Lecture 3

Hyperbolic surfaces

We take it back where we sort of left it at the end of Lecture 1. We had considered *conics* and some *cubics*, and showed that they admitted nice parametrisations.

- Conics are actually *rationally* biholomorphic to the Riemann sphere.
- Some cubics (actually all non-degenerate ones, which we didn't prove) can be *uniformised* by the complex plane $\mathbb{C}$. This means that they admit a natural holomorphic parametrisation by $\mathbb{C}$, which is also a covering map. This in particular implies that a cubic cannot, even locally, be parametrised by rational functions.

In this lecture we try to generalise these results to the case of higher degree algebraic curves. This will require a detour via *hyperbolic geometry*, where the *hyperbolic plane* replaces the complex plane and negative curvature geometry replaces Euclidean geometry.

3.1 Riemann surfaces

3.1.1 Formal definition

Definition 3.1.1 (Riemann surface). A Riemann surface is an atlas of charts on a topological surface, taking values in $\mathbb{C}$ and whose transition maps are holomorphic. In other words, a Riemann surface is 1-dimensional complex manifolds, a complex curve.

Algebraic curves are particular examples of Riemann surfaces. These objects came about as people gradually realised that for a lot of natural questions about algebraic curves, only the Riemann surface structure matters.

3.1.2 The genus of algebraic curves

A first awkward question is that if we are given a polynomial equation, there is no easy way of determining the topology of the associated surfaces. We had to grind to show that conics have genus 0 and (some) cubics have genus 1. We are not yet ready to give anything in the way of an answer to this question, however we are going to slightly spoil the final answer by revealing what happens in degree 4.

Theorem 3.1.2. *Let $\mathcal{C}$ be an algebraic surface*[1] *in $\mathbb{C}^2$ defined by a quadratic equation $P(z,w)=0$. Then $\mathcal{C}$ has genus* 3.

3.2 Hyperbolic geometry

We recall important facts about hyperbolic geometry in dimension 2. AN interesting feature of hyperbolic geometry is that it can be treated from two almost distinct (but complementary) view points: that of Riemannian geometry and that of complex analysis. We quickly introduce both here.

Definition 3.2.1 (Hyperbolic space, the Riemann surface). The hyperbolic plane is the Riemann surface

$$\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

Definition 3.2.2 (Hyperbolic space, the Riemannian surface). The hyperbolic plane $\mathbb{H}$ is the unique (up to isometry) complete, simply-connected Riemannian 2-manifolds of constant curvature -1.

This second definition contains a non-trivial statement: that such a Riemannian surface exists and is unique. The existence is easy: just check that $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ together with the metric $\frac{dx^2+dy^2}{y^2}$ with $z = x+iy$ is such an example. The uniqueness is a delicate result in Riemannian geometry, which we will admit.

Proposition 3.2.3. *The conformal structure of $\mathbb{H}$ the Riemannian surface is the same as that of $\mathbb{H}$ the Riemann surface.*

Proposition 3.2.4. *The group of biholomorphisms of $\mathbb{H}$ is equal to the group of isometries of $\mathbb{H}$.*

The proofs of these two Proposition are left as exercises.

[1] that is a surface in the topological sense, algebraic geometer would probably call it a smooth (or regular) surface

3.3 Hyperbolic surfaces

3.3.1 Fuchsian groups

Definition 3.3.1 (Fuchsian groups). A Fuchsian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R}) \simeq \mathrm{Iso}^+(\mathbb{H})$.

Definition 3.3.2 (Hyperbolic surface, analytic definition). A hyperbolic surface is the quotient of $\mathbb{H}$ by a Fuchsian group Γ which acts on $\mathbb{H}$ without fixed points.

