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A Nonlinear Model of the Helmholtz Resonator with a Movable Wall

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Abstract—A nonlinear model of the Helmholtz resonator with a movable wall is developed. The energy reflectance, transmittance, and absorbance, as well as the energy distribution, are calculated as functions of frequency with allowance for Stokes friction and the friction of an acoustic boundary layer. Characteristics necessary for such a system to be used as a sound absorber or sound insulator are considered.

At high levels of sound pressure, the response of a concentrated oscillatory system—a Helmholtz resonator (HR)—to an external stimulus in the form of an acoustic wave is known to become nonlinear [1]. This nonlinearity gives rise to harmonics, combination frequencies [2] and amplitude-dependent shifts of the maximum of the HR frequency response [3]. Most treatises on resonance sound absorbers assume a fixed back wall of the resonator. Rudenko and Khizhnykh [4] have proposed a nonlinear model of the HR that describes all said nonlinear phenomena in systems where HRs may be used to absorb intense acoustic or shock waves.

Rzhevkin *et al.* [5, 6] have studied resonators with a compliant front wall and the radiation of a panel coated with sound absorber. In real situations with an oscillating back wall of the resonator, one have to cope not only with a reflected wave but also with a transmitted wave. This model not only alters the characteristics of the HR as a sound absorbing system, but also compels one to consider, unlike [4], a concomitant sound insulation problem.

In this paper, we extend the model of Rudenko and Khizhnykh [4] sketched in Fig. 1. We consider an oscillatory system consisting of a throat (*1*) made as a cylindrical tube of radius a, and a cavity (2) with a movable back wall (*3*). The throat is filled with a viscous incompressible liquid of density ρ_l, the cavity is filled with air of density ρ_0. The system is irradiated by a sound wave incident from the left. This wave sets in motion the liquid in the throat and, via the compressible air in the cavity, the back wall of the HR. A part of energy of the incident wave is absorbed due to the viscosity of the throat liquid, the friction of the oscillating wall, and reradiation into the reflected and transmitted waves.

Let us construct a mathematical model that describes the oscillations of the wall and the throat liquid. The equation of motion for the wall is

$$m\frac{d^2z}{dt^2} = -kz + p(x = L, t)S' - p_{ac2}(z = 0, t)S', \quad (1)$$

where z is the travel of the wall from the equilibrium position, m is the mass of the wall, and p and p_{ac2} are the pressure at the input into the cavity and the acoustic pressure behind the wall. We assume that the oscillations of the back wall are bending plate oscillations, and we will consider only their "piston" mode.

The pressure at the resonator input may be defined using the adiabaticity of forcing a liquid into the cavity:

$$p(x = L, t) = -c_0^2\rho\frac{V'}{V_0} = \frac{c_0^2\rho_0}{V_0} \times\int_0^a \xi(x = L, r, t)2\pi r dr - \frac{c_0^2\rho S'}{V_0}z, \quad (2)$$

where V' is the increment of the volume in the cavity, and ξ is the displacement of liquid in the throat.

Using (2) and differentiating (1) with respect to time, we obtain the equation for the wall velocity

$$\frac{d^2u}{dt^2} + \left[\frac{k}{m} + \frac{c_0^2\rho_0 S'^2}{V_0 m}\right]u = \frac{c_0^2\rho}{V_0 m} \times\int_0^a V_x(x = L, r, t)2\pi r dr - \frac{S'}{m}\frac{dp_{ac2}}{dt}(z = 0, t). \quad (3)$$

In order to describe the motion of liquid in the throat, we use simplified equations of quasi-uniform flow [7, 8]

$$\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_r\frac{\partial V_x}{\partial r} - \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) = -\frac{1}{\rho_l}\frac{\partial}{\partial x}p(x, t), \quad (4)$$

$$\frac{\partial V_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0, \quad (5)$$

where V_x and V_r are the longitudinal and transverse velocity components of the liquid and ν is the shear viscosity.

