Definition 2.1. *In general, we define the derivative of a function f at x as*

$$f'(x) = \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{2.2}$$

provided that the limit exists. If the limit exists, we say f is differentiable at x. If we simply say f is differentiable, we mean f is differentiable at all values of x. In this case, $f'(x)$ is also a function of x.

Comments:

1. We interpret the derivative as the instantaneous rate of change, or geometrically as the slope of the tangent line.
2. Equivalently,

$$f'(x) = \lim_{x_0 \to x} \frac{f(x_0) - f(x)}{x_0 - x}, \tag{2.3}$$

since if you put $h = x_0 - x$, then $h \to 0 \iff x_o \to x$ and $f(x_0) = f(x+h)$.

Example 2.4. An example of a function which is not differentiable at a certain point:

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

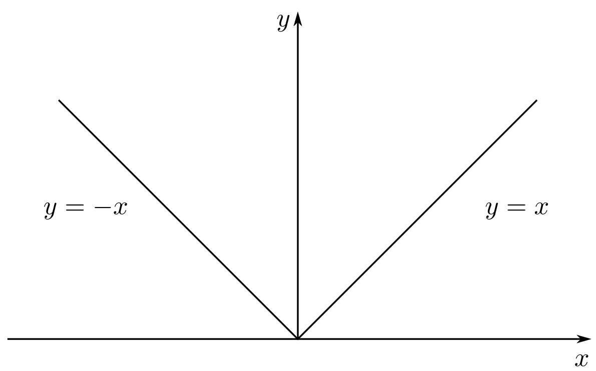


Figure 2.5: *Graph of $y = |x|$.*

At $x = 0$, $f(x)$ is continuous but not differentiable, since through the point $(0, 0)$, you can draw many, many tangent lines. We can also show

$$\text{for} \quad h > 0, \quad \frac{f(0+h) - f(0)}{h} = \frac{h - 0}{h} = 1,$$

$$\text{for} \quad h < 0, \quad \frac{f(0+h) - f(0)}{h} = \frac{-h - 0}{h} = -1,$$

i.e.

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

doesn't exist! Taking the limit from both sides must give the same answer.

Example 2.5. What is the derivative of x^n for a positive whole number n? Let $f(x) = x^n$, then

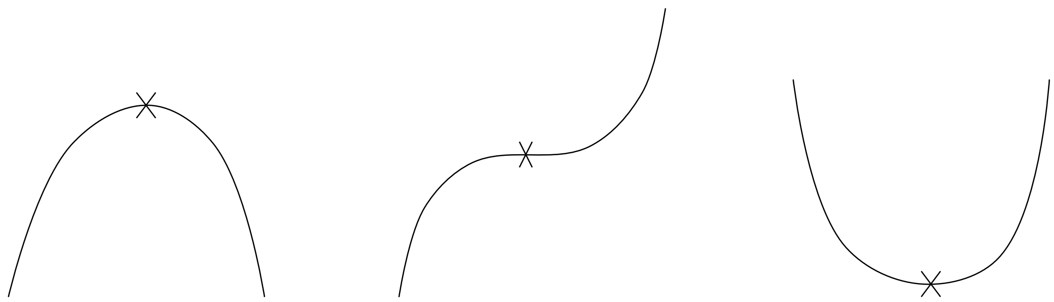
$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n\} - x^n}{h} \\
&= \lim_{h\to 0} \{nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^{n-1}\} \\
&= nx^{n-1}.
\end{aligned}$$

i.e.

$$f(x) = x^n, \quad \text{then} \quad f'(x) = nx^{n-1}. \tag{2.4}$$

We could now go on and find derivatives of "all" algebraic functions by definition. But it is too time consuming and impractical.

Important: If $f'(\alpha) = 0$, then the tangent to the curve f at $x = \alpha$ is parallel to the x-axis.



(a) *Local maximum.* (b) *Point of inflection.* (c) *Local minimum.*

Figure 2.6: *Different options for when $f'(x) = 0$.*

Often, f will have a local minimum or maximum at some $x = \alpha$.

2.1.2 Rules for differentiation

Some simple functions: x^a, a^x, $\sin x$, $\cos x$.

Complicated functions can be derived from these simple ones, by addition, multiplication and composition.

Example 2.6.

$$x + x^2, \quad xa^x, \quad x\sin x, \quad x^3 - x^4 = x^3 + (-x^4), \quad \frac{\sin x}{x} = \frac{1}{x} \cdot \sin x, \quad \cos(x^2).$$

So it is too time consuming to differentiate each individual function we can think of by first principle, i.e. using the definition.

Instead, we want to build a machine to help us differentiate various functions. The machine should contain three bits:

1. Sum rule,
2. Product rule,
3. Chain rule.

The idea of the machine is to tell us how to differentiate functions which are built from simpler pieces as long as we know how to differentiate the smaller pieces.

BONUS: you can still relate the derivative of the entire function to those of the smaller pieces, even if you don't know what the small pieces are.

<u>The sum rule:</u>

If f and g are differentiable, then

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right) = f'(x) + g'(x)$$

Example 2.7. Consider the function $f(x) = \left(x^3 + x^4\right)$, then using the above we have

$$\frac{d}{dx}\left(x^3 + x^4\right) = \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(x^4\right) = 3x^2 + 4x^3.$$

If you repeatedly apply the sum rule, you have

$$\frac{d}{dx}\left(f_1(x) + f_2(x) + \cdots + f_n(x)\right) = \frac{d}{dx}\left(f_1(x)\right) + \frac{d}{dx}\left(f_2(x)\right) + \cdots + \frac{d}{dx}\left(f_n(x)\right).$$

<u>The product rule:</u>

If f and g are differentiable, then

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x). \tag{2.5}$$

Example 2.8.

$$\begin{aligned}\frac{d}{dx}\left[(x^2+1)(x^3-1)\right] &= 2x(x^3-1) + (x^2+1)3x^2 \\ &= 2x^4 - 2x + 3x^4 + 3x^2 \\ &= 5x^4 + 3x^2 - 2x.\end{aligned}$$

Here we have put

$$f(x) = x^2 + 1 \quad \Longrightarrow \quad f'(x) = 2x,$$

and

$$g(x) = x^3 - 1 \quad \Longrightarrow \quad g'(x) = 3x^2.$$

Example 2.9. Consider the derivative of x^5, so

$$\begin{aligned} \frac{d}{dx}\left(x^5\right) &= \frac{d}{dx}\left(x^4 \cdot x\right) \\ &= 4x^3 \cdot x + x^4 \cdot 1 \\ &= 5x^4, \end{aligned}$$

as expected, since

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

Here we have put

$$f(x) = x^4 \quad \Longrightarrow \quad f'(x) = 4x^3,$$

and

$$g(x) = x \quad \Longrightarrow \quad g'(x) = 1.$$

This shows that differentiation can be approached in different ways, using what you feel most confident with.[2]

[2]End Lecture 6.