Noise-Induced Phase Transitions in Nonlinear Oscillators

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Abstract. Examples of nonequilibrium noise-induced phase transitions of the second kind in nonlinear oscillators are considered. These transitions reveal themselves in the excitation of relatively ordered oscillations under influence of noise, multiplicative or additive, or a combination of both.

INTRODUCTION

In the last few years great progress has been made in studying strong influence of small noises on the behaviour and properties of systems. Works concerning so-called large fluctuations, when escapes of a system away from its stable equilibrium state occur due to small noise (see, for example, (1–9)), should be noted primarily. This problem is of great importance because it is associated with irreversibility of physical processes. Among other important problems that should be mentioned are so-called fluctuational transport (10–16), when small noise causes directional motion of a Brownian particle, and noise-induced phase transitions, when the state of a system changes qualitatively as the intensity of noise acting upon the system increases. The latter can manifest themselves in the appearance of new extrema in the system probability distribution or dissappearance of old ones (17), in occurrence of multistability (17), in stabilization or destabilization of some system equilibrium states (18,19), in the appearance of so-called mean field (20,21), in excitation of oscillations (18,22–24,19) and so on.

In this paper we consider only noise-induced phase transitions revealing themselves in excitation of oscillations. It should be noted that the mechanisms of such excitation can be different.

NOISE-INDUCED PHASE TRANSITIONS IN A PENDULUM WITH A RANDOMLY VIBRATING SUSPENSION AXIS. THE INFLUENCE OF ADDITIVE NOISE

The motion equation for a pendulum with suspension axis randomly vibrating in a certain direction making an angle γ with the vertical (Fig. 1) can be written as

$$\ddot{\varphi} + 2\beta\left(1 + \alpha\dot{\varphi}^2\right)\dot{\varphi} + \omega_0^2 \sin\varphi + \omega_0^2\xi(t)\sin(\varphi + \gamma) = 0, \qquad (1)$$

where φ is the pendulum angular deviation from the equilibrium position, ω_0 is the natural frequency of small free pendulum oscillations, β is the damping factor, α is the coefficient of nonlinear friction, $\omega_0^2\xi(t)$ is the quantity proportional to the suspension axis acceleration that is assumed to be a comparatively wide-band random process with nonzero power spectrum density at the frequencies ω_0 and $2\omega_0$.

We assume that the intensity of the suspension axis vibration is moderately small, so that pendulum oscillations can be considered small to an extent that φ can be substituted in place of $\sin\varphi$. In this case Eq. (1) becomes

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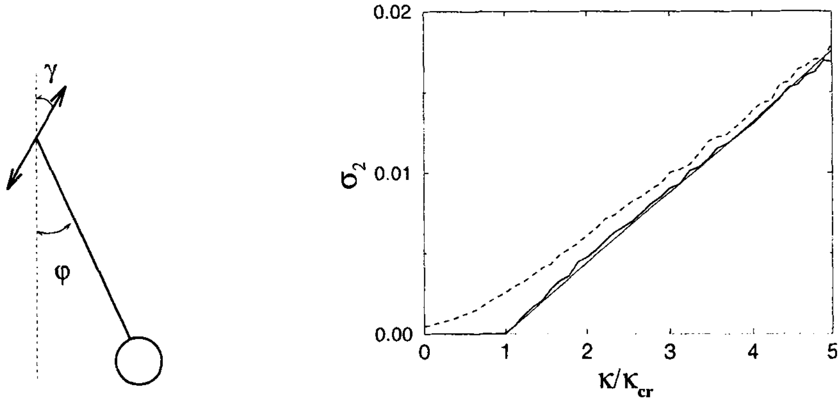


FIGURE 1. Schematic image of a pendulum with randomly vibrating suspension axis (left). Examples of the dependencies (right) of the variance of φ on multiplicative noise intensity found by numerical simulation of Eq. (1) in the presence of additive noise (dashed line; $\kappa_{\xi_2}(\omega_0)/\kappa_{cr} = 0.0005$) and in its absence (solid line).

$$\ddot{\varphi} + 2\beta\left(1 + \alpha\dot{\varphi}^2\right)\dot{\varphi} + \omega_0^2\Big(1 + \xi_1(t)\Big)\varphi = \omega_0^2\xi_2(t), \tag{2}$$

where $\xi_1(t) = \xi(t)\cos\gamma$ is the multiplicative component of the suspension axis random vibration, and $\xi_2(t) = -\xi(t)\sin\gamma$ is its additive component. Thus, if the vibration direction is not vertical then the vibration causes not only multiplicative noise, but additive noise as well.

An approximate analytical solution of the problem can be obtained on the assumptions that $\beta/\omega_0 \sim \epsilon$ and $\xi(t) \sim \sqrt{\epsilon}$, where ϵ is a certain small parameter which should be put equal to unity in the final results. With these assumptions Eq. (2) can be solved by the Krylov–Bogolubov method; to do this we set $\varphi = A(t)\cos\psi(t) + \epsilon u_1 + \ldots$, where $\psi(t) = \omega_0 t + \phi(t)$,

$$\begin{aligned} \dot{A} &= \epsilon f_1 + \ldots, \\ \dot{\phi} &= \epsilon F_1 + \ldots, \end{aligned} \tag{3}$$

$u_1, \ldots, f_1, \ldots, F_1, \ldots,$ are unknown functions. By using the Krylov–Bogolubov technique for stochastic equations (see (48)) we find the expressions for the unknown functions f_1 and F_1. Substituting these expressions into Eqs. (3) we obtain

$$\dot{A} = -\beta\left(1 + \frac{3}{4}\alpha\omega_0^2A^2\right)A + \omega_0\overline{g_1(A, \psi, \xi_1, \xi_2)}, \tag{4}$$

$$\dot{\phi} = \omega_0\overline{g_2(A, \psi, \xi_1, \xi_2)}, \tag{5}$$

where

$$\begin{aligned} g_1(A, \varphi, t) &= \frac{A}{2}\xi_1(t)\sin 2(\omega_0 t + \varphi) + \xi_2(t)\sin(\omega_0 t + \varphi), \\ g_2(A, \varphi, t) &= \xi_1(t)\cos^2(\omega_0 t + \varphi) + \frac{1}{A}\xi_2(t)\cos(\omega_0 t + \varphi), \end{aligned} \tag{6}$$

the bar over the expression signify its time averaging.

