Math 6103 / Math 6500

Problem Sheet 1: Model solutions.

1) (i) $D = \mathbb{R}$ then $f(x)$ not a function. When $x^2 - 9 = 0$ ie $x = \pm 3$, $\frac{1}{x^2-9}$ not well defiined. Try $D = \mathbb{R} / \{-3, +3\}$.

(ii) $D = \{x \in \mathbb{R} : x \geqslant 1\}$. since $x \neq 0$ then f is well defined. It is a function.

(iii) $f(x) = \sqrt{x^3 + 2x}$. Alway require $x^3 + 2x \geqslant 0$ ie $x(x^2+2) \geqslant 0$. ie $x \geqslant 0$. For any $x < 0$, $x^3 + 2x < 0$. Hence f not function on $D = \mathbb{R}$. Try $D = \mathbb{R}^+$.

(iv) since $x^2 > 0$ $\forall x \in \mathbb{R}$, then $x^2 + 5 > 0$ $\forall x \in \mathbb{R}$. Then $f = x/\sqrt{x^2+5}$ is well defined on $x \in D = \mathbb{R}$. It's a function.

2) Definitions: $f(-x) = f(x) \Rightarrow$ even

$f(-x) = -f(x) \Rightarrow$ odd.

(i) $$f(-x) = \frac{1}{(-x)^2 - 9} = \frac{1}{x^2-9} \Rightarrow \text{even}$$
$$= f(x).$$

(ii) $$f(-x) = -x + \frac{1}{-x} = -\left(x + \frac{1}{x}\right) = -f(x) \Rightarrow \text{odd}.$$

(iii) $$f(-x) = \sqrt{(-x)^3 - 2x} = \sqrt{-x^3 - 2x} = \sqrt{-(x^3+2x)}$$
$$\neq -f(x)$$

$\Rightarrow$ Neither.

(iv) $$f(-x) = \frac{-x}{((-x)^2+5)^{1/2}} = -\frac{x}{(x^2+5)^{1/2}} = -f(x)$$

$\Rightarrow$ odd.

3) (i) $f(x) = x^3 - 2x^2 - 5x + 6$ (*)

$f(1) = 1 - 2 - 5 + 6 = 7 - 7 = 0$

so $(x-1)$ is a factor of $f(x)$.

write $f(x) = (x-1)(x^2 + ax + b)$.

expand:

$$f(x) = x^3 + (a-1)x^2 + (b-a)x - b. \quad (**)$$

compare coefficient of $(**)$ with $(*)$

$\therefore$ have $\quad a-1 = -2 \Rightarrow a = -1$

$\qquad b - a = -5$

$\qquad -b = 6 \Rightarrow b = -6$

so $f(x) = (x^{\cancel{3}}-1)(x^2 - x - 6)$

$$x^2 - x - 6 = (x+2)(x-3)$$

Hence $f(x) = (x-1)(x+2)(x-3)$.

$f(x) = 0$ then $\quad x = 1$

$\qquad x = -2$

$\qquad x = 3$.

(ii) $f(x) = x^4 - 1$.

$x = 1$ is solution to $f(x) = 0$

so $f(x) = (x-1)(x^3 + ax^2 + bx + c)$

expand:

$$f(x) = x^4 + (a-1)x^3 + (b-a)x^2 + (c-b)x - c$$

compare coefficients:

$$a - 1 = 0 \Rightarrow a = 1.$$

$$-c = -1 \Rightarrow c = 1$$

$$b - a = 0 \Rightarrow b = a = 1$$

$$f(x) = (x-1)\underbrace{(x^3 + x^2 + x + 1)}_{q(x)}.$$

$q(x) = 0$ has solution $x = -1$

$$\therefore q(x) = (x+1)(x^2 + \tilde{b}x + \tilde{c}).$$

expand: $q(x) = x^3 + (\tilde{b}+1)x^2 + (\tilde{b}+\tilde{c})x + \tilde{c}$

compare coeff: $\tilde{b} + 1 = 1 \Rightarrow \tilde{b} = 0$

$\tilde{c} = 1$

$$q(x) = x^2 + 1$$

$$\therefore f(x) = (x-1)(x+1)(x^2+1).$$

$$f(x) = 0 \Rightarrow x = +1$$
$$x = -1$$

[unseen]: $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1} = \pm i$

$z = \alpha + \beta i$ is complex number.

(iii) $f(x) = \dfrac{x^3 - 3x + 2}{x^3+5x^2+8x+4} = \dfrac{g(x)}{h(x)}$.

$g(1) = 0 \Rightarrow g(x) = (x-1)(x^2+ax+b)$

$= x^3 + (a-1)x^2 + (b-a)x - b$

comparing coeff: $a-1 = 0 \Rightarrow a = 1$

$b-a = -3 \Rightarrow b = -2$

$g(x) = (x-1)(x^2+x-2) = \underline{(x-1)(x-1)(x+2)}$

$h(-1) = 0 \Rightarrow h(x) = (x+1)(x^2+\alpha x+\beta)$.

$h(x) = x^3 + (\alpha+1)x^2 + (\beta+\alpha)x + \beta$

compare coeff.

$\alpha+1 = 5 \Rightarrow \alpha = 4$

$\beta = 4$.

so $h(x) = (x+1)(x^2+4x+4) = \underline{(x+1)(x+2)(x+2)}$

$\therefore f(x) = \dfrac{(x-1)^2\cancel{(x+2)}}{(x+1)(x+2)^{\cancel{2}}} = \dfrac{(x-1)}{(x+1)(x+2)}$.