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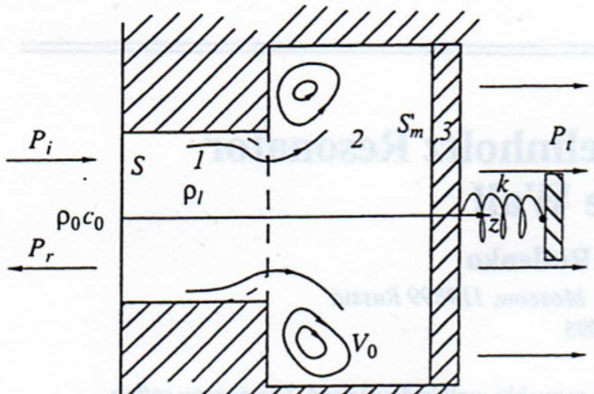

Fig. 1. Helmholtz resonator with a movable wall.

These equations suggest that the pressure must be a linear function of x: $p(x, y) = p_{ac2}(t) + x\beta(t)$ [4]; at $x = 0$ it must be equal to the pressure at the input to the throat $p_{ac}(x = 0, t)$, and at $x = L$, it must be $p(x = L, t)$. Now, we easily obtain the right side of equation (4). We construct an approximate solution of the equations for flows in the throat by introducing the longitudinal velocities averaged over the cross section and over the volume of the throat:

$$\overline{V} = \frac{1}{S}\int_0^a V_x 2\pi r dr, \quad \overline{\overline{V}} = \frac{1}{SL}\int_0^L dx\int_0^a V_x 2\pi r dr, \tag{6}$$

where $S = \pi a^2$. We note that, from the continuity, it follows that $\overline{V} = \overline{V}(t)$; a flow over the cross-section does not depend on the coordinate x. This conclusion agrees with the fact that we seek a solution in the low-frequency region.

Combining equations (3)–(6) and averaging over the volume, we obtain

$$\begin{aligned}\frac{d^2\overline{\overline{V}}}{dt^2} + \frac{c_0^2\rho_0}{V_0\rho_l L}\overline{\overline{V}} + D_1(\overline{\overline{V}}) + D_2(\overline{\overline{V}}) + \frac{1}{L}\frac{d}{dt}\\ \times[\overline{V}^2(L, t) - \overline{V}^2(0, t)] = \frac{1}{\rho_l L}\frac{dp_{ac}(t)}{dt} + \frac{c_0\rho_0 S'}{\rho_l L V_0}\dot{u},\end{aligned} \tag{7}$$

where [7]

$$D_1(\overline{\overline{V}}) = \frac{1}{\pi a}\sqrt{\frac{\nu}{\pi}}\frac{d}{dt}\int_{-\infty}^{t}\frac{d\overline{\overline{V}}}{dt'}\frac{dt'}{\sqrt{t-t'}} \tag{8}$$

is the dissipative term responsible for the acoustic boundary layer, and

$$D_2(\overline{\overline{V}}) = \nu a_2 n\frac{d\overline{\overline{V}}}{dt} \tag{9}$$

is a second dissipative term responsible for the Stokes friction.

If the throat is filled with a material that has a surface considerably exceeding that of the pipe, then πa in (8) should be replaced with a constant of length dimension: $a_1 = a/N$, where N takes into account the increase of the surface of the boundary layer, a_2 is a second empirical constant with a magnitude of the order of the typical dimension of elements passed by the stream, and n is the volume concentration of these elements (the physical meaning of these constants is discussed in depth in [4, 9, 10]). For simplicity, we further omit the overbars and write v as V.

