REVERSIBILITY FOR DIFFUSIONS VIA QUASI-INVARIANCE

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ABSTRACT. Why is the drift coefficient b associated with a reversible diffusion on $\mathbb{R}^d$ given by a gradient? Our explanation is inspired by Handa's recent results on reversibility and quasi-invariance of the invariant measure.

1. INTRODUCTION

We look at the problem of reversibility for operators of the form

$$Lf(x) = \frac{1}{2}\Delta f(x) + \sum_{i=1}^{d} b_i(x)\partial_i f(x), \quad f \in C_c^\infty(\mathbb{R}^d). \tag{1}$$

For simplicity we assume that the function b_i is smooth for every $1 \le i \le d$.

We call L *reversible* if there exists a measure m on $\mathbb{R}^d$ so that

$$\int (Lf)(x)g(x)\, m(dx) = \int (Lg)(x)f(x)\, m(dx), \quad f, g \in C_c^\infty(\mathbb{R}^d).$$

This terminology derives from the time reversal property of the corresponding diffusion process $X = (X_t)$, with initial state chosen randomly using the measure m. If the generator L of X is reversible, and $\mathcal{L}(X_0) = m$, then for any $T > 0$, the two finite horizon processes $(X_t)_{0 \le t \le T}$ and $(X_{T-t})_{0 \le t \le T}$ have identical finite-dimensional distributions. That is, X_t has the same probabilistic properties whether the time parameter t runs forwards or backwards.

A classical result of Kolmogorov [5] tells us that a diffusion process in $\mathbb{R}^d$ with infinitesimal generator L is reversible if and only if the vector field $b(x) = (b_1(x), \ldots, b_d(x))$ is conservative, i.e., given by a gradient. In this note, we offer an alternative proof using the concept of quasi-invariance.

Here is some notation and terminology that we use throughout the paper. A measure m is a non-zero Borel measure that is finite on compact sets. By "transformation" we will mean a measurable bijection with measurable inverse. The space $C^\infty(\mathbb{R}^d)$ is the space of smooth functions on $\mathbb{R}^d$, while $C_c^\infty(\mathbb{R}^d)$ are the smooth functions with compact support. The brackets $\langle x, y\rangle$ will refer to the usual inner product on $\mathbb{R}^d$.

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2. Quasi-invariant measures

Let $\mathbb{R}^d$ be equipped with its Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. Let $S = \{S_v\}_{v\in V}$ denote a group of transformations on $\mathbb{R}^d$, indexed by a vector space V. In other words, for each $v \in V$, the mapping $S_v : \mathbb{R}^d \to \mathbb{R}^d$ is a transformation, and the following properties hold:

$$S_{u+v}(x) = S_u(S_v(x)), \quad S_0(x) = x.$$

Since these mappings are bimeasurable, we can define the image measure by $m \circ S_v(B) := m(S_v(B))$. This image measure is characterized by the fact that for all $g \in C_c^\infty(\mathbb{R}^d)$

$$\int g(S_v(x))\,(m \circ S_v)(dx) = \int g(x)\,m(dx). \tag{2}$$

Definition 1. *Let $S = \{S_v\}_{v\in V}$ be a transformation group on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. A measure m is called S-quasi-invariant if, for each $v \in V$, the measures m and $m \circ S_v$ are equivalent.*

If m is S-quasi-invariant, then we can write the density of $m \circ S_v$ with respect to m, for some measurable function $\Lambda(v, x)$, as

$$\frac{d(m \circ S_v)}{dm}(x) = e^{\Lambda(v,x)}, \quad m\text{-a.s.},$$

and we say that m is S-quasi-invariant with cocycle Λ. This terminology is explained by the proposition below, where (3) is called the cocycle identity. See [3], [4], and [6] for similar results in other contexts and [1] for more information on cocycles.