3(iv) $f(x) = x^3 - 7x - 6$.

$f(-1) = 0 \implies (x+1)$ is a factor.

$f(x) = (x+1)(x^2 + ax + b)$.

$= x^3 + (a+1)x^2 + (b+a)x + b$.

compare coeff:

$a+1 = 0 \implies a = -1$.

$b = -6$.

$f(x) = (x+1)(x^2 - x - 6)$.

$f(x) = (x+1)(x+2)(x-3)$.

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4). $f(x) = \dfrac{1}{1+2x}$.

$f(x+h) = \dfrac{1}{1+2(x+h)}$.

$f(x+h) - f(x) = \dfrac{1}{1+2(x+h)} - \dfrac{1}{1+2x}$.

$\therefore f(x+h) - f(x) = \dfrac{\cancel{1}+\cancel{2x} - (\cancel{1} + 2(\cancel{x}+h))}{(1+2(x+h))(1+2x)}$.

$$\therefore f(x+h)-f(x) = \frac{-2h}{(1+2(x+h))(1+2x)}$$

using the definition for the derivative

$$f'(x) = \lim_{h\to 0}\left\{\frac{1}{h}\cdot\left(\frac{-2h}{(1+2(x+h))(1+2x)}\right)\right\}$$

$$= \lim_{h\to 0}\left\{\frac{-2}{(1+2(x+h))(1+2x)}\right\}$$

$$= -\frac{2}{(1+2x)(1+2x)} = -\frac{2}{(1+2x)^2}.$$

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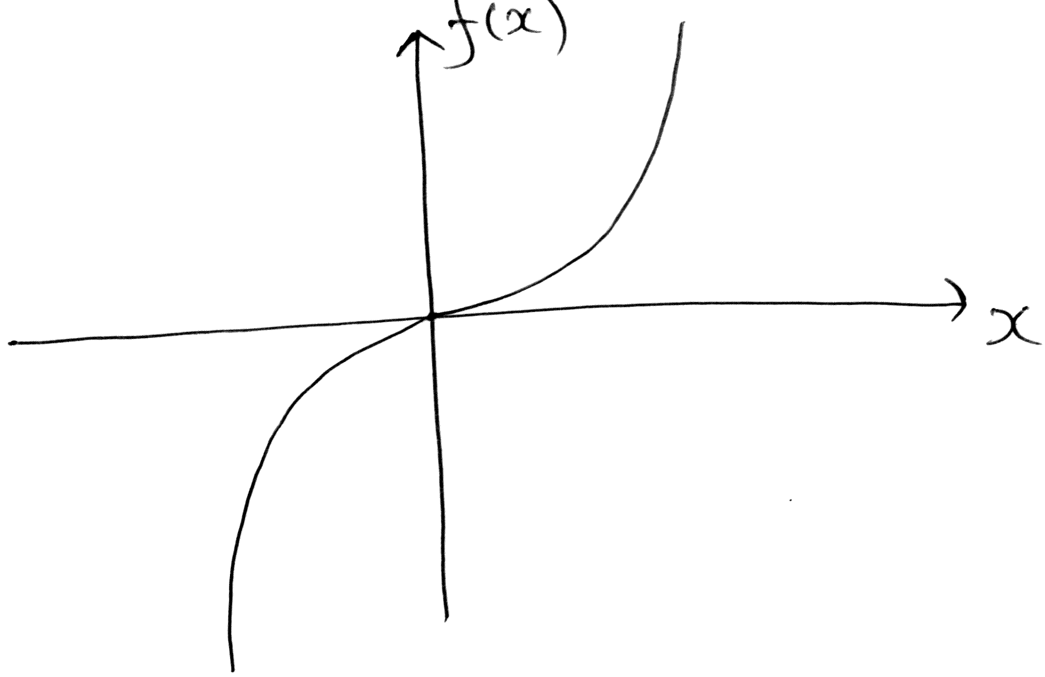
5). $$f(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0. \end{cases}$$

Graph:



Finding derivatives

(i) $x > 0$.

$$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}.$$

$$= \lim_{h \to 0} \left\{ \frac{(x+h)^2 - x^2}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{x^2 + 2xh + h^2 - x^2}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ 2x + h \right\} = 2x.$$

(ii) $x < 0$

$$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{-(x+h)^2 - (-x^2)}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{-x^2 - 2xh - h^2 + x^2}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{-2x - h}{1} \right\} = -2x.$$

To check if $f(x)$ is differentiable at $x=0$, must check the left hand limit $[f'(0^-)]$ and right hand limit $[f'(0^+)]$ match.

$$\lim_{h\to 0^-}\left\{\frac{f(h)-f(0)}{h}\right\} = \lim_{h\to 0^-}\left\{\frac{h^2}{h}\right\} = 0$$

$$\lim_{h\to 0^+}\left\{\frac{f(h)-f(0)}{h}\right\} = \lim_{h\to 0^+}\left\{\frac{-h^2}{h}\right\} = 0.$$

$\Rightarrow$ $f'(0)$ exist [f differentiable at 0]

From part (i) and (ii) clearly

$$f'(x) = 2|x| = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0. \end{cases}$$

$f'(x)$ not differentiable at zero by same argument for function $|x|$ [see lecture notes]. Hence $f''(0)$ does not exist. [set $g(x)=f'(x)$ and differentiate $g(x)$ to get $f''(x)$].