Such a "hyperbolic surface" naturally inherits

1. a Riemann surface structure, as that of $\mathbb{H}$ passes to the quotient since elements of Γ act via biholomorphisms;

2. a *complete* [2] Riemannian metric of constant curvature -1; as the Riemannian metric $\frac{dx^2+dy^2}{y^2}$ on $\mathbb{H}$ passes to the quotient since elements of Γ act via isometries.

What is less clear is whether one can work the other way around:

1. If one is given a Riemann surface, is it (biholomorphic to) a quotient of $\mathbb{H}$ by a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$?

2. If one is given a Riemanniann metric of constant curvature -1, is it (isometric to) a quotient of $\mathbb{H}$ by a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$?

We are going to see that the answer to both question is essentially yes. For the first, it is the very deep *uniformisation theorem*, which we won't discuss in this Lecture. For the second, it derives from the non-trivial (but arguably less deep) fact from Riemannian geometry, which we discuss below.

3.3.2 Metrics of constant curvature

We start by recalling the following theorem.

Theorem 3.3.3. *A **complete**, simply-connected Riemannian surface of constant curvature -1 (respectively 0, 1) is isometric to the hyperbolic space (respectively $\mathbb{R}^3$ with the Euclidean metric, S^2 with the round metric).*

[2] The completeness hypothesis is very important to be able to invert this construction. It is too often inconspicuous and its importance not appreciated.

We give a few hints as to how to prove this theorem in the Exercises section. (I think it's fair to say that if you can understand the proof of this theorem, which is not very difficult, you have achieved a respectable understanding of basic Riemannian geometry). We can now give an equivalent definition of a hyperbolic surface, based on Riemannian geometry.

DEFINITION 3.3.4 (Hyperbolic surface, metric version). A hyperbolic surface is a smooth surface endowed with a **complete** Riemannian metric of class at least $\mathcal{C}^2$ of constant curvature -1.

By Theorem 3.3.3, this definition is equivalent to the analytic definition that we have given earlier.

This is a general fact that the universal cover picture of a manifold can be geometrised. If one has a Riemannian manifolds (M, g), one can pull the metric g back to $\tilde{M}$ the universal cover of M to a metric which denote $\tilde{g}$. In the case of (Σ, g) a hyperbolic surface, one readily checks that

- $(\widetilde{\Sigma}, \tilde{g})$ is complete;
- $(\widetilde{\Sigma}, \tilde{g})$ has constant curvature -1;
- $\pi_1(\Sigma)$ acts on $(\widetilde{\Sigma}, \tilde{g})$ by isometries.

In the particular, by Theorem 3.3.3, $(\widetilde{\Sigma}, \tilde{g})$ is isometric to $(\mathbb{H}, \frac{dx^2+dy^2}{y^2}$ and $\pi_1(\Sigma)$ acts via a fixed-point free Fuchsian group Γ.

3.4 Building a hyperbolic surface

After doing away with all the abstract nonsense, we can start asking the only question that should matter at this stage: *are there such things as hyperbolic surfaces?*

THEOREM 3.4.1. *There exists a compact hyperbolic surface of arbitrary genus $g \geq 2$, that is a compact surface endowed with a Riemannian metric of curvature -1.*

THEOREM 3.4.2. *For all $g \geq 2$, there exists a discrete group $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ acting on $\mathbb{H}$ without fixed points such that the quotient $\mathbb{H}/\Gamma$ is homeomorphic to a genus g surface.*

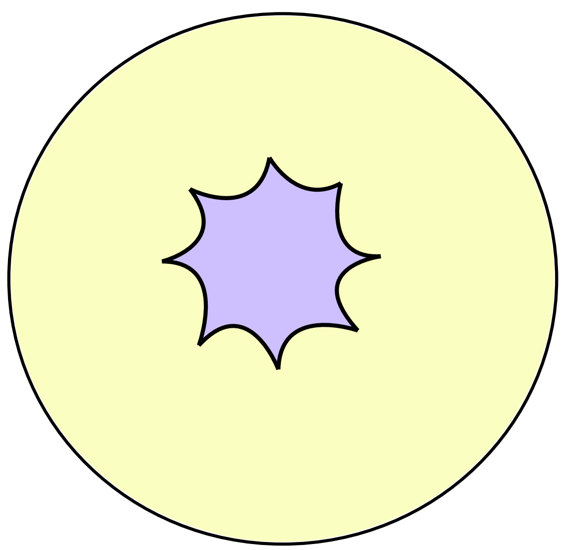
3.4.1 Hyperbolic polygons and the group associated to it

WARNING: What follows is a difficult construction. That for two reasons.