Proposition 1. *If m is S-quasi-invariant with cocycle Λ, then for any $u, v \in V$, we have:*

$$\Lambda(u + v, x) = \Lambda(u, S_v(x)) + \Lambda(v, x), \quad m\text{-a.s.} \tag{3}$$

Proof. For every $v \in V$ the measures m and $m \circ S_v$ are mutually absolutely continuous. Then on one hand, from the definition of Radon-Nikodym density we have

$$dm \circ S_{u+v}(x) = e^{\Lambda(u+v,x)}dm(x).$$

On the other hand, using the transformation group properties we have

$$\begin{aligned} dm \circ S_{u+v}(x) &= dm \circ S_u(S_v(x)) \\ &= e^{\Lambda(u,S_v(x))}dm(S_v(x)) \\ &= e^{\Lambda(u,S_v(x))}e^{\Lambda(v,x)}dm(x) \\ &= e^{\Lambda(u,S_v(x))+\Lambda(v,x)}dm(x). \end{aligned}$$

Hence $e^{\Lambda(u+v,x)} = e^{\Lambda(u,S_v(x))+\Lambda(v,x)}$ m-a.s., which gives the result. □

3. Reversibility and Quasi-invariance

In the remainder of the paper we take the group S of transformations to be $S_v(x) := x + v$ for $v \in \mathbb{R}^d$. The next proposition provides a host of examples of quasi-invariant measures on $\mathbb{R}^d$.

Proposition 2. *Let $U : \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function. The measure $m(dx) = e^{U(x)}dx$ is S-quasi-invariant with cocycle*

$$\Lambda(v,x) = \int_0^1 \langle \nabla U(S_{tv}(x)), v\rangle \, dt. \tag{4}$$

Proof. Define $\Lambda(v,x) = U(S_v(x)) - U(x)$, so that for any $v \in \mathbb{R}^d$,

$$(m \circ S_v)(dx) = e^{U(S_v(x))}dx = e^{U(S_v(x))-U(x)}e^{U(x)}dx = e^{\Lambda(v,x)}m(dx),$$

so m is S-quasi-invariant with cocycle Λ. Note that

$$U(S_v(x)) - U(x) = U(S_v(x)) - U(S_0(x)) = \int_0^1 \frac{d}{dt}U(S_{tv}(x))dt.$$

We finish by using the chain rule to obtain $\frac{d}{dt}U(S_{tv}(x)) = \langle \nabla U(S_{tv}(x)), v\rangle$. □

Let us comment on how this result helps our intuition for the reversible case. Consider an operator L as in (1), and assume that the measure m is reversible for L. Formally, m should be given by $m(dx) = e^{U(x)}dx$ for some "potential function" U. Then by the result above m is S-quasi-invariant with cocycle (4). Furthermore, we know from Kolmogorov's criterion that in the reversible case we have $b(x) = \frac{1}{2}\nabla U(x)$, where $b(x)$ is the drift of the operator L. So we expect that

$$\Lambda(v,x) = 2\int_0^1 \langle b(S_{tv}(x)), v\rangle \, dt. \tag{5}$$

Our main theorem, Theorem 1, makes this intuition rigorous by showing that *a measure m on $\mathbb{R}^d$ is a reversible measure for the operator L if and only if m is quasi-invariant under the group $\{S_v\}_{v\in\mathbb{R}^d}$ of translations with cocycle (5).*

The following three technical lemmas will be used in proving Theorem 1.

Lemma 1. *Let Λ be given by (5). Fix an arbitrary $v \in \mathbb{R}^d$. For any given $g \in C_c^\infty(\mathbb{R}^d)$ and $t \in \mathbb{R}$, define*

$$g_t(x) = g(S_{tv}(x)) \exp\{\Lambda(tv,x)\}.$$

Then $g_t \in C_c^\infty(\mathbb{R}^d)$ for all $t \in \mathbb{R}$, and

$$\frac{d}{dt}g_t(x) = 2\langle b(x), v\rangle g_t(x) + \langle v, \nabla g_t(x)\rangle. \tag{6}$$

