# Chapter 2

# Differentiation

## 2.1 Rates of change

Suppose we drive from London to Birmingham (100 miles). We plot a graph of the distance traveled against elapsed time. We want to measure how fast we traveled. The average speed of the trip is calculated as follows:

$$\frac{100 \text{ miles}}{2 \text{ hrs}} = 50 \text{ mph}.$$

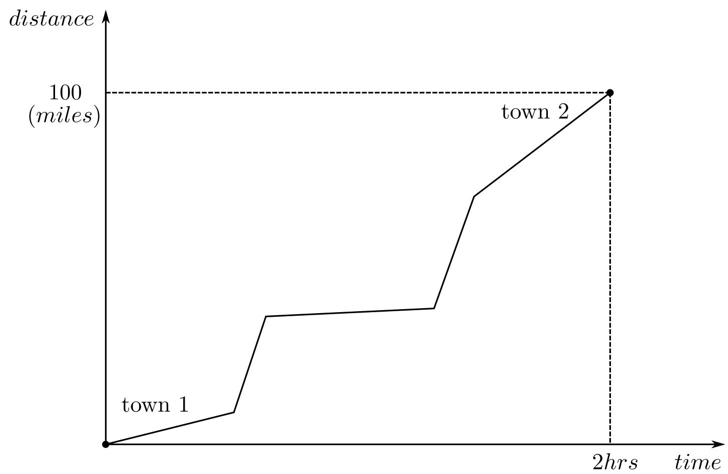


Figure 2.1: *Graph showing distance travelled against time, from town 1 to town 2.*

However, when travelling on the motorway you do not stick to one speed, sometimes you do more than 50 mph, sometimes much less. The reading on your speedometer is your *instantaneous* speed. This corresponds to the *gradient* (slope) "at a point" of the graph given in Fig. 2.1.

In the following, we want to find the slopes of curves.

### 2.1.1 What is the slope of a curve at a point?

The slope of a curve at a point is the slope of the *tangent* line at that point.

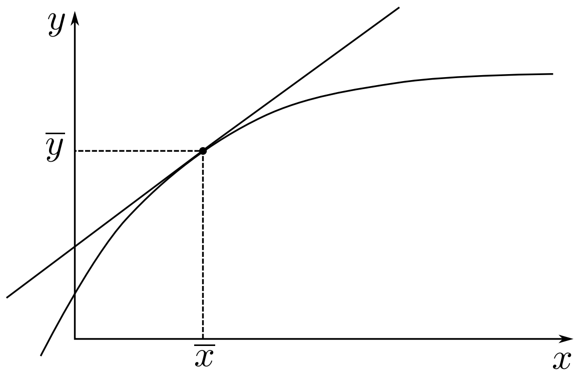


Figure 2.2: *Curve $y = f(x)$ with tangent line at $(\overline{x}, \overline{y})$.*

A straight line touching the edge of a curve (but not cutting across it) is called the tangent line at the touching point on the curve.

We hope that for mathematical curves, we find slopes "algebraically".

**Example 2.1.** Let us start with the example of the curve $y = S(x) = x^2$.

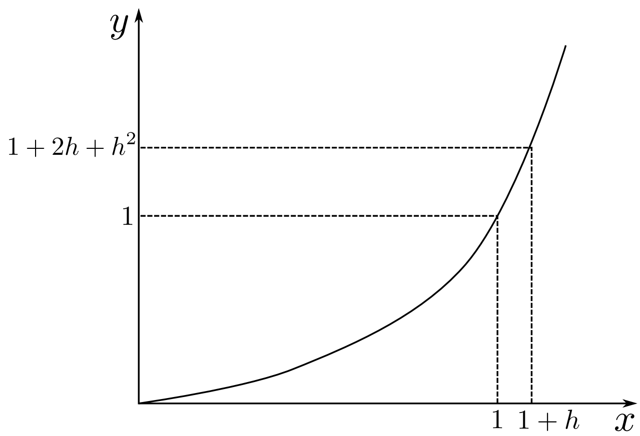


Figure 2.3: *Curve $y = S(x) = x^2$ displaying small increment about $x = 1$.*

Look at the point $(1, 1)$ on the curve. We want to understand how this function behaves when $x$ is close to 1. We therefore choose some $x$ close to 1, i.e. $x = 1 + h$ where $h \ll 1$. So we have

$$S(1 + h) = (1 + h)^2 = 1 + 2h + h^2 = S(1) + 2h + h^2.$$

This tells us two things:

1. Near 1, the value of $S(1)$ is also near 1, since if $h$ is small, then $2h+h^2$ is also small. This is the concept of continuity, which says:

$$f(x_0) \to f(x) \quad \text{as} \quad x_0 \to x,$$

for a continuous function $f$.