As follows from (48), the Fokker–Planck equation associated with Eqs. (4, 5) is

$$\frac{\partial w(A,\phi)}{\partial t} = -\frac{\partial}{\partial A}\left(\left(-\beta\left(1 + \frac{3}{4}\alpha\omega_0^2A^2\right)A + \omega_0^2\int_{-\infty}^{0}\left(\left\langle\overline{\frac{\partial g_1(A,\psi,t)}{\partial A}g_1(A,\psi,t+\tau)}\right\rangle + \right.\right.\right.$$

$$\left.\left.\left.\left\langle\overline{\frac{\partial g_1(A,\varphi,t)}{\partial \varphi}g_2(A,\varphi,t+\tau)}\right\rangle\right)d\tau\right)w(A,\phi)\right) -$$

$$\omega_0^2 \int\limits_{-\infty}^{0} \left(\left\langle \overline{\frac{\partial g_2(A,\psi,t)}{\partial A} g_1(A,\psi,t+\tau)} \right\rangle + \left\langle \overline{\frac{\partial g_2(A,\varphi,t)}{\partial \varphi} g_2(A,\varphi,t+\tau)} \right\rangle \right) d\tau \, \frac{\partial w(A,\phi)}{\partial \varphi} +$$

$$\frac{\omega_0^2}{2} \left(\frac{\partial^2}{\partial A^2} \left(\left(\frac{K_{11}}{4} A^2 + K_{12} \right) w(A,\phi) \right) + \left(K_{21} + \frac{K_{22}}{A^2} \right) \frac{\partial^2 w(A,\phi)}{\partial \phi^2} \right), \tag{7}$$

where the angular brackets signify averaging over statistical ensemble,

$$K_{11} = \frac{1}{2}\kappa_{\xi_1}(2\omega_0) = \frac{\cos^2\gamma}{2}\kappa_\xi(2\omega_0), \quad K_{12} = \frac{1}{2}\kappa_{\xi_2}(\omega_0) = \frac{\sin^2\gamma}{2}\kappa_\xi(\omega_0), \tag{8}$$

$$K_{21} = \frac{\cos^2\gamma}{4}\left(\kappa_\xi(0) + \frac{1}{2}\kappa_\xi(2\omega_0)\right), \quad K_{22} = \frac{\sin^2\gamma}{4}\left(\kappa_\xi(0) + \frac{1}{2}\kappa_\xi(\omega_0)\right), \tag{9}$$

$$\kappa_\xi(\omega) = \int\limits_{-\infty}^{\infty} \langle \xi(t)\xi(t+\tau)\rangle \cos\omega\tau \, d\tau \tag{10}$$

is the power spectrum density of the process $\xi(t)$ at the frequency ω.

Let us calculate now the integrals in Eq. (7) taking into account the expressions for g_1 and g_2. In consequence of this we obtain:

$$\int\limits_{-\infty}^{0} \left(\left\langle \overline{\frac{\partial g_1(A,\psi,t)}{\partial A} g_1(A,\psi,t+\tau)} \right\rangle + \left\langle \overline{\frac{\partial g_1(A,\varphi,t)}{\partial \varphi} g_2(A,\varphi,t+\tau)} \right\rangle \right) d\tau =$$

$$\frac{3A}{8} \int\limits_{-\infty}^{0} \langle \xi_1(t)\xi_1(t+\tau)\rangle \cos 2\omega_0\tau \, d\tau + \frac{1}{2A} \int\limits_{-\infty}^{0} \langle \xi_2(t)\xi_2(t+\tau)\rangle \cos \omega_0\tau \, d\tau = \frac{3K_{11}}{8}A + \frac{K_{12}}{2A},$$

$$\int\limits_{-\infty}^{0} \left(\left\langle \overline{\frac{\partial g_2(A,\psi,t)}{\partial A} g_1(A,\psi,t+\tau)} \right\rangle + \left\langle \overline{\frac{\partial g_2(A,\varphi,t)}{\partial \varphi} g_2(A,\varphi,t+\tau)} \right\rangle \right) d\tau =$$

$$\frac{1}{4} \int\limits_{-\infty}^{0} \langle \xi_1(t)\xi_1(t+\tau)\rangle \sin 2\omega_0\tau \, d\tau \equiv M. \tag{11}$$

The value of M depends on the characteristic of the random process $\xi(t)$: if it is white noise then $M = 0$, but if $\xi(t)$ has a finite correlation time, as, for example, its power spectrum density is $\kappa_\xi(\omega) = (a_1^2\kappa_\xi(2\omega_0))/((\omega - 2\omega_0)^2 + a_1^2)$, then $M = -(a_1\omega_0\cos^2\gamma\kappa_\xi(2\omega_0))/(4\left(16\omega_0^2 + a_1^2\right))$.

It should be noted firstly that the value of M is determined only by the multiplicative component of noise, whereas the additive component does not contribute to M; secondly, M is negative, resulting in the decrease of mean oscillation frequency. This decrease is the more considerable the larger is the intensity of the noise multiplicative component $\xi_1(t)$.

The following Langevin equations can be related to the Fokker–Planck equation (7) in view of (11), (8):

$$\begin{aligned} \dot{A} &= \left(\frac{3\omega_0^2}{8} K_{11}\eta - \frac{3\omega_0^2}{4}\beta\alpha A^2\right) A + \frac{\omega_0^2}{2A} K_{12} + \frac{\omega_0}{2} A\zeta_{11}(t) + \omega_0\zeta_{12}(t), \\ \dot{\phi} &= \omega_0^2 M + \omega_0\left(\zeta_{21}(t) + \frac{\zeta_{22}(t)}{A}\right), \end{aligned} \tag{12}$$

where $\eta = 1 - 8\beta/3\omega_0^2 K_{11}$, $\zeta_{11}(t)$, $\zeta_{12}(t)$, $\zeta_{21}(t)$, and $\zeta_{22}(t)$ each are white noise with zero mean value noncorrelated with A. The intensities of these noises are K_{11}, K_{12}, K_{21}, and K_{22}, respectively.

First of all let us consider the case when $\gamma = 0$, that means the absence of additive noise component. We note that even in this case Eqs. (12) differ by numerical coefficient of η from that derived in (49,48). The reason is that authors, using the variable $u = \ln A$ in place of A, implicitly ignored correlation between the noise $\xi(t)$ and the amplitude A. The same mistake was also made by us in (22,18,19).

For $\gamma = 0$ the steady-state solution of Eq. (7), satisfying to the condition for the probability flux to be equal to zero, is

$$w(A,\phi) = \frac{C}{2\pi A^2} \exp\left\{3\left(\eta \ln A - \frac{a(1-\eta)A^2}{2}\right)\right\}, \tag{13}$$

where $a = 3\alpha\omega_0^2/4$ is the nonlinear parameter. The constant C is determined from the normalization condition

$$\int_0^{2\pi}\int_0^{\infty} w(A,\phi) A\, dA\, d\phi = 1. \tag{14}$$

Upon integrating (13) with respect to ϕ and calculating the integrals under the exponential symbol, we find the expression for the probability density of the amplitude of oscillations

$$w(A) = C A^{3\eta - 1} \exp\left(-\frac{3a(1-\eta)A^2}{2}\right). \tag{15}$$

From the normalization condition we find

$$C = 2\begin{cases} \left(\dfrac{3a(1-\eta)}{2}\right)^{3\eta/2} \dfrac{1}{\Gamma(3\eta/2)} & \text{for } \eta \geq 0 \\ 0 & \text{for } \eta \leq 0. \end{cases} \tag{16}$$

Hence,

$$w(A) = 2\begin{cases} \left(\dfrac{3a(1-\eta)}{2}\right)^{3\eta/2} \dfrac{1}{\Gamma(3\eta/2)} A^{3\eta-1} \exp\left(-\dfrac{3a(1-\eta)A^2}{2}\right) & \text{for } \eta \geq 0 \\ \delta(A) & \text{for } \eta \leq 0. \end{cases} \tag{17}$$

The fact that for $\eta \leq 0$ the probability density of the amplitude turns out to be δ-function is associated with the absence of additive noise (see below).