Proof. Set $F_t(x) = \Lambda(tv,x)$ and rewrite $g_t(x) = g(S_{tv}(x))e^{F_t(x)}$. From (5) we see that $x \mapsto \Lambda(v,x)$ is smooth so that $F_t \in C^\infty(\mathbb{R}^d)$ and $g_t \in C_c^\infty(\mathbb{R}^d)$. The product rule gives us

$$\frac{d}{dt}g_t(x) = \left[\frac{d}{dt}g(S_{tv}(x))\right] e^{F_t(x)} + g_t(x)\frac{d}{dt}F_t(x) \tag{7}$$

$$\nabla g_t(x) = [\nabla g(S_{tv}(x))] \, e^{F_t(x)} + g_t(x)\nabla F_t(x). \tag{8}$$

By the chain rule

$$\frac{d}{dt}g(S_{tv}(x)) = \langle v, \nabla g(S_{tv}(x))\rangle. \tag{9}$$

Take the inner product of (8) with v, and subtract from (7) (using (9)) to get

$$\frac{d}{dt}g_t(x) - \langle v, \nabla g_t(x)\rangle = g_t(x)\left(\frac{d}{dt}F_t(x) - \langle v, \nabla F_t(x)\rangle\right). \tag{10}$$

So we need to analyze the function F_t. A change of variables gives

$$F_t(x) = 2\int_0^t \langle b(S_{rv}(x)), v\rangle\, dr. \tag{11}$$

Using the auxiliary function $h(x) := \langle b(x), v\rangle$, we differentiate (11) to get

$$\nabla F_t(x) = 2\int_0^t \nabla(h\circ S_{rv})(x)\, dr.$$

Then using the same calculation as in (9) with h instead of g we get

$$\begin{aligned}\langle v, \nabla F_t(x)\rangle &= 2\int_0^t \langle v, \nabla(h\circ S_{rv})(x)\rangle\, dr = 2\int_0^t \frac{d}{dr}h(S_{rv}(x))\, dr \\ &= 2\left[h(S_{tv}(x)) - h(S_0(x))\right] = 2\left[\langle b(S_{tv}(x)), v\rangle - \langle b(x), v\rangle\right] \\ &= \frac{d}{dt}F_t(x) - 2\langle b(x), v\rangle.\end{aligned}$$

Substituting this back into (10) gives the result. □

Lemma 2. *Let L be an operator of the form (1), and let m be a measure on $\mathbb{R}^d$. If m is reversible for L, then*

$$\int_{\mathbb{R}^d} Lf(x)\, m(dx) = 0, \quad f\in C_c^\infty(\mathbb{R}^d). \tag{12}$$

Proof. Let $f\in C_c^\infty(\mathbb{R}^d)$ be arbitrary. Since both f and Lf have compact support, we can find an open ball $B_r(0)$ centered at the origin with radius big enough to contain the supports of both these functions. Take a function $g\in C_c^\infty(\mathbb{R}^d)$ such that $g = 1$ on $B_r(0)$. We have

$$\int_{\mathbb{R}^d} Lf(x)\, m(dx) = \int_{\mathbb{R}^d} Lf(x)g(x)\, m(dx) = \int_{\mathbb{R}^d} f(x)Lg(x)\, m(dx) = 0,$$

where in the last step we use the fact that $Lg = 0$ on $B_r(0)$. □

Lemma 3. *Let L be an operator of the form (1), and let m be a measure on $\mathbb{R}^d$. Then m is a reversible measure for L if and only if, for any $f, g\in C_c^\infty(\mathbb{R}^d)$,*

$$\int (Lf)(x)g(x)\, m(dx) = -\frac{1}{2}\int \langle\nabla f(x), \nabla g(x)\rangle\, m(dx). \tag{13}$$

Proof. Suppose that m is reversible for L, and fix $f, g\in C_c^\infty(\mathbb{R}^d)$, so that

$$\int (Lf)(x)g(x)\, m(dx) = \int f(x)(Lg)(x)\, m(dx). \tag{14}$$

Now, a direct computation shows that

$$(Lf)(x)g(x) + f(x)(Lg)(x) - L(fg)(x) = -\langle\nabla f(x), \nabla g(x)\rangle. \tag{15}$$

Also, $fg\in C_c^\infty(\mathbb{R}^d)$, so $\int L(fg)(x)\, m(dx) = 0$ by (12). Then integrating both sides of (15) with respect to m we get (13).