The nonlinear term in equation (7) may be approximated by a power expansion [4, 9, 10]

$$I = (\gamma_1 V + \gamma_2 V^2)dV/dt. \tag{10}$$

Equations (3) and (7) describe two oscillatory circuits: the throat and wall; however, for a complete problem statement, we need to introduce boundary conditions that will match the hydrodynamic and acoustic parts of the problem. For $x < 0$, the acoustic field is constituted by the incident and reflected waves:

$$\begin{aligned}p_{ac}(x, t) &= p_i(t - x/c_0) + p_r(t + x/c_0),\\ u_{ac}(x, t) &= \frac{1}{\rho_0 c_0}[p_i(t - x/c_0) - p_r(t + x/c_0)],\end{aligned} \tag{11}$$

where u_{ac} is the oscillatory velocity.

Now, with the matching conditions:

$$u_{ac}(0, t) = V(0, t), p_{ac2}(z = 0, t) = \rho_0 c_0 u,$$

the equations of system motion take the form

$$\frac{d^2u}{dt^2} + 2\delta_1\frac{du}{dt} + \omega_{01}^2 u = \kappa_1 V, \tag{12}$$

$$\begin{aligned}\frac{d^2V}{dt^2} + 2\delta_2\frac{dV}{dt} + \omega_{02}^2 V + \frac{1}{a_1}\sqrt{\frac{\nu}{\pi}}\frac{d}{dt}\int_{-\infty}^{t}\frac{dV}{dt'}\frac{dt'}{\sqrt{t-t'}}\\ + (\gamma_1 V + \gamma_2 V^2)\frac{dV}{dt} = \frac{2}{\rho_l L}\frac{dp_i(t)}{dt} + \kappa_2 u,\end{aligned} \tag{13}$$

where

$$\omega_{01}^2 = k/m + c_0^2\rho_0 S'^2/V_0 m \text{ and } \omega_{02}^2 = c_0^2\rho_0 S/\rho_l V_0 L$$

are the eigenfrequencies of the wall and throat in the absence of nonlinearity and friction,

$$\delta_1 = \rho_0 c_0 S'/2m \text{ and } \delta_2 = \nu a_2 n/2 + \rho_0 c_0/2\rho_l L$$

are the losses due to friction and radiation (the wall friction may be taken into account by adding a friction factor to δ_1), and

$$\kappa_1 = c_0^2\rho_0 SS'/mV_0 \text{ and } \kappa_2 = c_0^2\rho_0 S'/\rho_l V_0 L$$

are the coefficients of coupling between two oscillatory circuits.