- It looks deceptively easy, when a lot of the reasons why it works are fairly subtle. Pay close attention to the angle condition.
- It's a very geometric construction, with a lot of it being "intuitive" geometric/topological steps (isometric glueing, taking a copy of an object and shifting it in space, glueing sides preserving the orientation,etc.). It's usually not too hard to make these rigorous, but it's formally painful. We do a fair amount of hand-waving, which is fine provided one knows that what hides behind it can easily be made rigorous.

The method to construct such object is to start from a standard topological construction of surface : consider a topological $4g$-gon and glue pairs of sides together in the way that make the resulting surface genus g. We have seen two such examples in the previous lecture.

A *hyperbolic polygon* is the object one gets when joining a finite number of consecutive points by the hyperbolic line segments from one point to the next (see Figure below).



A natural way to try and make the topological construction geometric is to glue side a hyperbolic polygon using isometries. To this end, we use the following fact.

Proposition 3.4.3. *Let A, B and P, Q two pairs of distinct points in $\mathbb{H}$. If $d_{\mathbb{H}}(A,B) = d_{\mathbb{H}}(P,Q)$, there exists a* ***unique*** *orientation-preserving isometry of $\mathbb{H}$ mapping A to P and B to Q.*

For the rest of this discussion we consider a hyperbolic polygon $\mathcal{P}$.

- The consecutive oriented sides are labelled $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, \cdots, a_g, b_g, a_g^{-1}, b_g^{-1}$.
- We assume that for all i, a_i and a_i^{-1} on one side, and b_i and b_i^{-1} on the other, have same length.
- We denote by γ_i the hyperbolic isometry mapping a_i to a_i^{-1} and μ_i the one mapping b_i to b_i^{-1}.

Exercise 3.4.4. Show that such a polygon exists.

The group associated to $\mathcal{P}$ We have defined above the elements $\gamma_1, \cdots, \gamma_g, \mu_1, \cdots, \mu_g$.

Definition 3.4.5. Let Γ be the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by $\gamma_1, \cdots, \gamma_g, \mu_1, \cdots, \mu_g$.

The main result of this lecture is the following theorem, due to Poincaré.

Theorem 3.4.6 (Poincaré). *If the sum of the interior angles of $\mathcal{P}$ is equal to*[3] 2π, *then the group Γ is a Fuchsian group.*

In particular, the quotient $\mathbb{H}/\Gamma$ is a compact hyperbolic surface of genus 2.

3.4.2 The quotient surface

We actually start by describing what the quotient $\mathbb{H}/\Gamma$ in Poincaré's theorem *should* look like. We now consider the (for now topological) surface Σ obtained by identifying sides of P via the γ_i and μ_is.

- It is a genus g surface.
- The vertices project onto the same point $p \in \Sigma$.
- The sides projects onto $2g$ simple closed curves all based at p, which we call $a_1, b_1, \cdots, a_g, b_g$.

We now define a Riemannian metric on Σ the following way.

[3] The optimal hypothesis here is "the sum of the interior angles of $\mathcal{P}$ is equal to $\frac{2\pi}{n}$ for some integer n".

1. We define it to be the push-forward of the hyperbolic metric on the interior of $\mathcal{P}$. This defines a Riemmannian metric on $\Sigma \setminus \bigcup_i a_i \cup b_i$.