Using (17) we can find $\langle A \rangle$ and $\langle A^2 \rangle$:

$$\begin{gathered} \langle A \rangle = \begin{cases} \sqrt{\dfrac{3}{2a(1-\eta)}} \dfrac{\Gamma\left((3\eta+1)/2\right)}{\Gamma(3\eta/2+1)} \eta & \text{for } \eta \geq 0 \\ 0 & \text{for } \eta \leq 0 \end{cases} \\ \langle A^2 \rangle = \begin{cases} \dfrac{\eta}{a(1-\eta)} & \text{for } \eta \geq 0 \\ 0 & \text{for } \eta \leq 0 \end{cases} \end{gathered} \tag{18}$$

It is seen from this that for $\eta > 0$ the parametrical excitation of pendulum oscillations occurs under the effect of multiplicative noise. This manifests itself in the fact that the mean values of the amplitude and the amplitude squared become different from zero. The availability of the parametrical excitation implies the transition of the system into a new state, which is to say that the phase transition occurs in the system. Thus, the condition $\eta = 0$ gives the onset of the phase transition. It follows from this condition that in the

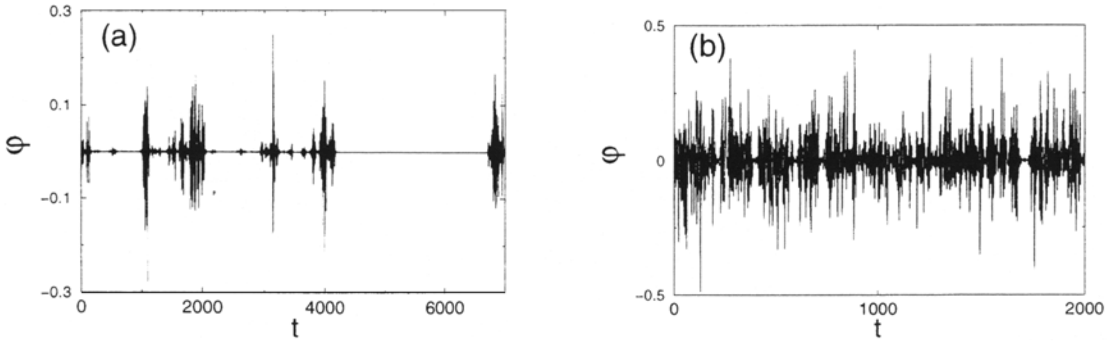


FIGURE 2. The time dependencies of the pendulum angular deviation $\varphi(t)$ for $\omega_0 = 1$, $\beta = 0.1$, $\alpha = 100$, $\gamma = 0$, and different values of noise intensity: (a) $\kappa(2\omega_0)/\kappa_{cr} = 1.01$, (b) $\kappa(2\omega_0)/\kappa_{cr} = 3.0$.

absence of additive noise the critical value of the multiplicative noise intensity is $\kappa_\xi^{cr}(2\omega_0) \equiv \kappa_{cr} = 16\beta/(3\omega_0^2)$. The parameter η characterises the extent to which the intensity of multiplicative noise component exceeds its critical value.

In the case that γ is nonzero, i.e. additive noise component is different from zero, the steady-state solution of Eq. (7), satisfying to the condition for the probability flux to be equal to zero, is conveniently written as

$$w(A,\phi) = \frac{Ca}{2\pi(aA^2+q)} \exp\left\{ \int \frac{3\Big(\eta - a(1-\eta)A^2\Big)aA^2 + q}{(aA^2+q)A}\, dA \right\}, \tag{19}$$

where $q = 4aK_{12}/K_{11}$ is the value characterizing the ratio between the intensities of additive and multiplicative noise components.

The constant C is determined from the normalization condition (14).

Upon integrating (19) with respect to ϕ and calculating the integral under the exponential symbol, we obtain

$$w(A) = CA^2\left(A^2 + \frac{q}{a}\right)^{3(1-\eta)(q-1)/2} \exp\left(-\frac{3a(1-\eta)A^2}{2}\right). \tag{20}$$

It follows from the normalization condition that

$$C^{-1} = \int_0^\infty A^2\left(A^2 + \frac{q}{a}\right)^{3(1-\eta)(q-1)/2} \exp\left(-\frac{3a(1-\eta)A^2}{2}\right) dA. \tag{21}$$

The integral in the right-hand side of (21) can be expressed in terms of a Whittaker function (54). As a result we find

$$C = \frac{4\Big(3(1-\eta)\Big)^{\mu+1/2} a^{2\mu} q^{1/2-\mu}}{2^\mu\sqrt{2\pi}W_{\mu-1,\mu}\left(3q(1-\eta)/2\right)} \exp\left(-\frac{3q(1-\eta)}{4}\right), \tag{22}$$

where $\mu = 3(\eta + q - q\eta)/4$.

To make sure that, as $q \to 0$, the expression (22) turns in the expression (16) obtained in the absence of additive noise component, we can use the representation of the Whittaker function $W_{\lambda,\mu}(z)$ in terms of two other Whittaker functions $M_{\lambda,\mu}(z)$ and $M_{\lambda,-\mu}(z)$ (54):

$$W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\lambda)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\lambda)} M_{\lambda,-\mu}(z). \tag{23}$$

By using further the expansion of the functions $M_{\lambda,\mu}(z)$ and $M_{\lambda,-\mu}(z)$ each in a series with respect to z (54) we find that for $q \to 0$ the expression (22) does coincide with (16).

Using (20), (22) we can find $\langle A \rangle$ and $\langle A^2 \rangle$. For example, for $\langle A^2 \rangle$ we obtain

$$a\langle A^2\rangle = \sqrt{\frac{3q}{2(1-\eta)}}\,\frac{W_{\mu-3/2,\mu+1/2}\Big(3q(1-\eta)/2\Big)}{W_{\mu-1,\mu}\Big(3q(1-\eta)/2\Big)}. \tag{24}$$

Carrying out the same manipulation as for the normalization constant C we can assure ourselves that, as $q \to 0$, the expressions for $\langle A \rangle$ and $\langle A^2 \rangle$ coincide with (18).

Taking into account the following recursion relation (54) $W_{\lambda,\mu}(z) = \sqrt{z}\, W_{\lambda-1/2,\mu+1/2}(z) + (1/2-\lambda-\mu)\, W_{\lambda-1,\mu}(z)$, the expression (24) is conveniently rewritten as

$$a\langle A^2\rangle = \frac{1}{1-\eta}\left(1-\left(\frac{3}{2}-2\mu\right)\frac{W_{\mu-2,\mu}\Big(3q(1-\eta)/2\Big)}{W_{\mu-1,\mu}\Big(3q(1-\eta)/2\Big)}\right). \tag{25}$$

In an explicit form the expression for $\langle A^2 \rangle$ can be obtained only in the limiting case, when the multiplicative noise intensity is comparatively large, so that $q \ll 1$. In this case we can use the representation (23) and subsequent expansion of the functions $M_{\lambda,\mu}(z)$ and $M_{\lambda,-\mu}(z)$ each in a series with respect to z. With the constraint $0 < \mu \ll 1$ we find for $W_{\mu-2,\mu}\Big(3q(1-\eta)/2\Big)/W_{\mu-1,\mu}\Big(3q(1-\eta)/2\Big)$ the following approximate expression:

$$\frac{W_{\mu-2,\mu}\Big(3q(1-\eta)/2\Big)}{W_{\mu-1,\mu}\Big(3q(1-\eta)/2\Big)} \approx \frac{2}{3}\left(1-\frac{4}{3}\,\mu\left(\frac{3q(1-\eta)}{2}\right)^{3\eta/2}\right). \tag{26}$$