Conversely, assume that (13) holds. Since the right hand side of this equation is symmetric in f and g, this implies that the left hand side is also symmetric; *i.e.*, (14) holds. ☐

Theorem 1. *Let L be an operator of the form (1), and let m be a measure on $\mathbb{R}^d$. Then m is a reversible measure for L if and only if m is quasi-invariant under the group $\{S_v\}_{v\in\mathbb{R}^d}$ of all translations with cocycle*

$$\Lambda(v,x) = 2\int_0^1 \langle b(S_{tv}(x)), v\rangle\, dt. \tag{16}$$

Proof. Let us take $g \in C_c^\infty(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$ arbitrary. Integrating both sides of (6) with respect to m we get

$$\int \langle b(x), v\rangle g_t(x)\, m(dx) + \frac{1}{2}\int \langle v, \nabla g_t(x)\rangle\, m(dx) = \frac{1}{2}\int \frac{d}{dt} g_t(x)\, m(dx).$$

Consider a ball $B_r(0)$ big enough to contain the supports of all the functions g_t, for $0 \le t \le 1$ (see Lemma 1 for the definition of g_t), and take $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi(x) = 1$ for $x \in B_r(0)$. Now define the function $f(x) := \langle x, v\rangle\varphi(x)$. We may regard this kind of function f as a "truncated polynomial" of first degree. Note that for $x \in B_r(0)$ we have $f(x) = \langle x, v\rangle$, $\nabla f(x) = v$, and $Lf(x) = \langle b(x), v\rangle$. For such f

$$\int (Lf)(x) g_t(x)\, m(dx) + \frac{1}{2}\int \langle \nabla f(x), \nabla g_t(x)\rangle\, m(dx) = \frac{1}{2}\frac{d}{dt}\int g_t(x)\, m(dx). \tag{17}$$

We know that $g_t \in C_c^\infty(\mathbb{R}^d)$ by Lemma 1.

If m is a reversible measure for L, then the left-hand side of (17) vanishes by (13). Therefore $\int g_t(x)\, m(dx)$ is constant in t and in particular $\int g_1(x)\, m(dx) = \int g_0(x)\, m(dx)$, or equivalently

$$\int g(S_v(x)) \exp\{\Lambda(v,x)\}\, m(dx) = \int g(x)\, m(dx).$$

This gives us (2) with the appropriate density, and therefore m is quasi-invariant under S with the desired cocycle.

Conversely, assume that the measure m is quasi-invariant under S with the given cocycle Λ. Fix an arbitrary $g \in C_c^\infty(\mathbb{R}^d)$, and consider equation (17) for g and $f(x) = \langle x, v\rangle\varphi(x)$ for some $v \in \mathbb{R}^d$. It is clear that the right-hand side of (17) vanishes (see Lemma 1 and equation (2)). Thus (13) holds in this case.

Next we note that, for fixed $g \in C_c^\infty(\mathbb{R}^d)$, the set of functions $f \in C_c^\infty(\mathbb{R}^d)$ that satisfy equation (13) is closed under multiplication. To see this, assume that

$f_1, f_2 \in C_c^\infty(\mathbb{R}^d)$ satisfy (13). Then using (15) with $f_1 f_2$ replacing f we have

