The Determination of the Boltzmann Constant from the Thermal Noise generated by a Resistor

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Abstract: An experiment was conducted to investigate the effect of varying the resistance and temperature on an amplified signal through a resistive circuit, alongside determining the equivalent noise bandwidth to verify a value for the Boltzmann constant, k. The intrinsic noise of the low noise amplifier was taken into account and from this graphs were plotted of corrected noise amplitude as a function of resistance and temperature respectively. The equivalent noise bandwidth was then determined by measuring the gain over a large range of frequencies. The area of the gain curve alongside the gradients of the first two graphs was used to experimentally determine the Boltzmann constant, k. The mean value for k obtained was \( k = (1.1 \pm 0.1) \times 10^{-23} \text{JK}^{-1} \), compared to the accepted value \( k = 1.3806488 \times 10^{-23} \text{JK}^{-1} \) [1], indicating that the experiment conducted was accurate.

I. INTRODUCTION

At any temperature above absolute zero the thermal motion of charge carriers within a conducting medium, such as a resistor, will induce a randomly fluctuating voltage across its external terminals, known as Johnson noise. Johnson noise is an intrinsic characteristic of any conducting medium generated when a voltage is passed through it. The magnitude of the voltage noise is given by (1).

\[
\bar{e}_n^2 = 4kRT\Delta f
\]  

(1)

Where \( \bar{e}_n^2 \) is the mean square voltage induced across a resistor \( R \) at temperature \( T \), within a frequency range \( f \) to \( f + \Delta f \), and where \( k \) is the Boltzmann constant. In this investigation the noise generated by a warm resistor is amplified electronically to a measurable level, as in practice the resistor noise is small however will still limit the accuracy and sensitivity of measurements. Introducing an amplifier establishes an additional noise, intrinsic to the amplifier itself, into the amplified signal. Therefore, the signal produced from the amplifier can be represented as an equivalent noiseless amplifier in conjunction with a series voltage noise source, \( V_n^2 \). The total equivalent input noise can be expressed by (2).

\[
V_{in}^2 = e_n^2 + V_n^2 + (i_nR)^2
\]  

(2)

Where \( e_n^2 \) is the mean square voltage induced across a resistor (combined resistor and amplifier noise), \( V_n^2 \) is the series voltage noise source and \( i_n \) is the current noise, which for this investigation can be neglected. The amplifier gain is a function of frequency; hence the noise bandwidths will be dependent on the total frequency response of the amplifier and voltmeter. Therefore, the mean square noise voltage as measured on the voltmeter can be represented by (3) [5].

\[
V_{n}^2(R) = 4kTR\int_0^{\infty} G^2(f)df
\]  

(3)

Where \( k \) is the Boltzmann Constant, \( T \) is the temperature, \( R \) the resistance and the quantity \( \int_0^{\infty} G^2(f)df \) is the equivalent noise band width of the system. The gain in amplification of the signal is determined using (4).

\[
G = \frac{V_o}{V_{in}A}
\]  

(4)

Where \( V_{in} \) is the input voltage, \( V_o \) is the corrected output voltage calculated by subtracting the intrinsic amplifier noise from the root mean square (RMS) voltage and \( A \) is known as the attenuation constant calculated using (5).

\[
A = \frac{V_{out}}{V_{in}}
\]  

(5)

Where \( V_{out} \) is the output voltage and \( V_{in} \) is the input voltage through the attenuator shown in Figure 3.

II. EXPERIMENTAL METHOD

An experiment was conducted to determine how the noise of a resistor varies as a function of resistance and temperature respectively. The apparatus used is shown in Figure 1. Prior to taking measurements the oscilloscope was calibrated to view the waveform of the induced noise across the resistor.
The ambient temperature was measured to be $T=(293.4\pm0.1)K$, and the apparatus in Figure 1a was set up. It was determined that there was negligible interference from the 50Hz mains electricity supply, as there was no noticeable sinusoidal envelope around the noise signal of the low noise amplifier on the oscilloscope screen.

