So we have proven that
$$\frac{d}{dx}(x^n) = nx^{n-1},$$
is true when $n$ is any integer and also when $n = 1/2$.

It turns out that for all possible powers of $p$:
$$\frac{d}{dx}(x^p) = px^{p-1}.$$

WARNING: If $p$ is a fraction, such as $p = 1/2$, then we require $x > 0$. We don't want to take the root of negative numbers, i.e. $\sqrt{\text{-m}}$, where $m > 0$.

**Example 2.15.** Consider the functions $g(x) = x^4 - x^2$, $f(u) = u^{-\frac{4}{3}}$ and so $g'(x) = 4x^3 - 2x$, $f'(u) = -\frac{4}{3}u^{-\frac{7}{3}}$. The composition gives $f \circ g(x) = (x^4 - x^2)^{-\frac{4}{3}}$. So
$$\begin{aligned} \frac{d}{dx}\left[(x^4 - x^2)^{-\frac{4}{3}}\right] &= f'(g(x))g'(x) \\ &= -\frac{4}{3}(x^4 - x^2)^{-\frac{7}{3}}(4x^3 - 2x). \end{aligned}$$

Generalisation: Suppose we have three functions $f$, $g$ and $h$. Then the composition is
$$f \circ g \circ h(x) = f(g(h(x))),$$
and it's derivative is
$$\frac{d}{dx}\left[f(g(h(x))\right] = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

**Example 2.16.** Consider the function $y(x) = \left[(x^3 + x)^{\frac{1}{2}} + 1\right]^{\frac{1}{3}}$. The derivative is
$$\begin{aligned} \frac{d}{dx}\left[(x^3 + x)^{\frac{1}{2}} + 1\right]^{\frac{1}{3}} &= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) \\ &= \frac{1}{3}\left[(x^3 + x)^{\frac{1}{2}} + 1\right]^{-\frac{2}{3}} \cdot \frac{1}{2}(x^3 + x)^{-\frac{1}{2}} \cdot (3x^2 + 1). \end{aligned}$$

Here we chose the functions in the composition $y(x) = f \circ g \circ h(x)$ as follow:
$$\begin{aligned} h(x) = x^3 + x, &\qquad \Longrightarrow \qquad h'(x) = 3x^2 + 1, \\ g(u) = u^{\frac{1}{2}} + 1 &\qquad \Longrightarrow \qquad g'(u) = \frac{1}{2}u^{-\frac{1}{2}}, \\ f(w) = w^{\frac{1}{3}} &\qquad \Longrightarrow \qquad f'(w) = \frac{1}{3}w^{-\frac{2}{3}}. \end{aligned}$$

The composition works as
$$f(g(h(x))) = f(g(x^3 + x)) = f((x^3 + x)^{\frac{1}{2}} + 1) = \left[(x^3 + x)^{\frac{1}{2}} + 1\right]^{\frac{1}{3}}.$$

**Example 2.17.** Now consider the example where $y(x) = \left[(x^3 + x)^{\frac{1}{2}} + x\right]^{\frac{1}{3}}$. It is similar to the above example, however, when replacing "1" by "$x$" in the square bracket, finding a nice composition becomes a little harder. Instead of trying to work out what this function is as a composition, we simply apply the chain rule using the method "differentiate outer bracket, work inwards". When the derivative is performed on $y(x)$ it follows the steps:

$$\begin{aligned}\frac{d}{dx}\left[(x^3+x)^{\frac{1}{2}}+x\right]^{\frac{1}{3}} &= \frac{1}{3}\left[(x^3+x)^{\frac{1}{2}}+x\right]^{\frac{1}{3}-1}\frac{d}{dx}\left[(x^3+x)^{\frac{1}{2}}+x\right] \\ &= \frac{1}{3}\left[(x^3+x)^{\frac{1}{2}}+x\right]^{-\frac{2}{3}}\left[\frac{1}{2}(x^3+x)^{-\frac{1}{2}}\frac{d}{dx}(x^3+x)+1\right] \\ &= \frac{1}{3}\left[(x^3+x)^{\frac{1}{2}}+x\right]^{-\frac{2}{3}}\left[\frac{1}{2}(x^3+x)^{-\frac{1}{2}}(3x^2+x)+1\right].\end{aligned}$$

Note that in addition to the chain rule, the sum rule has also been applied.