Substituting (26) in (25) and taking into account that $2\mu = (3/2)(\eta + q - q\eta)$ we obtain

$$a\langle A^2\rangle \approx \frac{\eta+q-q\eta}{1-\eta}\left(1+\left(\frac{3q(1-\eta)}{2}\right)^{3\eta/2}\right). \tag{27}$$

If for $\kappa_{\xi_1}(2\omega_0) = \kappa_{cr}$, i.e. for $\eta = 0$, the condition (27) still stands then

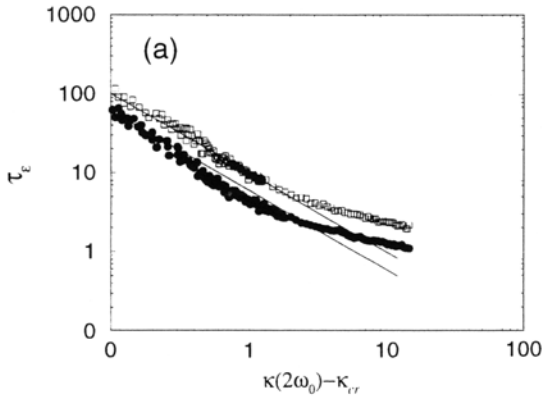
$$\langle A^2\rangle\Big|_{\kappa_{\xi_1}(2\omega_0)=\kappa_{cr}} \approx \frac{2q}{a} = \frac{16K_{12}}{\kappa_{cr}}. \tag{28}$$

For $q(1-\eta) \ll \eta$ and $\Big(3q(1-\eta)/2\Big)^{3\eta/2} \ll 1$ the dependence of $\langle A^2 \rangle$ on η coincides with the corresponding dependence as additive noise is absent.

It follows from the results obtained that slight additive noise results in smoothing the dependencies of the oscillation amplitude and amplitude squared on the noise intensity; they become without fractures inherent in a phase transition induced by only multiplicative noise.

In numerical experiment it is more convenient to calculate not the mean amplitude squared but the variance of the corresponding variable. It is evident that the dependencies of these values on noise intensity should be similar. Indeed, in the case that the amplitude A is a slow-changing function the variance is equal to $\langle A^2\rangle/2$. Examples of the dependencies of the variance of φ on multiplicative noise intensity found by numerical simulation of Eq. (1) in the presence of additive noise and in its absence are given in Fig. 1 (right). We see that nearby the phase transition boundary both of these dependencies can be approximated by a straight line intersecting the abscissa at point $\kappa_{\xi_1}(2\omega_0) = \kappa_{cr}$. Away from the phase transition boundary the growth rate of the variance somewhat decreases.

A distinguishing characteristics of the considered noise-induced phase transitions is that they occur via so-called on-off intermittency (23,38). This can be seen, for example, from Fig. 2, where examples of the pendulum oscillations found by numerical simulation of Eq. (2) are given. Close to the phase transition onset the pendulum oscillates in the immediate vicinity of its equilibrium position over prolonged periods (so called 'laminar' phases); these slight oscillations alternate with short strong bursts ('turbulent' phases). Away from the threshold the duration of laminar phases decreases and that of turbulent ones increases, and finally laminar phases disappear. The variance of the pendulum oscillations increases in this process.



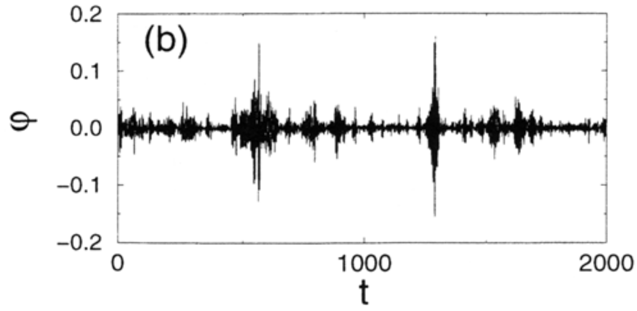


FIGURE 3. (a) The dependencies of τ_ϵ on the excess of the noise intensity over its critical value $\kappa(2\omega_0) - \kappa_{cr}$ for $\epsilon = 0.06$ (full circles) and $\epsilon = 0.1$ (squares) from numerical simulation. The corresponding theoretical dependencies are shown by solid lines;(b) On-off intermittency for subcritical values of the multiplicative noise intensity in the presence of the additive noise. The dependence $\varphi(t)$ for $\kappa_{\xi_2}(\omega_0)/\kappa_{cr} = 2 \cdot 10^{-6}$, $\kappa_{\xi_1}(2\omega_0)/\kappa_{cr} = 0.92$.

The term 'on-off intermittency' was recently introduced by Platt et al. (43), though a map associated with the similar type of intermittent behavior was first considered by Pikovsky (42) and then by Fujisaka and Yamada (34). In (35) it was found that on-off intermittency can take place not only in dynamical systems but in stochastic systems as well. In (35) statistical properties of on–off intermittency were obtained from the analysis of the map $x_{n+1} = a(1+z_n)x_n + f(x_n)$, where z_n is either a certain deterministic chaotic process or a random process, a is the bifurcation parameter, and $f(x_n)$ is a nonlinear function. For this map it was shown that the mean duration of laminar phases has to be proportional to a^{-1}.

Let us calculate the mean duration of laminar phases for the pendulum, using Eqs. (12) and the Fokker–Planck equation (7) associated with them. We restrict ourselves to the simplest case of $\gamma = 0$. We assume that the pendulum oscillates in a laminar phase if the oscillation amplitude A is not larger than a certain value ϵ. Then the mean duration of the laminar phase τ_ϵ is determined by the mean duration of a random walk-like motion of a representative point inside the circle of radius ϵ on the plane φ, $\dot{\varphi}$. This duration can be calculated (see, for example, (4) using the steady-state solution of Eq. (7)) with the boundary condition

$$w(A,\phi)|_{A=\epsilon} = 0. \tag{29}$$

Because the value of ϵ is assumed to be small, we can neglect the term $(3/4)\beta\alpha\omega_0^2 A^2$ in Eq. (7). In so doing the solution of Eq. (7) with the boundary condition (29) is

$$w(A,\phi) = \frac{3G_0(1-\eta)}{\beta(1-3\eta)A}\left(\left(\frac{A}{\epsilon}\right)^{3\eta-1} - 1\right), \tag{30}$$

where G_0 is the value of the probability flow

$$G = \frac{\omega_0^2 K_1}{8}\left(3\eta A w - \frac{d}{dA}(A^2 w)\right) \tag{31}$$

across any circumference inside the circle of radius ϵ.