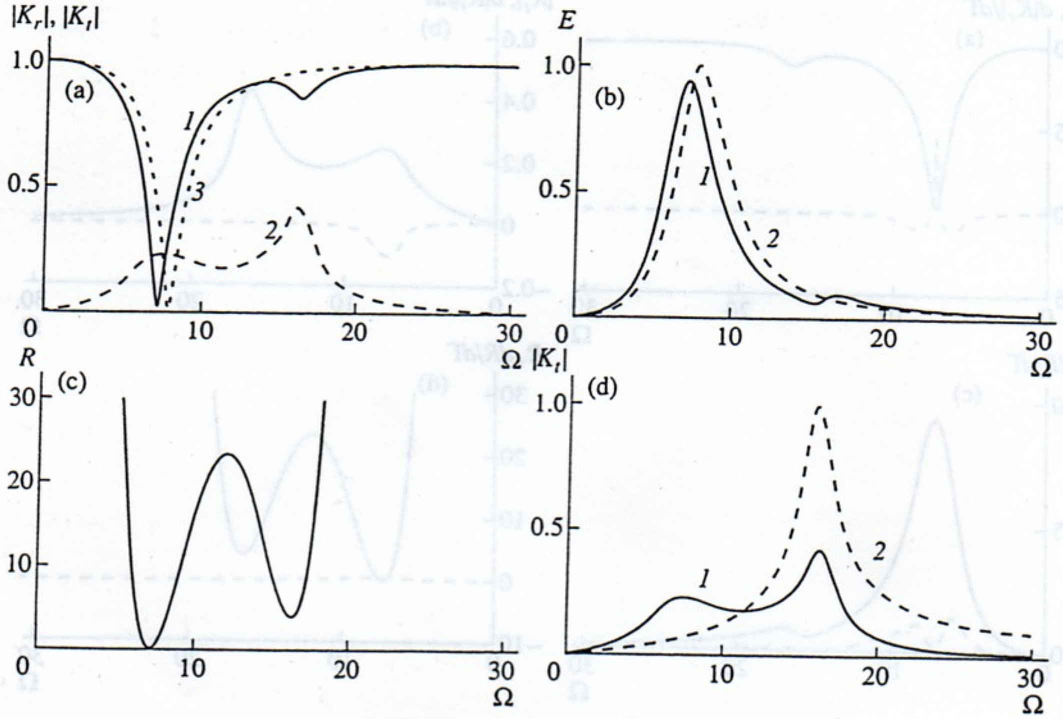

Fig. 2. Acoustic characteristics of the resonator.

Consider the simplest case of a harmonic incident wave with pressure

$$p_i = \frac{1}{2}(Pe^{i\omega t} + \text{c.c.})$$

and a weak nonlinearity. Then, to take into account nonlinear corrections to the amplitude–frequency response of the HR, it suffices to include the cubic nonlinearity. Now, for the frequency dependences of the absorption factor $K_r = p_r/p_i$ and the sound transmission factor $K_t = p_t/p_i$, we obtain

$$K_r = 1 - \{4i\Omega(\Omega_{01}^2 - \Omega^2 + 2i\Delta_1\Omega)\}/\{(\Omega_{01}^2 - \Omega^2 + 2i\Delta_1\Omega)(\Omega_{02}^2 - \Omega^2 + 2i\Delta_2\Omega - \Omega^{3/2} \times V(1-i) + i\Gamma F(\Omega)|\Pi_0|^2) - K_1K_2\},$$

$$K_t = \{4i\Omega K_1\}/\{(\Omega_{01}^2 - \Omega^2 + 2i\Delta_1\Omega)(\Omega_{02}^2 - \Omega^2 + 2i\Delta_2\Omega - \Omega^{3/2}V(1-i) + i\Gamma F(\Omega)|\Pi_0|^2) - K_1K_2\},$$

where

$$F(\Omega) = \frac{32\Omega^3((\Omega_{01}^2 - \Omega^2)^2 + 4\Delta_1^2\Omega^2)}{F'(\Omega)}, \quad (14)$$

$$F'(\Omega) = ((\Omega_{01}^2 - \Omega^2)(\Omega_{02}^2 - \Omega^2) - \Omega^{3/2} \times V(\Omega_{01}^2 - \Omega^2) - 4\Delta_1\Delta_2\Omega^2 - 2\Delta_1\Omega^{5/2} - K_1K_2)^2 + (2\Delta_2\Omega(\Omega_{01}^2 - \Omega^2) + 2\Delta_1\Omega(\Omega_{02}^2 - \Omega^2) + \Omega^{3/2}V(\Omega_{01}^2 - \Omega^2) - 2\Omega^{3/2}V\Delta_1\Omega)^2,$$

and the dimensionless variables

$$\tau = \frac{2\rho_l L}{\rho_0 c_0}, \quad \Omega = \omega\tau, \quad \Omega_{01} = \omega_{01}\tau,$$

$$\Omega_{02} = \omega_{02}\tau, \quad \Delta_1 = \delta_1\tau, \quad \Delta_2 = \delta_2\tau,$$

$$V = \sqrt{\frac{\nu\tau}{2a_1^2}}, \quad K_1 = \kappa_1\tau^2, \quad K_2 = \kappa_2\tau^2, \quad (15)$$

$$\Gamma = \frac{1}{8}\gamma_2 c_0^2\tau, \quad \Pi_0 = \frac{|P|}{\rho_0 c_0}.$$

The typical frequency responses of $|K_r|$ and $|K_t|$ are shown in Fig. 2a (curves *1* and *2*, respectively) for the following parameters $\Omega_{01} = 16$, $\Omega_{02} = 8$, $\Delta_1 = 1$, $\Delta_2 = 2$, $V = 0$, $\Gamma = 0$, $K_1 = 50$, and $K_2 = 50$. In the linear case, these dependencies exhibit a resonance behavior in the range of throat and wall eigenfrequencies. The reflected wave will have minimum near these frequencies, and the transmitted wave will have maximum.