$$\begin{aligned}\int (Lf_1 f_2)(x) g(x)\, m(dx) &= \int (Lf_1)(x) f_2(x) g(x)\, m(dx) \\ &\quad + \int f_1(x)(Lf_2)(x) g(x)\, m(dx) \\ &\quad + \int \langle \nabla f_1(x), \nabla f_2(x)\rangle g(x)\, m(dx) \\ &= -\frac{1}{2}\int \langle \nabla f_1(x), \nabla (f_2 g)(x)\rangle\, m(dx) \\ &\quad -\frac{1}{2}\int \langle \nabla f_2(x), \nabla (f_1 g)(x)\rangle\, m(dx) \\ &\quad + \int \langle \nabla f_1(x), \nabla f_2(x)\rangle g(x)\, m(dx) \\ &= -\frac{1}{2}\int f_2(x)\langle \nabla f_1(x), \nabla g(x)\rangle\, m(dx) \\ &\quad -\frac{1}{2}\int f_1(x)\langle \nabla f_2(x), \nabla g(x)\rangle\, m(dx) \\ &= -\frac{1}{2}\int \langle \nabla (f_1 f_2)(x), \nabla g(x)\rangle\, m(dx),\end{aligned}$$

so the product $f_1 f_2$ also satisfies (13).

Thus (13) can be extended to all functions $f(x) = \langle x, v_1\rangle \cdots \langle x, v_k\rangle \varphi(x)$ with $v_1, \ldots, v_k \in \mathbb{R}^d$. Using the linearity in f of (13) it follows that this expression must be true for all "truncated polynomials" f of arbitrary degree. A suitable approximation procedure (see *e.g.* [2, Appendix 7]) shows that (13) is valid for all $f \in C_c^\infty(\mathbb{R}^d)$. Since $g \in C_c^\infty(\mathbb{R}^d)$ was fixed arbitrarily, this establishes the reversibility of m. □

As consequence, we give now our explanation that an operator L as in (1) is reversible precisely when its drift b is of gradient form.

Corollary 1. *The operator L as in (1) has a non-zero reversible measure m, if and only if b has a potential,* i.e.*, there is a function $F : \mathbb{R}^d \to \mathbb{R}$ such that $b = \nabla F$.*

Proof. If $b = \nabla F$, then set $U(x) = 2F(x)$ and $m(dx) = e^{U(x)}dx$. Proposition 2 and Theorem 1 show that m is reversible for L.

Now suppose that L has a non-zero reversible measure m. Define Λ as in (16) using the function b associated with L. By Proposition 1 and Theorem 1, Λ satisfies the cocycle identity:

$$\Lambda(u + v, x) = \Lambda(u, x + v) + \Lambda(v, x), \quad m\text{-a.s.} \tag{18}$$

On the other hand, Theorem 1 also says that m is quasi-invariant with respect to shifts, therefore it has full support on $\mathbb{R}^d$. Since both sides of the equation (18) are continuous functions, the equation must be true for all $x \in \mathbb{R}^d$.

To show that b has a potential it is enough to prove that

$$\partial_v \langle b(x), u\rangle = \partial_u \langle b(x), v\rangle, \quad u, v \in \mathbb{R}^d, x \in \mathbb{R}^d. \tag{19}$$

For if (19) holds, then taking $u = e_i$, $v = e_j$ we get $\partial_j b_i(x) = \partial_i b_j(x)$, and this condition is sufficient for b to have a potential, since the domain $\mathbb{R}^d$ is simply connected. Now let us establish (19). From the cocycle identity we get

$$\Lambda(u, x+v) + \Lambda(v, x) = \Lambda(v, x+u) + \Lambda(u, x),$$

or equivalently

$$\Lambda(u, x+v) - \Lambda(u, x) = \Lambda(v, x+u) - \Lambda(v, x).$$

Then using the definition of Λ we get

$$\int_0^1 \langle b(x+v+tu) - b(x+tu), u\rangle\, dt = \int_0^1 \langle b(x+u+tv) - b(x+tv), v\rangle\, dt.$$

Replace u by δu and v by εv, for some $\delta, \varepsilon > 0$. Then

$$\int_0^1 \frac{\langle b(x+\varepsilon v+\delta tu) - b(x+\delta tu), u\rangle}{\varepsilon}\, dt = \int_0^1 \frac{\langle b(x+\delta u+\varepsilon tv) - b(x+\varepsilon tv), v\rangle}{\delta}\, dt,$$

and letting first $\delta \to 0$ and then $\varepsilon \to 0$ we get $\partial_v \langle b(x), u\rangle = \partial_u \langle b(x), v\rangle$. □

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