The value of G_0 is determined from the normalization condition by integrating the expression (30) over the circle of radius ϵ. As a result, we obtain

$$G_0^{-1} = \frac{2\pi(1-\eta)\epsilon}{\beta\eta}. \tag{32}$$

It was shown in (4) that the mean duration of reaching the circle boundary is equal to G_0^{-1}. Taking into account that the representative point touching the boundary of the circle can return back with a certain probability p, we obtain for the mean duration of the laminar phase τ_ϵ the following expression:

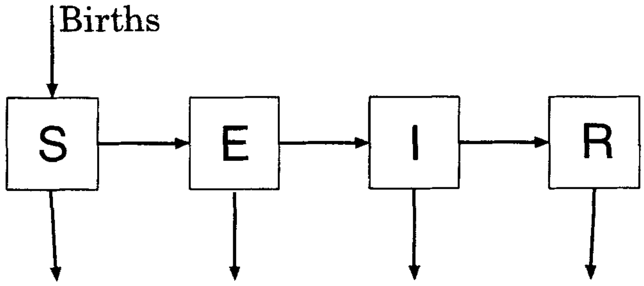


FIGURE 4. Diagram illustrating mutual relations between different components in the model of childhood epidemics.

$$\tau_\epsilon = G_0^{-1}(1-p)\sum_{j=1}^{\infty} jp^{j-1}. \tag{33}$$

Summarizing the series and having regard to the expression (32), we find

$$\tau_\epsilon = G_0^{-1}(1-p)^{-1} = \frac{2\pi(1-\eta)\epsilon}{3\beta\eta(1-p)} = \frac{32\pi\epsilon}{3\omega_0^2\Big(\kappa(2\omega_0)-\kappa_{cr}\Big)(1-p)}. \tag{34}$$

It is evident that $p < 1/2$ because the mean force which acts upon the representative point on the circle boundary for $\eta > 0$ is directed outside the circle. For small η and ϵ this force is very small and p is close to 1/2. In this case $\tau_\epsilon = 2G_0^{-1}$. As η and ϵ increase the value of p first decreases and then increases due to nonlinearity.

It follows from (34) that for small η and ϵ the mean duration of laminar phases has to be proportional to ϵ and inversely proportional to η. This result agrees quite well with (35).

Numerical simulation of Eq. (2) showed that for small η and ϵ the formula (34) with $p = 1/2$ is valid in a good approximation; whereas for larger ϵ some discrepancies were observed.

The dependencies of τ_ϵ on $\kappa(2\omega_0) - \kappa_{cr}$ for two values of ϵ calculated by numerical simulation are given in Fig. 3 a. The corresponding theoretical dependencies are shown by solid lines. A discrepancy between theoretical and numerical dependencies for large value of $\kappa(2\omega_0) - \kappa_{cr}$ is caused by the nonlinear dependence of p on $\kappa(2\omega_0) - \kappa_{cr}$ and by the fact that theoretical calculations are valid for small ϵ only.

In addition to smoothing effect, the presence of slight additive noise results in the fact that on-off intermittency (38) is observed for multiplicative noise intensity lesser then its critical value (Fig. 3 b).

As the intensity of additive noise increases the phase transition boundary becomes progressively less detectable and the phenomenon of intermittency disappears gradually too.

NOISE-INDUCED PHASE TRANSITION IN A MODEL OF CHILDHOOD INFECTIONS CAUSED BY RANDOM SEASONAL VARIATIONS OF THE CONTACT RATE

A standard epidemiological model for the description of the oscillations of childhood infections, such as chickenpox, measles, mumps and rubella, under the influence of seasonal variations of the contact rate of children suspective to infection with infective was first studied by Dietz (29). Dietz assumed that the contact rate varies periodically with the period equal to one year and found analytically periodic oscillations of the model variables. Later this model was studied by Olsen and Schaffer (41) and Engbert and Drepper (30) who showed that periodic variation of the contact rate can result not only in periodic oscillations of childhood epidemics but in chaotic as well. The model contains four components: (1) Susceptibles (S); (2) Exposed but not yet infective (E); (3) Infective (I); 4) Recovered and immune (R). Mutual relations between these components are illustrated schematically by Fig. 4. The relative number of children S susceptible to infection increases with total number of children and decreases both owing to that the part of them remains nonexposed and due to falling these children in the category of the exposed but not yet infective (E). A part of the children exposed remains noninfective, whereas another part falls in the category of the infective (I). In its turn, a part of the children infective does not fall sick and another part, having had the disease, become recovered, falling in the forth category (R). Taking into account that the total number of children is constant, the model's equations can be written as

$$\dot{S} = m(1-S) - bSI, \quad \dot{E} = bSI - (m+a)E, \quad \dot{I} = aE - (m+g)I, \tag{35}$$

$$\dot{R} = gI - mR, \tag{36}$$

where $1/m$ is the average expectancy time, $1/a$ is the average latency period, $1/g$ is the average infection period, b is the contact rate (the average number of susceptibles contacted yearly with infective). Let us note that Eqs. (35) do not contain the variable R; hence these equations can be considered independently of Eq. (36).

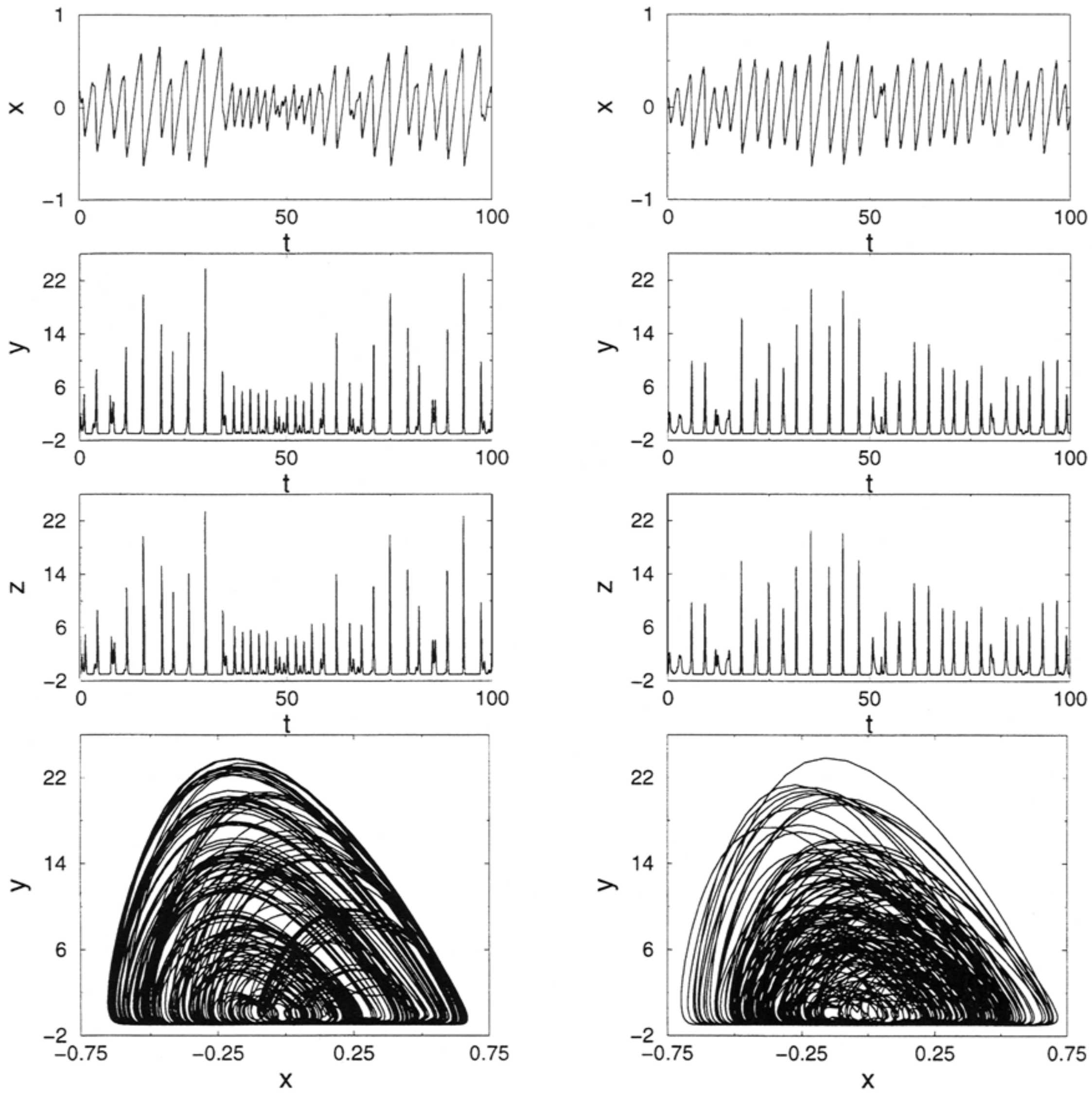
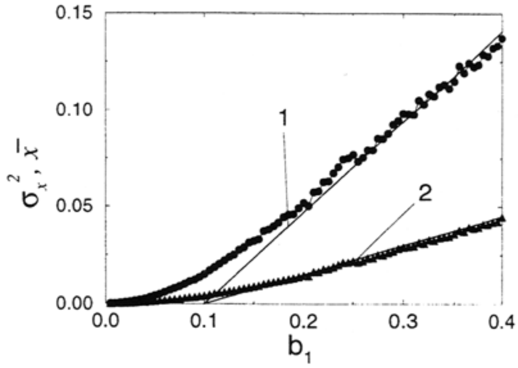