Initially, the intrinsic amplifier noise was measured by setting the resistance of the switched resistor box to $R=0\Omega$. A weighted mean calculation was computed due to random fluctuations in voltage noise and the amplifier noise was obtained to be $V_n=(9.7\pm0.2)mV$. A sample of twenty readings of RMS voltage was taken, due to random fluctuations in voltage noise, for each of the ten resistances of the switched resistor box varying from $R=0\Omega-25.5k\Omega$. The mean RMS voltage was computed using statistical analysis techniques and the corrected output noise was computed by subtracting the intrinsic amplifier noise, $V_m$, from the mean RMS voltage noise. It was ensured that the RMS voltmeter was set to DC in order to read the true RMS voltage. All uncertainties and hence standard errors were propagated and a graph of corrected noise amplitude as a function of resistance was plotted, shown in Figure 4.

The apparatus is Figure 1b was then set up. The intrinsic noise of the low noise amplifier, $V_{an}$, was assumed to remain constant. The active end of the resistive probe, of resistance $R=(10.0\pm0.1)k\Omega$, was immersed into cold water of a temperature $T=(293.7\pm0.1)K$ and a sample of twenty readings of RMS voltage was measured, whilst monitoring the temperature of the water using a type K thermocouple.

The above procedure was repeated for materials of different temperatures including hot water $T=(364.0\pm0.5)K$, ice $T=(273.3\pm0.1)K$, dry ice $T=(195.4\pm0.1)K$ and liquid nitrogen $T=(79.6\pm0.1)K$, taking a sample of twenty readings for each medium.

The mean RMS voltage was computed from the sample and the corrected output noise was again calculated by correcting the measured voltage by subtracting the amplifier noise, $V_m$. All uncertainties and standard errors were propagated and a graph of corrected noise amplitude as a function of temperature was plotted, as shown in Figure 5.

The final stage was to investigate the frequency response of the amplifier by determining the equivalent noise bandwidth, $\int_0^\infty G^2(f)df$. This was done with the apparatus configuration shown in Figure 2.

Initially, the function generator was set to produce a sinusoidal wave at a frequency of 5kHz and the input voltage of the generator, $V_{in}$, was adjusted until the RMS output voltmeter, $V_o$, read approximately 2V. The input voltage and output voltage of the generator was recorded and the output was corrected for the amplifier noise, $V_{an}$. These measurements were then repeated for $V_o$ and $V_{in}$ over the frequency range 20Hz-50kHz in small increasing increments, for which the low noise amplifier had a significant non-zero gain.

Knowledge of the attenuation constant, A, enables the gain to be calculated for each frequency given by (4). The attenuator used is shown in Figure 3 by a potential divider circuit. It was used to lower the input voltage signal to a suitable level, as well as to improve impedance matching, when the signal was then amplified. This provided efficient power transfer and maximum gain in amplification.

From Figure 3, $V_{in}$ and $V_{out}$ across the attenuator were then determined using the potential divider relation and hence the dimensionless attenuation constant was obtained to be $A=(2.1\pm0.1)x10^{-3}$ using (5). The gain was...
then calculated for each frequency using the relation given in (4), all uncertainties were propagated and hence a graph of $G^2$ as a function of frequency was plotted shown in Figure 6.

### III. RESULTS & ANALYSIS

The intrinsic amplifier noise was found to be $V_a=(9.7\pm0.2)\text{mV}$, which was obtained using a weighted mean calculation due to fluctuations in the voltage noise as well as to reduce the proportion of random error in the measurement. The amplifier noise was used to correct the noise readings from the RMS voltmeter in order to eliminate the systematic error and obtain a true result for the noise of the resistor throughout the investigation.

The random fluctuations in the RMS voltage were related by a Gaussian probability distribution, hence a sample of twenty readings of the RMS voltage were measured. This enabled accurate inferences to be made about the parent distribution of noise. Nonetheless, the precision may be improved by increasing the sample size, which would ensure greater precision and accuracy in the mean noise result.

