, and suppose that there exists an element $T \in \mathcal{B}(X, \mathbb{R})$ that does *not* attain its supremum on the closed unit ball of X. Let

$$M \;=\; \ker T \;=\; \{x \in X \colon T(x) = 0\} \,.$$

It is easy to see that M is closed (why?) and proper (why?). Prove that there does *not* exist $x \in X$ with $\|x\| = 1$ such that $d(x, M) \geq 1$.

Do you see how part (c) subsumes parts (a) and (b)? What are the maps T in these two cases?

Remark: It follows from a deep theorem due to R.C. James, Characterizations of reflexivity, *Studia Math.* **23**, 205–216 (1964) that, for a Banach space X, the following are equivalent:

(a) X has the "improved F. Riesz property": that is, for every proper closed linear subspace $M \subset X$, there exists $x \in X$ with $\|x\| = 1$ and $d(x, M) = 1$.

(b) Every $T \in \mathcal{B}(X, \mathbb{R})$ attains its supremum on the closed unit ball of X.

(c) X is *reflexive*. [This is a property that we will define later in this course. We will see that, for instance, ℓ^p is reflexive for $1 < p < \infty$ but nonreflexive for $p = 1, \infty$. This explains the behavior in part (b) above.]

It is not hard to show that (c) $\Longrightarrow$ (a) $\Longleftrightarrow$ (b); the difficult part of James' theorem, which is far beyond the scope of this course, is to show that (a) or (b) implies (c).