For analysis of the acoustic characteristics of such a system, it is convenient to use the coefficient of energy absorbed in the resonator, E, and the energy ratio of reflected and transmitted waves, $R = |K_r|^2/|K_t|^2$. To define E, we assume that the energy is conserved:

$$|K_r|^2 + |K_t|^2 + E = 1, \quad (16)$$

where E is normalized to the energy of the incident wave.

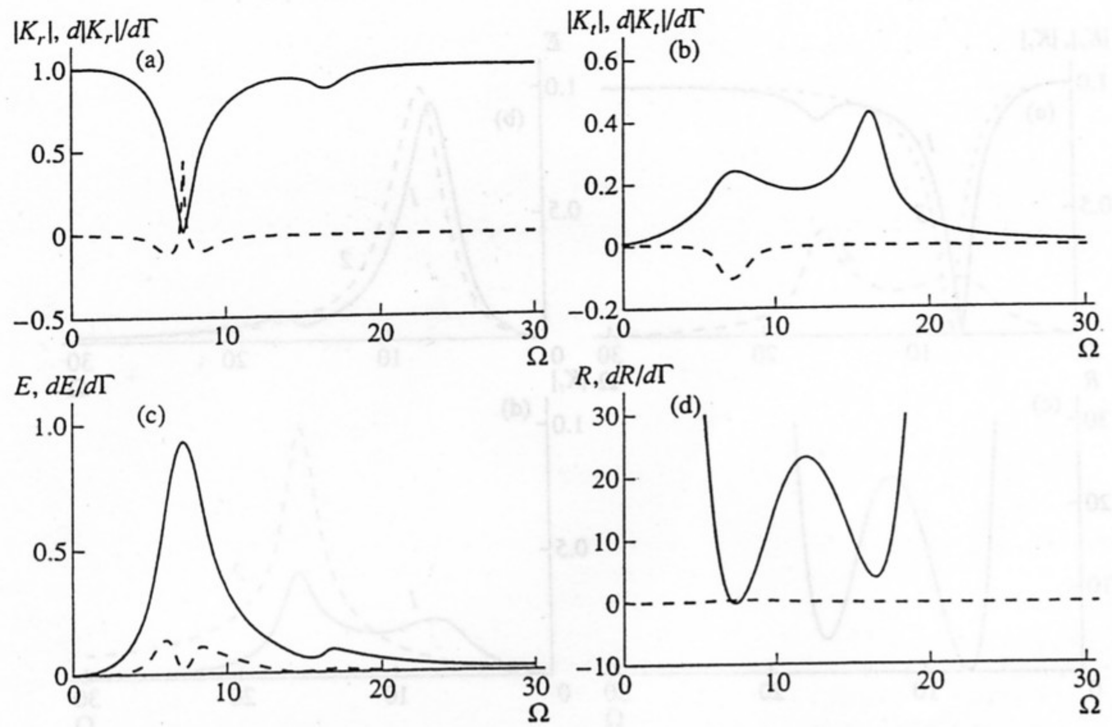

Fig. 3. Partial derivatives with respect to Γ.

The profiles of E and R as functions of the frequency of the incident wave with the same parameters as $|K_r|$ and $|K_t|$ are presented in Figs. 2b (curve *1*) and 2c. From the plot for E, one may define the bands of best energy absorption: they are near the frequencies Ω_{01} and Ω_{02}. Far from eigenfrequencies, only the reflected wave is present and R tends to infinity; whereas near these frequencies, R has a minimum; and between these frequencies, R has a local maximum.