The mean and standard error were calculated ensuring an average value for the noise was obtained for each resistance. Figure 4 shows the plot of the variation of mean noise as a function of resistance.

A linear relationship was obtained between the mean noise amplitude and resistance, which was in agreement with the theory given by (3). Referring to Figure 4, the gradient was determined to be $4kR\int_0^\infty G^2(f)df = (454\pm3)\times10^{-9} \text{V}^2\Omega^{-1}$, obtained using the least squares fitting method [3]. Here, $R=(10.0\pm0.1)\text{k}\Omega$, is the resistance of the resistive probe.

The precision in both gradients of Figures 4 and 5 may be improved by taking a larger sample size ($>>20$), as well as taking more readings of resistance and temperature respectively. It was assumed that the ambient temperature and the intrinsic amplifier noise remained constant throughout the investigation.

The equivalent noise bandwidth was then determined from the area of the plot in Figure 6.

### Results

![Graph of $G^2$ as a function of frequency](image1)

**Figure 4** Variation of noise generated by a resistor plotted as a function of resistance, at constant temperature $T=(293.4\pm0.1)\text{K}$ [2]. Note that the errors in resistance and mean corrected noise amplitude are negligible in comparison to the scale. A linear relationship was obtained between the mean noise and temperature, which was in agreement with the theory given by (3). The gradient of Figure 5 was determined to be $4kR\int_0^\infty G^2(f)df = (14.3\pm0.1)\times10^{-6} \text{V}^2\text{K}^{-1}$, obtained using the least squares fitting method [3]. Here, $R=(10.0\pm0.1)\text{k}\Omega$, is the resistance of the resistive probe.

The precision in both gradients of Figures 4 and 5 may be improved by taking a larger sample size ($>>20$), as well as taking more readings of resistance and temperature respectively. It was assumed that the ambient temperature and the intrinsic amplifier noise remained constant throughout the investigation.

The equivalent noise bandwidth was then determined from the area of the plot in Figure 6.

![Graph of G2 as a function of frequency](image2)

**Figure 5** Variation of noise generated by a resistor plotted as a function of temperature, for the resistive probe (resistance $R=(10.0\pm0.1)\text{k}\Omega$) immersed in media of differing temperature [2]. Note that the errors in temperature and mean corrected noise amplitude are negligible in comparison to the scale. A linear relationship was obtained between the mean noise and temperature, which was in agreement with the theory given by (3). The gradient of Figure 5 was determined to be $4kR\int_0^\infty G^2(f)df = (14.3\pm0.1)\times10^{-6} \text{V}^2\text{K}^{-1}$, obtained using the least squares fitting method [3]. Here, $R=(10.0\pm0.1)\text{k}\Omega$, is the resistance of the resistive probe.

The precision in both gradients of Figures 4 and 5 may be improved by taking a larger sample size ($>>20$), as well as taking more readings of resistance and temperature respectively. It was assumed that the ambient temperature and the intrinsic amplifier noise remained constant throughout the investigation.

The equivalent noise bandwidth was then determined from the area of the plot in Figure 6.
The frequency response of the circuit, which may be due to the fact the signal was attenuated and then amplified. Whereas, the uncertainties in the switched resistor box of 0.1% and the digital thermometer of 0.5% would not have a significant effect in the propagated uncertainties.

The two values for the Boltzmann constant were determined; at constant temperature k was obtained to be $(1.1\pm0.1)\times10^{-23}$ JK$^{-1}$ and at constant resistance k was found to be $(1.0\pm0.1)\times10^{-23}$ JK$^{-1}$. Both of these values are in agreement with each other and when combined through a weighted mean calculation, resulting in a mean value of $k=(1.1\pm0.1)\times10^{-23}$ JK$^{-1}$.

Evaluating this against the accepted value of k given by [1] it can be seen that the value for k obtained experimentally was accurate as its upper bound of 3 uncertainty ranges lied within the accepted value. The precision may be improved by taking a larger sample size ($>>20$) as well as determining a more precise way to measure the RMS output voltage.

REFERENCES