2. It is easily seen that this metric extends to a Riemannian metric on $\Sigma \setminus \{p\}$. This is because A_i and $A_i^{|1}$ are segments glued via an isometry, thereby gluing two half-spaces along a geodesic line via an isometry, which is a way of a reconstructing the hyperbolic space.

We have thus constructed a Riemannian metric on $\Sigma \setminus \{p\}$ which is everywhere locally isometric to the hyperbolic plane.

Near p, the surface looks like a number of angle bisectors glued together around p, whose total angle is the sum of the angles is equal to the sum of the interior angles of $\mathcal{P}$. This finishes to prove the

Proposition 3.4.7. *The restriction of the hyperbolic metric to $\mathcal{P}$ induces a hyperbolic metric on Σ if and only if the sum of the interior angles of $\mathcal{P}$ is equal to 2π.*

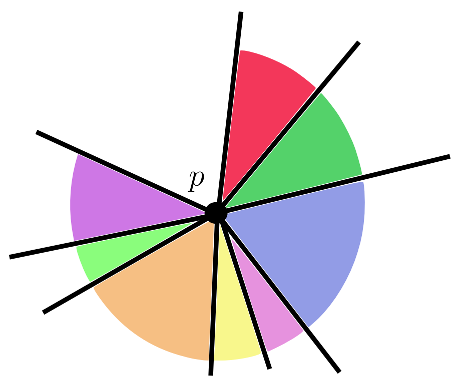


Figure 3.4.1: Neighbourhood of point p, where the two red half-lines are identified, for a polygon whose interior angles don't necessarily sum up to 2π.

3.4.3 The group Γ and its action

One could argue that what we have done in the previous section is not a a hundred percent rigorous. That's only because we have not written down the tedious detail of defining the metric near the sides and the point p, but it does not present any conceptual difficulty. However, we give a second construction based on the action of the group generated by the maps $\gamma_1, \cdots, \gamma_g, \mu_1, \cdots, \mu_g$.

Definition 3.4.8. Let Γ be the subgroup of $\mathrm{PSL}(2, \mathbb{R})$

The claim that we are going after is that Γ acts nicely on $\mathbb{H}$ (properly discontinuously that is).

- If a group Γ does not act properly discontinously, there is a sequence of non-trivial elements ($\gamma_n \in \Gamma$) which tends to identity.
- In particular, we can find an element $\gamma \neq \mathrm{id} \in \Gamma$ such that $\gamma(\mathcal{O})$ intersects itself.

This is an invitation to look at what happens to translates of $\mathcal{P}$ by elements of Γ.

First generation We first look at what we get when we apply the generators of Γ (which are the maps that we used to glued sides of $\mathcal{P}$ to build Σ earlier). We get something that looks like this:

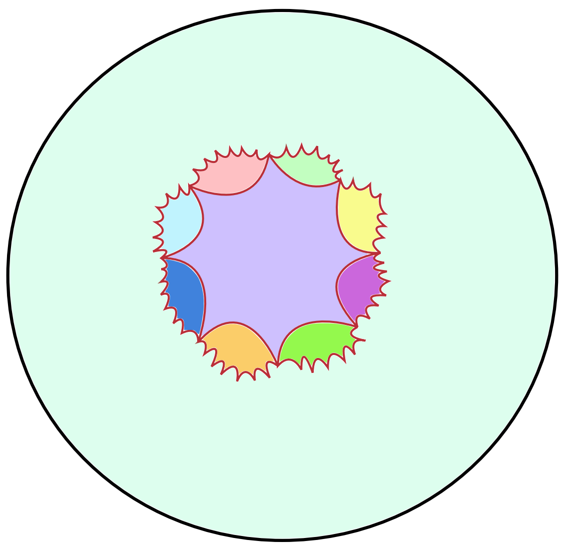


Figure 3.4.2: The polygon $\mathcal{P}$ and its images under the generators of Γ.