Incorporation of the movable wall alters the reflection characteristics. This can be seen from Fig. 2a, which compares the reflectance for the resonator with a movable wall (curve *1*) with the reflectance of the same resonator with a fixed wall (curve *3*). The fixed-wall frequency profile of E also has a different form (Fig. 2b, curve *2*). For an ideal HR (with a fixed wall at $V = 0$ and $\Gamma = 0$), the condition of total absorption is defined as $\Delta_2 = 2$ at the frequency of the incident wave $\Omega_2 = \Omega_{02}^2$. In order to find the corrections Δ and α to these values for the considered system, we substitute $\Delta_2 = 2 + \Delta$ and $\Omega^2 = \Omega_{02}^2 + \alpha$ into the system

$$\mathrm{Re}(K_r) = 0, \quad \mathrm{Im}(K_r) = 0. \tag{17}$$

Solving this system for relatively small quantities Δ and α and neglecting their second and higher powers, we obtain the corrections to resonator parameters

$$\Delta = \frac{-2\Delta_1 K_1 K_2 + VB + V^2 C}{2[(\Omega_{01}^2 - \Omega_{02}^2)^2 + 4\Delta_1^2\Omega_{02}^2 + VA]} \tag{18}$$

and to frequency

$$\alpha = -\frac{(\Omega_{01}^2 - \Omega_{02}^2)K_1 K_2 - VD}{(\Omega_{01}^2 - \Omega_{02}^2)^2 + 4\Delta_1^2\Omega_{02}^2 + VA}, \tag{19}$$

where A, B, C, and D are frequency multipliers:

$$A = 3/4\Omega_{01}^4\Omega_{02}^{-1/2} - 5/2\Omega_{01}^2\Omega_{02}^{3/2} + 2\Delta_1\Omega_{01}^2\Omega_{02}^{1/2} + 7/4\Omega_{02}^{7/2} + 3\Delta_1^2\Omega_{02}^{3/2},$$

$$B = 2\Omega_{01}^2\Omega_{02}^{5/2} - \Omega_{01}^4\Omega_{02}^{1/2} - \Omega_{02}^{9/2} - 2\Delta_1\Omega_{02}^{7/2} - 4\Delta_1^2\Omega_{02}^{5/2} + 1/4\Omega_{01}^2\Omega_{02}^{-3/2}K_1K_2 - 5/4\Omega_{02}^{1/2}K_1K_2,$$

$$C = \Omega_{01}^4 + \Omega_{01}^2\Omega_{02}^2 - 2\Delta_1\Omega_{01}^2\Omega_{02} - 1/2\Omega_{02}^4 - 3/4\Delta_1\Omega_{02}^3 + 2\Delta_1^2\Omega_{02}^2, \tag{20}$$

$$D = \Omega_{01}^4\Omega_{02}^{3/2} - 2\Omega_{01}^2\Omega_{02}^{7/2} + \Omega_{02}^{11/2} + 4\Delta_1\Omega_{02}^{9/2}.$$

Figure 2d illustrates the possibility of using an HR with a movable wall for sound insulation. Curves *1* and *2* represent the frequency dependencies of the transmission factor of the considered system and the wall with the same parameters except for the sound insulating coat. It is obvious that the insulating coat reduced the amplitude of the transmitted wave. The performance of this system as a sound muffler will be decided by the frequency profile of K_t.

Each of the frequency dependencies for $|K_r|$, $|K_t|$, E, and R is determined by six parameters: Δ_1, Δ_2, K_1, K_2,

V, and Γ. To analyze the effect of each of them on the characteristics of a specific resonator with fixed values for these parameters, one can consider the partial derivatives of $|K_r|$, $|K_t|$, E, and R with respect to these parameters as functions of the frequency of incident waves.

An analysis of these partial derivatives indicates that the parameter Δ_2, which is responsible for losses due to Stokes friction in the throat and radiation of the reflected wave, broadens the profile of the reflection coefficient and the profile of the absorbed energy (energy increases) near Ω_{02}, reduces the transmission coefficient (especially at eigenfrequencies of the throat and wall), and almost does not affect the distribution of energy between the reflected and transmitted waves.