FIGURE 5. The time dependences of x, y, z, and the projection of the phase trajectory on the x, y–plane for $m = 0.02\,\text{year}^{-1}$, $a = 35.84\,\text{year}^{-1}$, $g = 100\,\text{year}^{-1}$, $b_0 = 1800\,\text{year}^{-1}$; *left column:* $f(t) = \cos(2\pi t)$, $b_1 = 0.28$; *right column:* $f(t) = \chi(t)$, $b_1 = 0.24$.



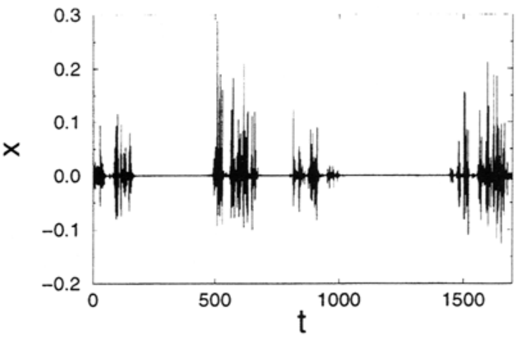


FIGURE 6. Left: The dependencies of $\sigma_x^2 = \overline{x^2} - \overline{x}^2$ (*label 1*) and $\overline{x}$ (*label 2*) on b_1 for $m = 0.02\,\text{year}^{-1}$, $a = 35.84\,\text{year}^{-1}$, $g = 100\,\text{year}^{-1}$, $b_0 = 1800\,\text{year}^{-1}$; the solid straight line labelled *1* is described by the equation $\sigma^2 = 0.52(b_1 - 0.1)$, and *2* is described by the equation $\overline{x} = 0.15(b_1 - 0.1)$; right: The solution of Eqs. (38) for $b_1 = 0.1$ in the case that additive constituent of the noise is absent and the term $-0.1(m+a)y^3$ is added in the right-hand side of the second equation.

It is easily shown that Eqs. (35) for $b = b_0 = \text{const}$, and for any values of the remaining parameters, have one aperiodically unstable singular point with coordinates $S = 1$, $E = I = 0$ and one stable singular point with coordinates

$$S_0 = \frac{(m+a)(m+g)}{ab_0}, \quad E_0 = \frac{m}{m+a} - \frac{m(m+g)}{ab_0}, \quad I_0 = \frac{am}{(m+a)(m+g)} - \frac{m}{b_0}. \tag{37}$$

If the parameter b varies with time then the variables S, E and I will oscillate, and these oscillations will be executed about the stable singular point with coordinates (37). Therefore, it is convenient to substitute into Eqs. (35) new variables $x = S/S_0 - 1$, $y = E/E_0 - 1$, and $z = I/I_0 - 1$. Putting $b = b_0(1 + b_1 f(t))$, where $f(t)$ is a function describing the shape of the contact rate variation, we rewrite Eqs. (35) in the variables x, y, z:

$$\begin{aligned} &\dot{x} + mx = -b_0 I_0 \left((1 + b_1 f(t))\,(x + z + xz) + b_1 f(t) \right), \\ &\dot{y} + (m+a)y = (m+a) \left((1 + b_1 f(t))\,(x + z + xz) + b_1 f(t) \right), \\ &\dot{z} + (m+g)z = (m+g)y. \end{aligned} \tag{38}$$

In Eqs. (38) the term $b_1 f(t)$ can be considered as an external action upon the system. We see from (38) that this action is not only multiplicative, i.e. parametric, but additive as well.

In (29,41) it was assumed that owing to seasonal variations of environmental conditions the contact rate b varies periodically in time with the period equal to one year, *viz.*, $f(t) = \cos(2\pi t)$).

It was shown that the periodic variation of the parameter b, depending on the amplitude b_1, causes the appearance either periodic or chaotic oscillations of the variables S, E, and I. The transition from periodic to chaotic oscillations as the parameter b_1 increases occurs via the sequence of period-doubling bifurcations. Chaotic oscillations were observed by Olsen and Schaffer (41) for the following values of the parameters: $m = 0.02\,\text{year}^{-1}$, $a = 35.84\,\text{year}^{-1}$, $g = 100\,\text{year}^{-1}$, $b_0 = 1800\,\text{year}^{-1}$, $b_1 = 0.28$. These parameters correspond to estimates made for childhood diseases in first world countries.

For these values of the parameters the time dependences of b, x, y, z, and the projection of the phase trajectory on the x, y–plane found by numerical simulation of Eqs. (38) are shown in Fig. 5. It can be see from this figure that the mean frequency of oscillations of the system variables is somewhat lower than one-half the frequency of periodic variation of the parameter b. This suggests that parametric mechanism of the oscillation excitation plays a dominant role.

It should be noted that the oscillations found in their form closely resemble the experimental data presented in (44). This fact seemingly justified the model. However, as shown below, similar results can be also obtained for a somewhat different model. It would appear more reasonable that the parameter characterising the contact rate of children suspective to infection with infective would be a random process for which the spectral density peaks at the frequency corresponding to the time period equal to one year. Starting from this assumption we have numerically simulated Eqs. (38) with $f(t) = \chi(t)$, where $\chi(t)$ is a random process which is a solution of the equation

$$\ddot{\chi} + 2\pi\dot{\chi} + 6\pi^2\chi = k\xi(t), \tag{39}$$

$\xi(t)$ is white noise, k is the factor which we choose so that the variance of $\chi(t)$ would be equal to 1/2. It is easy shown that the spectral density of $\chi(t)$ peaks at the frequency $\omega = 2\pi$.

