UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH6103**

MODULE NAME : **Differential And Integral Calculus**

DATE : **11-May-07**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATH6103A-001-EXAM-25

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Differentiate the following with respect to x:

   (a) $(x^2 - x + 1)^{-\frac{1}{3}}$, (b) $\exp(x)\cos(x)$, (c) $\exp(\cos(x))$,

   (d) $\cos(\exp(x))$, (e) $\ln\left(x^5(1+x^2)\right)$, (f) $x\ln(x^2)\sin(x)$.

   Express your answers in their simplest forms.

2. Write down the chain rule for differentiating a composition

$$f \circ g.$$

   If f and g are inverse functions, use the chain rule to obtain a relationship between their derivatives at appropriate places.

   Now let f and g be the functions $f(x) = e^x$ and $g(u) = \ln(u)$. Using the fact that $f'(x) = e^x$, but **without** using knowledge of the derivative of the logarithm, find g'.

3. Consider a litter bin made out of sheet metal that consists of a circular cylinder of radius r and height h, with a closed bottom and open top. Write down the expressions for its capacity (i.e. volume when full) and the total area of sheet metal in terms of r and h. If the bin is made of unit area of sheet metal, what is its maximum possible capacity, and what is the ratio $\frac{r}{h}$ when the maximum capacity is achieved?

   For unit surface area, sketch the graph of capacity against r.

   Why might a manufacturer of litter bins be interested in the results of this analysis?

4. Suppose $1Kg$ of an unknown radioactive substance is buried for 10 years, and after this time is found to have mass $0.9Kg$. If a simple exponential decay model is assumed determine the mass of the sample at the following times after burial:

- 5 year,
- 20 years,
- 30 years.

How long after burial would the sample have a mass exactly half of its initial value?

For the purposes of this question, you are given the values of the logarithm: $\ln 2 = 0.693, \ln 3 = 1.099, \ln 5 = 1.609$.

5. Compute the following integrals:

$$(a) \quad \int_1^2 (\ln x)^2\,dx, \qquad (b) \quad \int_0^1 x^3 \exp(-x^4)\,dx,$$

$$(c) \quad \int_0^{\sqrt{2}} \frac{2}{2+x^2}\,dx, \qquad (d) \quad \int_4^6 \frac{3x}{x^2+2x-8}\,dx.$$

6. Show that the integral

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

can not be approximated directly by the trapezium method due to a singularity. Find a suitable substitution to eliminate the singularity and then write down an approximation to I using the trapezium method with 4 intervals of equal length.

7. Solve the following initial-value problems:

$(a) \quad y' - \dfrac{y}{x} = x, \quad y(1) = 2,$

$(b) \quad y'' + 5y' + 4y = -10\sin(2x), \quad y(0) = 0, \quad y'(0) = 0.$