This looks promising, what we have done is just glued some isometric copies of $\mathcal{P}$ to each side. We thus get a new hyperbolic polygon, but this one has $7 \times 8 = 56$ sides. Next gen is going to be trickier to handle (or even to define).

Second generation The key remarks are the following.

Proposition 3.4.9. *Consider, on Figure 3.4.2, one of the* 8 *copies of* $\mathcal{P}$ *obtained by shifting* $\mathcal{P}$ *by one of the generators of* Γ*. Let* $\mathcal{P}'$ *be one of these copies.*

1. $\mathcal{P}'$ *meets* $\mathcal{P}$ *in exactly one of the sides of* $\mathcal{P}$*.*

2. *The isometry which maps any side of $\mathcal{P}'$ to the one it is paired (via the pairing induced by that of $\mathcal{P}$) belongs to Γ.*

The first point is almost by definition of $\mathcal{P}'$. For the second point, if $\mathcal{P}' = \gamma_0(\mathcal{P})$ and that δ is the element identifying a pair of sides of $\mathcal{P}$, one easily checks that $\gamma_0 \circ \delta\gamma_0^{-1}$ identifies the corresponding pair of sides in $\mathcal{P}'$.

The second step corresponds to adding, using the identifying maps from Proposition 3.4.9, copies of $\mathcal{P}$ to all the outer edges of the polygon with 56 sides constructed at the first generation. We thus obtain a new, bigger polygon with $56 \times 7 = 392$ sides.

n-th generation One would be tempted to keep iterating the following process to obtain, at the n-th generation a polygon with 8×7^n sides. There are cases where it works like that, and we can easily get convinced that all the copies of the polygon $\mathcal{P}$ exactly correspond to a reduced word in the generators of Γ.

In our case, we run into trouble at the 8^{th} generation. To understand why, one has to keep their focus a what happens around one of the vertices of the original polygon $\mathcal{P}$. We will have glued , around this vertex 8 copies of $\mathcal{P}$, each corresponding to one of the vertices of $\mathcal{P}$. Because the sum of the angles at this vertex is 2π, at the 8^{th}-step the polygon that we will want to add at one of the sides of the $8 \times 7^7 = 6588344$-gon that we will have constructed will be $\mathcal{P}$-itself.

- This will have meant that this big polygon will have closed-in on itself in this place.
- This is not a problem, as the element in the group γ corresponding to the sequence of gluings described above, will be trivial (an isometry that maps a polygon $\mathcal{P}$ on itself must be the identity).

So even if we encounter such a loop, because of the angle assumption, we still get a polygon, formed of a number of copies of $\mathcal{P}$, each of which corresponding, at step n, to an element of Γ which can be written as the product of fewer than n elements of Γ.

Finishing up Iterating *ad infinitum*, we obtain a tilling of $\mathbb{H}$ by copies of $\mathcal{P}$. (A tilling is a decomoposition of $\mathbb{H}$ a the union of infinitely many copies of $\mathcal{P}$ such that two distinct copies intersect in at most a side).

- Each *tile* of the tiling is associated to an element of $\gamma \in \Gamma$, which the element such that the tile is equal to $\gamma(\mathbb{P})$.

- One moves to one tile to another applying one of the generators of Γ, $\gamma_1, \mu_1, \gamma_2, \mu_2$ and their inverses.

I ran out of patience with my drawing software and decided to look for a picture of the tiling on the Internet. I found what I at first thought was a good one. Then upon looking closely I realise it wasn't quite the one that we have been working with.