The losses for wall friction and radiation of the transmitted waves, Δ_1, weakly affect the reflected wave (increase reflection at the wall frequency). An increase of Δ_1 considerably attenuates the transmitted wave at this frequency, thus increasing the absorbed energy.

With increase in the losses in the acoustic throat boundary layer (parameter V), the profiles of reflectance and absorbed energy will be shifted towards lower frequencies, and the transmitted wave will be attenuated, especially at the frequencies Ω_{01} and Ω_{02}.

The coupling coefficients K_1 and K_2, defined by the wall and throat parameters, weakly affect the resonator behavior. However, a considerable increase in K_1 can result in a greater amplitude of the transmitted wave near the wall eigenfrequency.

With reference to the nonlinear effects controlled by the parameter Γ, the partial derivatives plotted (by dashed line) for the parameters used to draw the resonator coefficients are depicted in Fig. 3, to compare with $|K_r|$, $|K_t|$, E, and R (solid line).

The variation of this parameter strongly affects the resonator characteristics near the throat eigenfrequency. An increase of nonlinearity reduces reflectivity at frequency Ω_{02} and broadens the curve. The absorbed energy will increase (the curve also broadens), and the amplitude of the transmitted wave will decrease. Variation of the nonlinearity parameter weakly affects the distribution of energy between the reflected and transmitted waves.

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REFERENCES

1. Ingard, U. and Labate, S., Acoustic Circulation Effects and Nonlinear Impedance of Orifices, *J. Acoust. Soc. Am.*, 1950, vol. 22, no. 2, pp. 211–218.
2. Thurston, G.B., Hargrove, L.E., and Cook, B.D., Nonlinear Properties of Circular Orifices, *J. Acoust. Soc. Am.*, 1957, vol. 29, no. 9, pp. 992–1001.
3. Velizhanina, K.A. and Lebedeva, I.V., Study of Resonance Sound Absorbers at High Levels of Sound, *Akust. Zh.*, 1980, vol. 26, no. 5, pp. 667–672 [*Sov. Phys. Acoust.*, vol. 26, no. 5, pp. 377–379].
4. Rudenko, O.V. and Khizhnykh, K.L., Model of the Helmholtz Resonator for Absorption of Intense Sound, *Akust. Zh.*, 1990, vol. 36, no. 3, pp. 527–534 [*Sov. Phys. Acoust.*, vol. 36, no. 3, pp. 293–297.]
5. Rzhevkin, S.N., Resonance Sound Absorber with a Compliant Wall, *Zh. Tech. Fiz.*, 1946, vol. XVI, no. 4, pp. 381–394.
6. Ivanov-Schitz, K.M., Rschevkin, S.N., and Welischanina, K.A., Uber die Wirkung des Schallabsorbers, der eine schwingende Oberflache bedeckt, auf die Schallabstrahlung, *Acustica*, 1963, vol. 13.
7. Landau, L.D. and Lifshitz, E.M., *Fluid Mechanics*, Oxford: Pergamon, 1959.
8. Gusev, V.E. and Rudenko, O.V., Nonstationary Quasi-Unidimensional Acoustic Flows in Unbounded Volumes with Allowance for Hydrodynamic Nonlinearity, *Akust. Zh.*, 1979, vol. 25, no. 6, pp. 875–881 [*Sov. Phys. Acoust.*, vol. 25, no. 6, pp. 493–496].
9. Rudenko, O.V. and Khizhnykh, K.L., Resonance Absorption of Intense Sound, *Opt. Acoust. Rev.*, 1990–91, vol. 1, no. 2, pp. 141–153.
10. Rudenko, O.V., Nonlinear Methods of Intense Noise Control, *Noise Control in Russia*, Moscow: NPK Informatika, 1992, pp. 175–185.