2. If you increase $x$ from 1 to $1+h$, then you increase the value of $S$ from $S(1)$ to $S(1)+2h+h^2$, which is an increase of approximately $2h$, since if $h$ is small then $h^2$ is very, very small. (e.g. $h=0.001$, $h^2=0.00001$).

From (2), we know that as you increase the value of $x$ from 1 to $1+h$, $S$ in creases by approximately twice as much as $h$. Therefore the rate of change at 1, which is defined as the derivative of the function $S(x)=x^2$, is $S'(1)=2$.

What about when $x=c$? As $x$ changes from $c$ to $c+h$, $S(x)=x^2$ changes from $c^2$ to $(c+h)^2=c^2+2ch+h^2=S(c)+2ch+h^2$. The change is roughly $2ch$. That is, $x$ has changed by $h$ and $x^2$ has changed by $2c$ times as much. So $S'(c)=2c$. We say the derivative of $S$ at $c$ is $2c$. In general,

$$S'(x)=2x.$$

So, we have the idea that the derivative of a function $f$ at $x=c$ is the coefficient of $h$ in the expansion of $f(c+h)$.

**Example 2.2.** Let us consider the function $q(x)=x^3$. At $x=c+h$ we have

$$q(c+h)=(c+h)^3=c^3+3c^2h+3ch^2+h^3.$$

Therefore, given what we have learnt, we can say

$$q'(c)=3c^2 \quad \text{i.e.} \quad q'(x)=3x^2.$$

PROBLEM: Suppose we want to examine the function $r(x)=1/x$. How do we expand $r(c+h)=1/(c+h)$?

Before we tackle this problem, let us turn back to Ex. 2.2. First let us write $q(c)=c^3$. Now, to get the change from $q(c+h)$ we have

$$q(c+h)-q(c)=3c^2h+3ch^2+h^3.$$

We now look at the ratio

$$\frac{\text{change in } q}{\text{change in } x}=\frac{q(c+h)-q(c)}{(c+h)-c}=\frac{3c^2h+3ch^2+h^3}{h}=3c^2+3ch+h^2.$$

We now examine what happens as $h$ approaches zero. In this case

$$\lim_{h\to 0}\left\{3c^2+3ch+h^2\right\}=3c^2.$$

Geometrically, we calculate the slope of the chord joining points on the curve (e.g. $q(c)$ to $q(c+h)$).

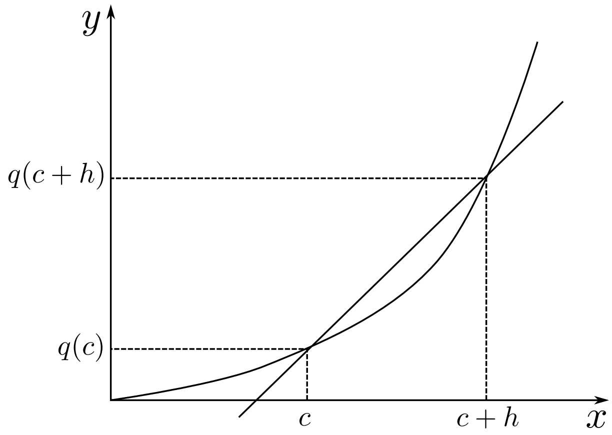


Figure 2.4: *Graph showing chord (line) joining the points* $(c, q(c))$ *and* $(c+h, q(c+h))$ *on the curve* $y = q(x)$.

The slope of the chord is

$$\frac{q(c+h) - q(c)}{h}, \tag{2.1}$$

and we watch what happens to the slope of the chord as $c + h$ gets closer and closer to $c$ (i.e. $h$ gets smaller and smaller) in the hope that the slope of the chord will approach the slope of the tangent line!

**Example 2.3.** Now let us consider the function $r(x) = 1/x$. In this case we have

$$r(c+h) - r(c) = \frac{1}{c+h} - \frac{1}{c}.$$

Now, let us consider the ratio

$$\frac{r(c+h) - r(c)}{h} = \frac{1}{h}\left(\frac{1}{c+h} - \frac{1}{c}\right) = \frac{1}{h}\left(\frac{-h}{c(c+h)}\right) = -\frac{1}{c(c+h)},$$

and as $h \to 0$, we have

$$r'(c) = -\frac{1}{c^2}, \quad \text{i.e.} \quad r'(x) = -\frac{1}{x^2}, \quad x \neq 0.$$

NOTE: $r(x) = 1/x$ is not well defined at $x = 0$ and in this case, nor is its derivative.[1]

[1]End Lecture 5.