The results of numerical simulation of Eqs. (38) with $f(t) = \chi(t)$ are shown in Fig. 5 (right column) for the same values of the parameters as in Fig. 5 (left column) and b_1 chosen so that the variance of $x(t)$ would be approximately the same as for $f(t) = \cos(2\pi t)$. It is seen from this figure that the noise-induced oscillations distinct very slightly in their form from these for the case of harmonic variation of the contact rate and from the experimental data.

As the parameter b_1 increases from 0 the variance of the variables $x(t)$, $y(t)$, and $z(t)$ increase too, and from a certain value of b_1 towards (for $b_1 > 0.2$) these dependencies become approximately linear (see Fig. 6 where the dependence of σ_x^2 on b_1 is presented). Prolonging the straight line approximating the dependence of σ_x^2 on b_1 for $b_1 > 0.2$ to intersect the abscissa, we find that the intersection point is $b_1 \approx 0.1$. It can be shown that this value of b_1 is critical for the appearance of the phase transition in the case of only multiplicative action. For this let us put artificially in Eqs. (38) the additive noise constituent to be equal to zero. Then we obtain that the oscillations are not excited to the point $b_1 = b_{1cr} \approx 0.1$. For $b_1 > 0.1$ instability appears, i.e. at a certain random instant the solution goes to infinity. To avoid this instability we add in the right-hand side of the second equation (38) the term $-0.1(m+a)y^3$. In so doing we see that for $b_1 = 0.1$ the on-off intermittency with very rare turbulent bursts takes place (Fig. 6).

NOISE-INDUCED PHASE TRANSITION IN A BONHOFFER–VAN-DER POL OSCILLATOR

The equations came to be known as equations for a Bonhoffer–van der Pol oscillator were suggested by Bonhoeffer for simulating neural pulses (25–28). They are the generalisation of the van der Pol equations for relaxation oscillations (52). These equations describe oscillations of the voltage across a neural membrane x having regard to refractoriness characterized by a certain value y. Later similar equations, but incorporating spatial diffusion, have come to be known as Fitz Hugh–Nagumo equations (31,39,32). The Bonhoeffer–van der Pol equations can be written as (40)

$$\dot{x} = x - \frac{x^3}{3} - y + I(t), \quad \dot{y} = c(x + a - by), \tag{40}$$

where a, b, and c are the membrane radius, the specific resistivity of the fluid inside the membrane, and the temperature factor, respectively; $I(t) = I_0 + F(t)$ is the current across the membrane with I_0 being the direct component of this current.

For $F(t) = 0$ and the parameters corresponding to real membranes (following, for example, (47) we set $a = 0.7$, $b = 0.8$, and $c = 0.1$) Eqs. (40) have a single singular point $x = x_0$, $y = y_0 = (x+a)/b$, where x_0 is a real root of the equation

$$\frac{x^3}{3} + \left(\frac{1}{b} - 1\right)x + \frac{a}{b} - I_0 = 0 \tag{41}$$

In the ranges $I_0 < 0.341$ and $I_0 > 1.397$ this point is a stable focus; whereas for $0.341 < I_0 < 1.397$ it is an unstable focus.

Setting $\xi = x - x_0$, $\eta = y - y_0$ we obtain for ξ and η the following equations:

$$\dot{\xi} = -\left(\frac{\xi^3}{3} + x_0\xi^2 + (x_0^2 - 1)\xi + \eta\right) + F(t), \quad \dot{\eta} = c(\xi - b\eta). \tag{42}$$

The case, when the system under consideration is not self-oscillatory, is of prime interest for the purpose of this paper. Therefore we restrict our consideration to the case of $I_0 = 0.2$. Eqs. (42) are fascinating by

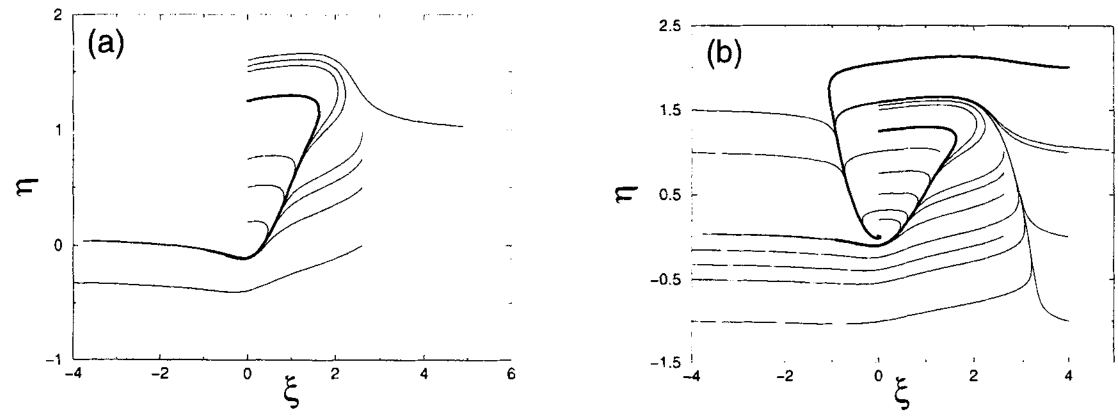


FIGURE 7. The transient repeller and neighbouring phase trajectories (a) and the phase portrait involving the transient repeller and transient attractor (b) for $a = 0.7$, $b = 0.8$, $c = 0.1$ and $I_0 = 0.2$.

the fact that they have two exceptional phase trajectories. One of these trajectories has positive Lyapunov exponent, i.e. it is unstable, whereas the other is stable. The first has a part which repels all neighbouring phase trajectories, and, contrary, the second has parts wich attract all neighbouring phase trajectories. Evidently, that these exceptional trajectories are not repeller and attractor in the strict sense, because the system described by Eqs. (42) for $I_0 = 0.2$ has no repellers and only a single attractor: the stable singular point. However, due to some similarity of these trajectories to repeller and attractor, in citeRabinovich:1998 they were called *transient repeller* and *transient attractor*, respectively. We note that there is variability in the name of these trajectories. For example, in (50) they are called *local separatrices* and *local attractors*, in (53) the attracting trajectory is called *phantom attractor* and so on.

To find numerically the transient repeller, we can reverse the direction of time. As a result, we obtain the picture on the phase plane shown in Fig. 7 a. The part which repels all neighbouring phase trajectories is shown as a thick solid line. A full phase portrait involving the trajectories with both attracting and repelling parts is given in Fig. 7 b. The attracting parts are shown as thick solid lines. We see that the transient repeller separates the regions of deviations from the equilibrium state corresponding to radically different transient processes.

