Since we know the derivatives of sin and cos, we should be able to find the derivative of tan since

$$\tan x = \frac{\sin x}{\cos x}.$$

Before we attempt to differentiate (using the product rule), there is another "so called" rule called the *quotient rule*, really, it is simply the product rule for rational functions. We derive the formula below, however if you do not remember the formula, it's best to work with the product rule.

First we write

$$\frac{f(x)}{g(x)} = f(x)\,[g(x)]^{-1},$$

as a product, then apply the product rule as follows:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = f'(x)\,[g(x)]^{-1} + f(x)\frac{d}{dx}\left\{[g(x)]^{-1}\right\}.$$

We need the derivative of $[g(x)]^{-1}$. This is obtained by the chain rule, if we consider $r(u) = 1/u$, then $r'(u) = -1/u^2$, so

$$\begin{aligned}
\frac{d}{dx}[g(x)]^{-1} &= \frac{d}{dx}[r(g(x))] \\
&= r'(g(x))g'(x) \\
&= -\frac{g'(x)}{[g(x)]^2}.
\end{aligned}$$

Finally, we have

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = f'(x)\,[g(x)]^{-1} - f(x)\frac{g'(x)}{[g(x)]^2},$$

finding a common denominator, we may write

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \tag{2.9}$$

This is known as the *quotient rule*.

Example 2.18. Consider the function $\tan x$. We will use the quotient rule where $f(x) = \sin x$ and $g(x) = \cos x$, therefore $f'(x) = \cos x$ and $g'(x) = -\sin x$. Hence

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x \\
&= 1 + \tan^2 x.
\end{aligned}$$

Exercise 2.2. Try find the derivatives of cosec$x$, $\sec x$ and $\cot x$.

Example 2.19. Consider the function $\cos(x^2 + 2x)$. Let us choose $f(g) = \cos(g)$ and $g(x) = x^2 - 2x$, i.e. $f'(g) = -\sin(g)$ and $g'(x) = 2x - 2$. Then, using the chain rule

$$\begin{aligned}\frac{d}{dx}\left[\cos(x^2+2x)\right] &= \frac{d}{dx}\left[f(g(x))\right] \\ &= f'(g(x))g'(x) \\ &= -(2x-2)\cdot(\sin(x^2-2x)).\end{aligned}$$

Example 2.20. Consider the function $\cos^4 x \sin x$. Here we will first apply the product rule, so that

$$\frac{d}{dx}\left[\cos^4 x \sin x\right] = \sin x \frac{d}{dx}(\cos^4 x) + \cos^4 x \frac{d}{dx}(\sin x).$$

The derivative of $\sin x$ is easy, we know this. All we need to do now is apply the chain rule on $\cos^4 x$. Here we will choose $f(u) = u^4$, therefore $f'(u) = 4u^3$ and $g(x) = \cos x$, whose derivative is $g'(x) = -\sin x$. Thus we have

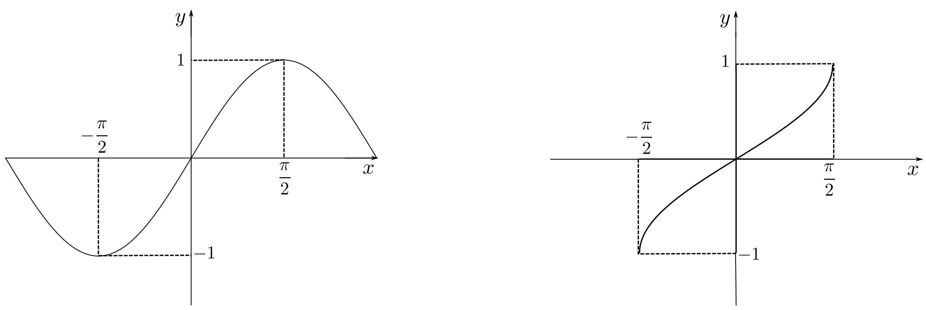
$$\begin{aligned}\frac{d}{dx}\left[\cos^4 x \sin x\right] &= \sin x \frac{d}{dx}(f(g(x))) + \cos^4 x \cdot \cos x \\ &= \sin x(f'(g(x))g'(x) + \cos^5 x \\ &= \sin x \cdot 4\cos^3 x \cdot (-\sin x) + \cos^5 x \\ &= -4\sin^2 x \cos^3 x + \cos^5 x.\end{aligned}$$

2.2.1 The derivatives of inverse trig functions

Now that we know the derivatives of the trigonometric functions, let us consider the inverses of these functions, i.e. $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.

NOTATION: $\sin^{-1} x$ is the inverse function of $\sin x$, whereas $(\sin x)^{-1} = 1/\sin x$.

First let us see how to define $\sin^{-1} x$. To define $\sin^{-1}$, we first must put a constraint on the domain of sin. Recall, a function which associates more than value of x for $f(x)$ cannot have an inverse. So we choose the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



(a) *Limiting* $\sin x$. (b) *Graph of* $\sin^1 x$.

Figure 2.14: *Reflecting* $\sin x$ *in* $y = x$ *gives* $\sin^{-1} x$. $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ *and* $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Let $f(u) = \sin u$ and $g(x) = \sin^{-1} x$, we need to work out $g'(x)$. We know that $f(g(x)) = x$, so differentiating on both sides we have

$$f'(g(x))g'(x) = 1 \qquad \Longrightarrow \qquad g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\sin^{-1} x)}.$$

If we let $\theta = \sin^{-1} x$, then we have $\sin\theta = x$. Now we construct a right angle triangle which satisfies $\sin\theta = x$.

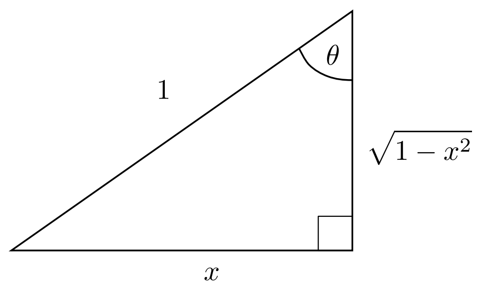


Figure 2.15: *If* $\sin(\theta) = x$, *one can construct such a right angle triangle using Pythagorus' theorem.*

Then we know from the triangle that

$$\cos\theta = \sqrt{1-x^2}.$$

Therefore

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

In the same way, we can work out

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}.$$

Exercise 2.3. Define $g(x) = \cos^{-1} x$ first and then work out $g'(x)$.[5]

[5] End Lecture 9.