Problem set: Single Variable Calculus

1. Compute the following limits,

   (a) $\lim_{x\to\infty} \frac{6x^3+4x^2-7}{x^3+x}$

   (b) $\lim_{x\to 5} \frac{2x^3+3x}{x}$

   (c) $\lim_{x\to 0} \frac{2x^3+3x}{x}$

   (d) $\lim_{x\to 1} \frac{1-x^2}{1-x}$

   (e) $\lim_{x\to 1} e^{\frac{4-5x}{1-x}}$

   (f) $\lim_{x\to 0} \frac{\ln x}{e^{2x}}$

2. Compute the derivatives of the following functions,

   (a) $f(x) = \sqrt{3x^2+x}$

   (b) $f(x) = e^{\sqrt{x}}$

   (c) $f(x) = \frac{4x^2+1}{x^3-x}$

   (d) $f(x) = \sqrt{8} + 2x^3$

   (e) $f(x) = \frac{7}{\sqrt[8]{9x^3}}$

   (f) $f(x) = x^7 \ln x$

   (g) $f(x) = \frac{\ln 4 + \sqrt{5}}{2x^7}$

3. Calculate the inverse of the following functions, identify their domain and compute the respective derivatives,

   (a) $f(x) = \frac{1}{x+1}$

   (b) $f(x) = e^{\sqrt{x}}$

   (c) $f(x) = x^3 + 3x + 1$

4. Sketch the graphic of the functions,

   (a) $f(x) = x^2 - 5x + 8$

   (b) $f(x) = \frac{x+1}{(x-2)^2}$

   (c) $f(x) = \frac{x^2+1}{x}$

5. If $f(x) = \ln(x)$, what is $f^{(n)}(x)$? Show that $\ln(1+x)$ can be written as a convergent series,

$$\ln(1+x) \quad = \quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

6. Prove that if f is differentiable at point x_0, it must be continuous at x_0 but the opposite implication is false.

7. *Mean Value Theorem:* Suppose f is $\mathcal{C}^1$ in [a,b]. Show that there exists a point $c \in [a, b]$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

8. Use the mean value theorem to derive the second order Taylor's series approximation of a C^2 function.

9. Suppose a firm's production function is:

$$Q = K^{1/2} L^{1/3}$$

where Q, K and L stand for output, capital and labour, respectively. Suppose further that the inputs follow the rules:

$$\begin{aligned} K &= 5 + 2t \\ L &= 2 + t \end{aligned}$$

(a) Find how Q changes with t.

(b) Show that the isoquants of the production function in terms of K and L are downward sloped.

(c) Show that the isoquants of the production function in terms of K and L are convex.

(d) Show that each of the isoquants of the production function in terms of K and L has two asymptotes.

Problem set: Integral Calculus

1. Compute the following integrals

   (a) $\int -\frac{1}{x} dx$.

   (b) $\int_a^x t dt$.

   (c) $\int x e^x dx$.

2. Show that if $f(x)$ is integrable over an interval $[a, b]$ then there exists a $c \in [a, b]$ such that $\int_a^b f(x)\, dx = (b-a) f(c)$.

3. Suppose $f(x)$ is continuous and $\int_a^b f(x) g(x)\, dx = 0$ for every continuous function g on [a,b]. Show that $f(x) = 0$ for all $x \in [a, b]$.

4. Use integration by substitution to calculate,

   (a) $\int_0^1 \frac{x^3}{x^4+1}\, dx$.

   (b) $\int_0^1 (2x+3)(x^2+3x-4)^{122}\, dx$.

5. Suppose the interest rate in continuous time t is $r(t)$.

   (a) Suppose a quantity $A(0)$ is invested at time $t = 0$ and the value of the investment at time t is $A(t)$. Explain that the following propositions are true:

      i. If Δt is a small increment in time, then:

      $$A(t + \Delta t) \quad \simeq \quad [1 + r(t)\Delta t]\, A(t)$$

      ii. $d \ln A(t) / dt = r(t)$

      iii. At a point $t = T$

      $$A(T) \quad = \quad A(0) \exp\left\{ int_0^T r(t) dt \right\}$$

   (b) Show that the present value at time 0 of the income stream $\{f(t), 0 \leqslant t \leqslant T\}$ is

   $$\int_0^T \exp\left\{ - \int_0^T r(s) ds \right\} f(t) dt$$

Problem set: Matrix algebra

1. Suppose that the matrix equation $Ax = b$ has a solution $x = x_0$ and the system $Ax = 0$ has a non-zero solution $x = c$. Show that $x = x_0 + kc$ is a solution to $Ax = b$ for any $k \in \mathcal{R}$.

2. If A and B are invertible, is $A + B$ necessarily invertible?

3. Evaluate $x'Ax$ when $A = [1 \;\; 3; 2 \;\; 6]$. Express $x_1^2 + 2x_2^2 - 3x_1x_2 + x_2x_3$ in the form $x'Ax$. Can you make A symmetric?