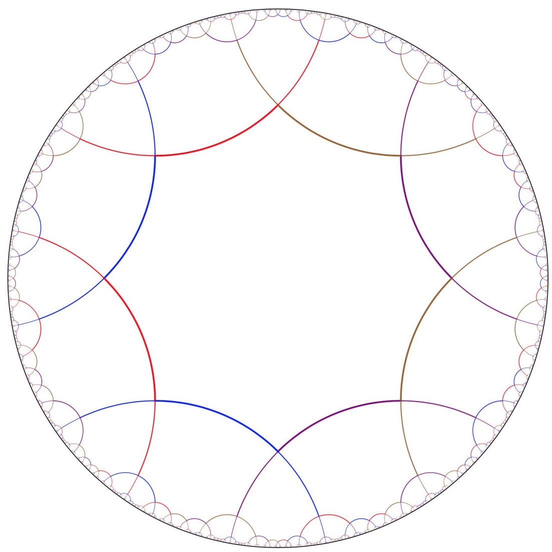


Figure 3.4.3: A tiling of $\mathbb{H}$ by translates of a polygon $\mathcal{Q}$.

Exercise 3.4.10. Work out the key differences between this tilling and the one coming from the genus 2 surface. What is the group associated to it?

3.4.4 How many surfaces did we build?

What we do in this paragraph is unrigorous, but could be made so with a bit of effort. However it shows the power of sometimes being a bit more liberal with rigour to give ourselves more space to take a step back.

To carry out our construction of a genus g hyperbolic surface, and the associated Fuchsian group, if was enough to have a hyperbolic $4g$-gon whose interior angles sum up to 2π, and whose paired sides had same length. How many such polygons to we have?

Proposition 3.4.11. *The set of hyperbolic surfaces of genus g can be locally parametrised by $6g - 6$ real parameters.*

Proof: (Sketch)

- To define an arbitrary $4g$-gon, it suffices to have $4g$ points in $\mathbb{H}$. That's exactly $8g$ real parameters.
- The fact that pairs of sides should have same length imposes $2g$ equations, which leaves us with $8g - 2g = 6g$ parameters.
- The sum of the angles is another equation, which leaves $6g - 1$ parameters.
- Finally, two polygons which are the image of one another via an isometry define the same surface. That's another 3 parameters gone, we're left with $6g - 4$ parameters.
- There is one last subtle equation. What our construction actually gives is a hyperbolic surface together with a **marked point** (the point onto which the vertices of the polygon projects). This marked point accounts for another 2 parameters. That's a total of $6g - 6$ parameters. ∎

3.4.5 Remarks about the construction

Universal cover of a higher genus surface In Lecture 2, we discuss the universal cover of genus g (topological) surfaces, and claimed that it was always homeomorphic to $\mathbb{R}^2$. A proof that one could give was to simply build on the fact that there are only two simply-connected surfaces, the sphere S^2 and $\mathbb{R}^2$.

The construction of the Fuchsian group Γ gives an alternative proof of this fact. Because the projection

$$\pi : \mathbb{H} \longrightarrow \mathbb{H}/\Gamma$$

is a covering map and $\mathbb{H}$ is simply-connected, by the uniqueness of the universal cover it is $\mathbb{H}/\Gamma$'s universal cover. But we have independently verified that $\mathbb{H}/\Gamma$ is homeomorphic to a genus 2 surface, so $\mathbb{H}$ (which is homeomorphic to $\mathbb{R}^2$) is the universal cover of the genus 2 surface.

Note that the exact same construction would work for a surface of arbitrary genus $g \geq 2$.

3.5 Back to algebraic surfaces

We go back to the case of quartics, which yield genus 3 surfaces. A degree 4 polynomial is given by an equation of the form

$$P(z,w) = \sum_{i,j \ i+j\leq 4} a_{i,j} z^i w^j$$

with $a_{i,j} \in \mathcal{C}$. That's exactly (count them yourself) 15 complex parameters (the coefficients $a_{i,j}$ with $i+j \leq 4$.

Now, when do two such polynomials give rise to the same Riemann surfaces?