If the current across the membrane contains an alternating component, for example, $F(t) = A\cos\omega t$, then from a certain critical value of A onward oscillations associated with the motion of the representative point along a phase trajectory involving the attracting parts excite (55,45,33). It follows from the results of numerical simulation of Eqs. (42) that the variance of the variables ξ and η increases nearly by jump in this process. The excitation of such oscillations accompanied by a drastic increase in the oscillation variance can be considered as a phase transition. An example of the dependence of the variance of the variable ξ on A for $\omega = 0.3$ is given in Fig. 8 a. We can conclude from this figure that the phase transition caused by harmonic component of the current across the membrane is of the first kind rather than of the second one. The critical value of A depends on the frequency ω. It is minimal for a certain value of the frequency which in turn depends on I_0. The reason of this dependence is resonance responce of the nonlinear oscillator to harmonic external force. As an example, the dependence of the critical value of the amplitude A on the frequency ω is shown in Fig. 8 b for $I_0 = 0.2$. We see that the critical value of the amplitude is minimal for $\omega = 0.27$, which is close to the frequency of small free oscillations about the equilibrium state equal to $\omega_0 \approx 0.3146$. For other values of the frequency the phase transition with increasing A occurs in a similar manner.

If alternating component of the current across the membrane $F(t)$ is a random process, for example white noise, then the transition to a new state occurs too, but it is of different character. The appearance of a 'limit cycle' induced by white noise was considered in (50,51,36,37). However, these works are mainly devoted to calculation of the probability distributions in the vicinity of the induced 'limit cycle'. We consider this phenomenon from the standpoint of the noise-induced transition of the system to a new state. The latter is associated with intersection of boundary on the phase plane (the transient repeller is such the boundary) by the representative point under action of noise. In principle, such intersection is possible for the noise intensity κ as small as is wished. Therefore the transition occurs smoothly from $\kappa = 0$ onward. Hence, in the strict

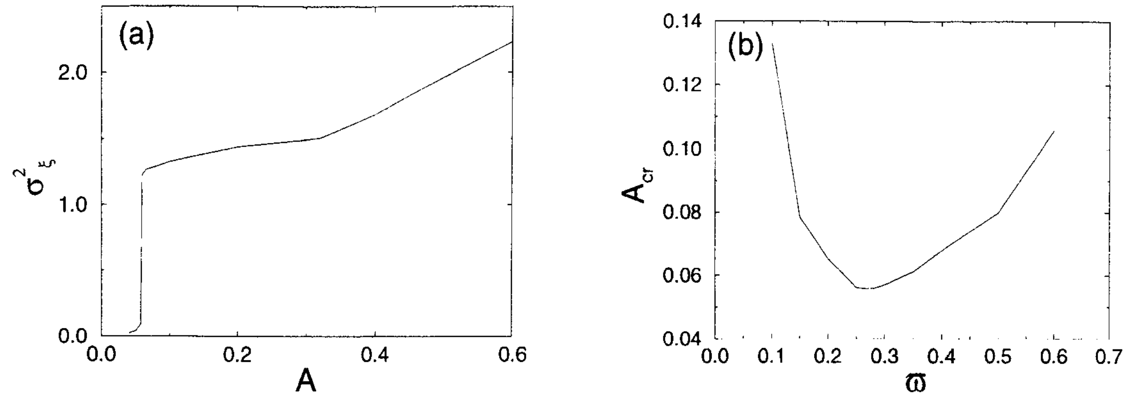


FIGURE 8. (a)The dependence of the variance of the variable ξ on the current alternating constituent amplitude for $\omega = 0.3$. (b)The dependence of the critical value of the current alternating constituent amplitude A_{cr} on the frequency ω.

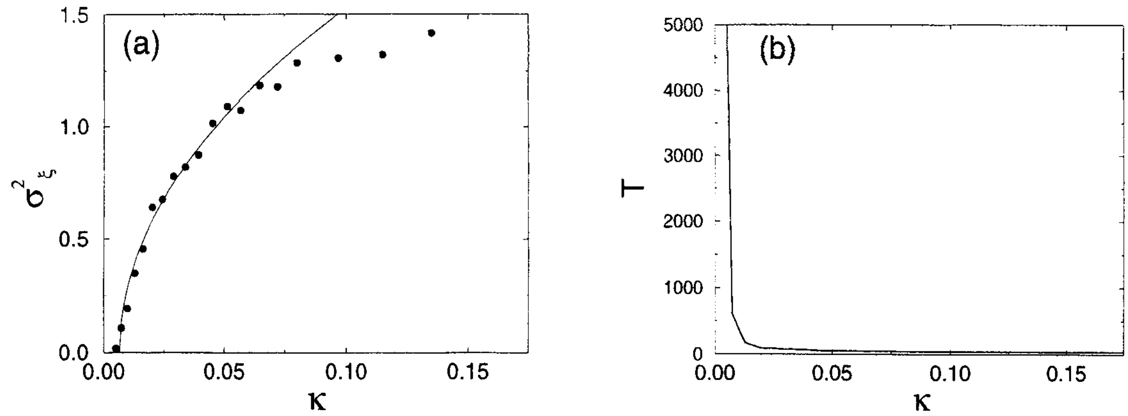


FIGURE 9. (a) The dependence of the variance of the variable ξ on the noise intensity κ. The dependence $\sigma_\xi^2 = 5\sqrt{\kappa - 0.0065}$ is shown as a solid line. (b)The dependence of the mean period T on the noise intensity κ. A solid line shows the dependence $T = 38\exp(0.02/\kappa)$.

sense this transition is not a phase transition. But it is closely similar to a noise-induced phase transition of the second kind. For example, the dependence of the variance of the variable ξ on the noise intensity κ, found by numerical simulation of Eqs. (42) and shown in Fig. 9 a, can be approximated on a certain interval by the formula $\sigma^2 \approx 5(\kappa - 0.0065)^{1/2}$. This formula is similar to that describes the dependence of an order parameter on temperature for ordinary phase transitions of the second kind with critical index equal to 1/2. Furthermore, it has been found that this transition occurs via a peculiar kind of on-off intermittency which is akin to that in the pendulum considered above. As for ordinary on-off intermittency taking place in the pendulum, close to the transition onset the representative point on the phase plane is walking in a certain ϵ-vicinity of the equilibrium state over prolonged periods (so called 'laminar phases'), and only from time to time escapes from this vicinity. Contrary to ordinary on-off intermittency, these escapes have not random but strictly specified shape of pulses, and the duration of each of them is unchanged as the noise intensity increases. That is why these escapes should not be called 'turbulent phases'. Away from the onset the duration of laminar phases decreases and the variance of the system variables increases. Because the duration of pulses is unchanged we can use the mean interpulse time (the mean period) in place of the mean duration of 'laminar phases'. The dependence of the mean period T on the noise intensity κ is shown in Fig. 9 b. It can be approximated by the formula $T \approx 38\exp(0.02/\kappa)$, which is typical for the mean time of the intersection a boundary (1,4). We see that the mean period decreases exponentially as the noise intensity increases. This result coincides qualitatively with the initial part of the corresponding dependence obtained in (50,51). However, we have not found the increase of the mean period with increasing the noise intensity which is obtained in (50,51).

CONCLUSIONS

In conclusion we note that the noise-induced phase transitions considered in the sections 1 and 2 are similar to phase transitions induced by periodic action on the same systems. All of them are nonequilibrium phase transitions of the second kind. In the Bonhoffer–van der Pol oscillator the transition induced by noise is not a phase transition but it is very similar to a phase transition of the second kind, whereas phase transitions induced by periodic force at one or other frequency are of the first kind.

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