4. Find

$$\det \begin{bmatrix} 4 & 3 & 1 & 9 & 2 \\ 0 & 3 & 2 & 4 & 2 \\ 0 & 3 & 4 & 6 & 4 \\ 1 & -1 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \end{bmatrix}$$

5. Show that if $A^2 = A$ then either $A = I$ or $|A| = 0$.

6. Show that I_n has rank n.

7. Show the if $Ax = b$ (A_{n*n}) has a unique solution then $\text{rank}(A) = n$.

8. Find the characteristic equation of a $2 * 2$ matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let the two eigenvalues be r_1 and r_2, not necessarily different. Show that,

$$\begin{aligned} \text{trace}(A) &= r_1 + r_2 \\ \det(A) &= r_1 r_2 \end{aligned}$$

9. Find the necessary and sufficient conditions on a, b, c, d for the matrix $A = [a \;\; b; c \;\; d]$ to be positive definite.

Problem set: Multivariate calculus

1. Find the partial derivatives of,
   (a) $y = 3x_1^2 + x_2^3 - 3x_1x_2$
   (b) $y = x_1^{x_2}$
   (c) $\ln(y) - x_1 - x_2 - 6 = 0$
   (d) $y^3x_1 + y^2x_2 + yx_3 = 2$

2. Find and study the turning points of
   (a) $y = 3x_1^2 + x_2^3 - 3x_1x_2$
   (b) $y = 6x_1^2 - 9x_1 - 3x_1x_2 - 7x_2 + 5x_2^2$
   (c) $y = 3x_1^2 + 2x_2^2 - 4x_2 + 1$
   (d) $y = x_1x_2x_3 - x_1x_2 + 2x_3 - x_3^2$

3. A firm producing two goods x and y has a profit function $\pi = 32x - x^2 + 2xy - 2y^2 + 16y - 8$. Find the profit-maximising output levels.

4. Find the total differential of
   (a) $y = 3x_1^2 + 2x_2^2 - 4x_2 + 1$
   (b) $y = x_1x_2x_3 - x_1x_2 + 2x_3 - x_3^2$
   (c) $z = 3x^2 + xy^2$

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as,

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \end{cases}$$

   Show that the cross partial derivatives exist at all values of (x,y) but are not continuous at $(0,0)$ and in fact are different at that point. Why does this happen?

6. Consider the CES production function

$$Q = A\left[\alpha K^\beta + (1-\alpha)L^\beta\right]^{1/\beta}$$

   where Q, K and L are the output, capital and labour, respectively, and $A > 0$, $0 < \alpha < 1$ and $\beta < 1$.
   (a) Show that the isoquants are negatively sloped and convex.
   (b) Suppose $\beta < 0$. Find the equation for the isquants and sketch them.
   (c) Suppose $0 < \beta < 1$. Find the points at which a isoquant $Q = c$ meets the axis. Sketch the isoquants.

Problem set: Static optimisation in $\mathbb{R}^n$

1. Suppose $x \in \text{int}\mathcal{D}$ is a local minimum of f. Suppose f is differentiable at x. Prove that $Df(x) = 0$ without making use of any theorems you might find on this.

2. Find the maxima and minima of $f(x, y) = 2+2x+2y-x^2-y^2$ on the set $\{(x, y) \in \mathbb{R} : x+y = 9\}$ by representing it as an unconstrained optimisation problem.

3. Show that the problem of maximising $f(x, y) = x^3 + y^3$ on the constraint set $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$ has no solution. What would happen if the Lagrangean method had been used? Why is there a problem?

4. A firm produces outputs y and z using a single input x. The set of attainable output levels, $H(x)$ is given by $H(x) = \{(y, z) \in \mathbb{R}^2_+ : y^2 + z^2 \leqslant x\}$. The maximum amount of input the firm has available is $x = 1$. Let (p_y, p_z) denote the market price of the two outputs. Determine the firm's optimal output mix.

5. A consumer has income $I > 0$ and faces a price vector $p \in \mathbb{R}^3_{++}$ for the three goods he consumes. All goods must be consumed in non-negative amounts. Moreover, she must consume at least 2 units of good 2 and cannot consume more than 1 unit of good 1. Assume $I = 4$ and $p = (1, 1, 1)$. Compute the optimal consumption bundle when utility is $u(x_1, x_2, x_3) = x_1x_2x_3$. What if $I = 6$ and $p = (1, 2, 3)$?

6. Study the following problem

$$\begin{aligned} \max_{(x,y)} \quad & (x+1)^2 + (y-1)^2 \\ s.t. \quad & x^2 + y^2 \leqslant 50 \\ & x \leqslant 1 \end{aligned}$$

7. Compute the maximum value function for the problem

$$\begin{aligned} \max_{(x,y)} \quad & f(x, y) = x^2y^2 \\ s.t. \quad & x^2 + y^2 = c \end{aligned}$$

Problem set: Differential equations

1. Solve the following differential equations

   (a) $y' - 4y = 0$ with $y(0) = 2$

   (b) $2y' - 1 = y$

   (c) $y' + 5y = 4e^{3t}$

   (d) $y' - \frac{y}{2t} = 3t$

   (e) $y' + \frac{y}{2t} = 3t$

2. Consider the second-order linear differential equation with constant coefficients $x'' + px' + qx = c$. Prove that if $p^2 > 4q$ the general solution is of the form,

$$x = d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t} + \frac{c}{q}$$

   State the condition for convergence to the equilibrium solution.

3. Find the integrating factor for the differential equation

$$2(t^3 + 1)dy + 3yt^2 dt = 0$$

4. Solve the following system of differential equations:

$$\begin{cases} y_1' = 4y_1 + y_2 \\ y_2' = 2y_1 + 3y_2 + 1 \end{cases}$$

   Find the solution when $y(0) = [7 \;\; 1]'$.