For this question, it is actually best to think of curves in $\mathbb{CP}^2$. Via the homogeneisation procedure, an arbitrary polynomial in $\mathbb{C}[X,Y]$ is canonically associated to a *homogeneous* polynomial in $\mathbb{C}[X,Y,Z]$. Homogeneous polynomial also form a (vector) space of complex dimension 15.

Projectively equivalent curves. There, one easily sees that two polynomials P_1 and P_2 give rise to isomorphic Riemann surfaces if one can obtain one from the other by a linear change of coordinates in $\mathbb{C}^3$. Let A be the matrix of such a linear change of coordinates, it induces a projective map on $\mathbb{CP}^2$, and the Riemann surface $\{P_1 = 0\} \subset \mathbb{CP}^2$ is the image of $\{P_2 = 0\}$ by the projective map induced by A (or its inverse, I didn't carefully check). Since the set of 3×3 invertible matrices is 9-dimensional, we obtain the following (loose) statement.

Proposition 3.5.1. *The space of quartics in* $\mathbb{C}^2$ *(or* $\mathbb{CP}^2$*) it (at most)* $15-9=6$ *(complex) dimensional.*

So what have we obtained.

- The space of hyperbolic surfaces of genus 3, by Proposition 3.4.11, is of real dimension 12.
- The space of quartics in $\mathbb{CP}^2$, which consists of Riemann surfaces of genus 3, also seems to be of real dimension 12 (because it is of complex dimension 6).

This sort of leaves us with two (plausible) possibilities[4].

1. Either the "space" of Riemann surfaces of genus 3 has (at least) two connected components, one consisting of algebraic curves and the other of hyperbolic surfaces.
2. Or hyperbolic surfaces and algebraic surfaces are (from the Riemann surface perspective) the same objects.

[4]If you think a bit you'll realise that some more exotic options are possible, but we leave it to these two to simplify the discussion

3.6 Higher dimensional hyperbolic geometry

To be written, maybe

3.7 Exercises

Exercise 3.7.1. Give an example of incomplete Riemannian surface with constant curvature -1. Show that its universal cover is NOT isometric to $\mathbb{H}$

Exercise 3.7.2. Show that for all $n \geq 5$, there exists a regular hyperbolic n-gon whose interior angles sum up to 2π.

Exercise 3.7.3. Show that the set of biholomorphisms of $\mathbb{CP}^1$ is exactly the set of homographies (maps of the form $z \mapsto \frac{az+b}{cz+d}$ with $ad - bc \neq 0$.

Exercise 3.7.4. As $\mathbb{H}$ is the upper-half plane, it can be seen as an open subset of $\mathbb{CP}^1$.

1. Show that any biholomorphism of $\mathbb{H}$ extends to $\mathbb{CP}^1$.
2. Derive from the previous question that $\mathrm{BiHol}(\mathbb{H}) = \{z \mapsto \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0\}$.

Exercise 3.7.5. Show that the set of orientation-preserving isometries of $\mathbb{H}$ for the metric $g = \frac{dx^2+dy^2}{y^2}$ in the upper-half plane model is exactly

$$\{z \mapsto \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0\}.$$

Exercise 3.7.6. Consider a subgroup $\Gamma < \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Iso}^+(\mathbb{H}) = \mathrm{BiHol}(\mathbb{H})$.

1. Show that Γ acts properly discontinuously on $\mathbb{H}$ if and only if it is discrete.
2. Show that $\mathbb{H}/\Gamma$ can be endowed with a metric of constant curvature -1 (or a structure of Riemann surface) if additionally Γ acts without fixed points (no elements apart from the identity has fixed points).

Exercise 3.7.7. **(Warning: difficult exercise!)** Show that a **complete**, simply-connected Riemannian surface of constant curvature -1 (respectively 0, 1) is isometric to the hyperbolic space (respectively $\mathbb{R}^3$ with the Euclidean metric, S^2 with the round metric).

You can consider the exponential map and write the metric in the coordinates of the exponential map, using Jacobi fields.