Finally, there is another way of writing the chain rule (or other derivatives). Let $w = g(x)$, $y = f(w) = f(g(x))$. Then we can write

$$\frac{dy}{dx} = \frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x) = \frac{dy}{dw}\cdot\frac{dw}{dx}.$$

Machine:

we have gained 3 basic rules of differentiation,

1. sum rule:
$$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{df}{dx}+\frac{dg}{dx},$$
2. product rule:
$$\frac{d}{dx}\left(f(x)g(x)\right) = g(x)\frac{df}{dx}+f(x)\frac{dg}{dx},$$
3. chain rule:
$$\frac{d}{dx}\left(f(g(x))\right) = \frac{df}{dg}\frac{dg}{dx}.$$

Extras:
$$\frac{d}{dx}(x^p) = px^{p-1}.$$

## 2.2 Differentiation of trigonometric functions

How do we find the derivative of $f(x) = \sin x$? By definition 2.1 (on pg. 22), the derivative of $f(x) = \sin x$ at the point $x = c$ is

$$\begin{aligned}\frac{df}{dx}(c) &= \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \\ &= \lim_{h\to 0}\frac{\sin(c+h)-\sin(c)}{h} \\ &= \lim_{h\to 0}\frac{\sin(c)\cos(h)+\sin(h)\cos(c)-\sin(c)}{h} \\ &= \lim_{h\to 0}\left\{\sin(c)\frac{\cos(h)-1}{h}+\cos(c)\frac{\sin(h)}{h}\right\}.\end{aligned}$$

If we know the limits of $\sin(h)/h$ and $(\cos(h)-1)/h$ as $h \to 0$, then we will know $f'(c)$, since $c$ is unrelated to $h$.

First let us consider $\sin(h)/h$. We draw a circular sector with a very small angle, where the curved side is of length $x$ (in radians). For small angle $x$ (in radians), $\sin x$ and $x$ are almost equal, i.e.

$$\frac{\sin x}{x} \approx 1.$$

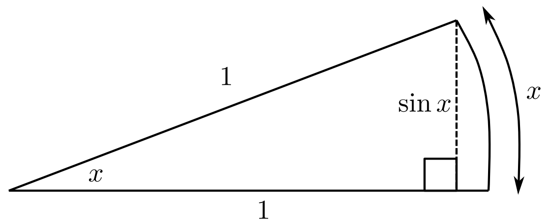
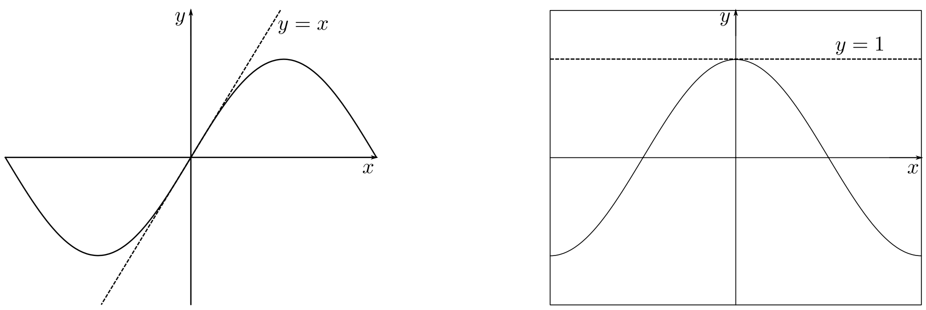



Figure 2.12: *Very small circular sector. Right angled triangle with hypotenuse and adjacent approximately equal i.e. two radii from circular sector.*

As $h \to 0$, $\sin(h)/h \to 1$, if you work in radians. Actually

$$\begin{aligned} \lim_{h\to 0}\frac{\sin(h)}{h} &= \lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h} \\ &= \left.\frac{d}{dx}(\sin x)\right|_{x=0} \\ &= \cos(0) \\ &= 1, \end{aligned}$$

that is, the derivative of $\sin x$ at $x=0$ is one, or the tangent line of $y=\sin x$ at $x=0$ is $y=x$.




(a) *Tangent* $\sin x$ *at* $x=0$. (b) *Tangent* $\cos x$ *at* $x=0$.

Figure 2.13: *Examining the tangents at $x=0$ gives an insight of the derivative at this point.*

Second, we notice that

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos(h)-1}{h} &= \lim_{h\to 0}\frac{\cos(0+h)-cos(0)}{h} \\
&= \frac{d}{dx}(\cos x)\bigg|_{x=0} \\
&= \sin(0) \\
&= 0,
\end{aligned}$$

i.e. the derivative of $\cos x$ at $x = 0$ is zero, since the tangent line of $y = \cos x$ at $x = 0$ is horizontal ($y = 1$). So

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0.$$

Therefore

$$f'(c) = \cos(c),$$

i.e.,

$$\frac{d}{dx}(\sin x) = \cos x.$$

Similarly, we can derive

$$\frac{d}{dx}(\cos x) = -\sin x.$$

NOTE: The change in sign when differentiating $\cos x$!

**Exercise 2.1.** Now that we know the derivatives of sin and cos, we should be able to calculate the derivative of $\tan x$. Try it yourself before next lecture![4]

[4]End Lecture 8.