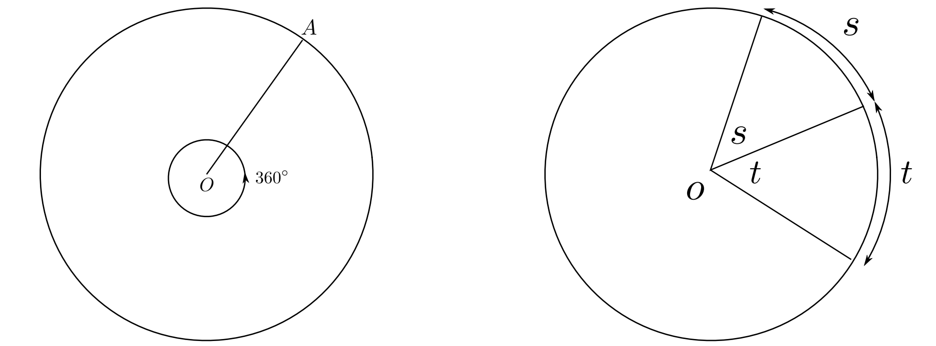
1.3.3 Trigonometric functions

The measure of an angle

Degrees: One full turn is 360°.

Radians: (Usually abbreviated as "rad" or the superscript " ^{c} " meaning "circular measure"). Suppose we have a disc of radius 1, we choose a couple of radii and want to measure the angle between them. To do so, we measure the length of the arc between the radii. Suppose the length is t. This is what we call the size of the angle, i.e. it is another measure for the length of the arc.



(a) 360° *is a full turn.* (b) *Addition of angles.*

Figure 1.8: *Some properties of angles.*

We can see immediately that a full turn is 2π rad because a circle of radius 1 has a circumference 2π. Therefore we have

$$\begin{aligned} 1 \text{ turn} &= 360^\circ = 2\pi \,\text{rad}, \\ \frac{1}{2} \text{ turn} &= 180^\circ = \pi \,\text{rad}. \end{aligned} \tag{1.11}$$

So,

$$\begin{aligned} 1\,\text{rad} &= \frac{180^\circ}{\pi}, \\ 1^\circ &= \frac{\pi}{180}\,\text{rad}. \end{aligned} \tag{1.12}$$

If the radius is not 1, then you need to take the ratio

$$\frac{\text{arc length}}{\text{radius}} = \text{angle (in radians)}. \tag{1.13}$$

Radians and degrees have one important property in common: if you follow one angle by another then the total angle is the sum e.g. $t + s$ (see Fig. 1.8(b)). This means that addition of angles has a geometric meaning.

Trignometric functions: cosine, sine & tangent

The values $\cos(\theta)$ and $\sin(\theta)$ (usually written $\cos\theta$, $\sin\theta$) are the horizontal and vertical coordinates of the point C (see Fig. 1.9).

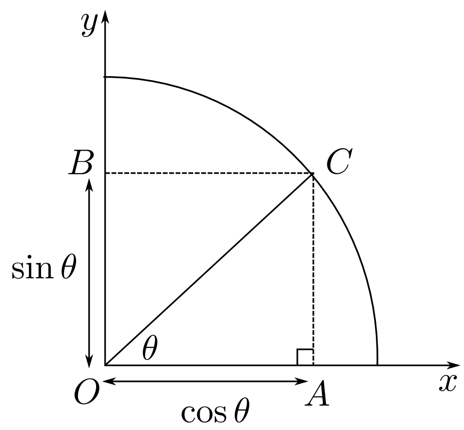


Figure 1.9: *Geometric definition of* cos *and* sin*, circle radius* $r = OC = 1$.

$$\cos\theta = \frac{OA}{OC} = OA, \tag{1.14a}$$
$$\sin\theta = \frac{AC}{OC} = AC, \tag{1.14b}$$
$$\tan\theta = \frac{AC}{OA} = \frac{\sin\theta}{\cos\theta}, \quad (\cos\theta \neq 0). \tag{1.14c}$$

A number of properties are clear:

1. $\cos^2\theta + \sin^2\theta = 1$

   $$\cos^2\theta + \sin^2\theta = \left(\frac{AC}{OC}\right)^2 + \left(\frac{BC}{OC}\right)^2 = \frac{AC^2 + BC^2}{OC^2} = \frac{OC^2}{OC^2} = 1.$$

   Note: $AC^2 + BC^2 = OC^2$ by Pythagoras' theorem.

2. cos and sin are periodic functions with period 2π, i.e. for any x, $\cos(x + 2\pi) = \cos x$, $\sin(x + 2\pi) = \sin x$.

   In general, if $f(x + T) = f(x)$ for all x, then we say that $f(x)$ is periodic function with period T.

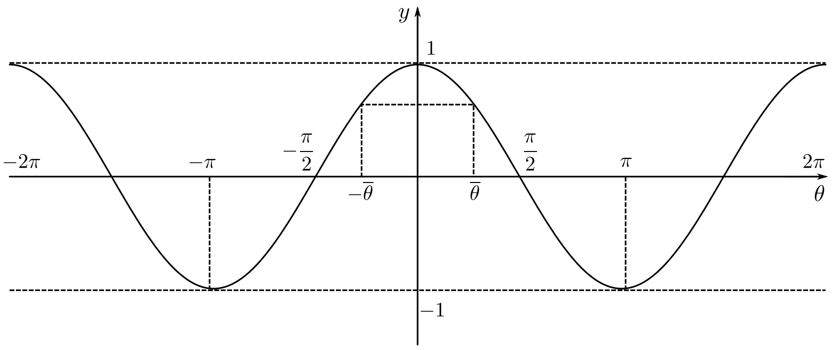


Figure 1.10: *Graph of* $\cos\theta$*.*

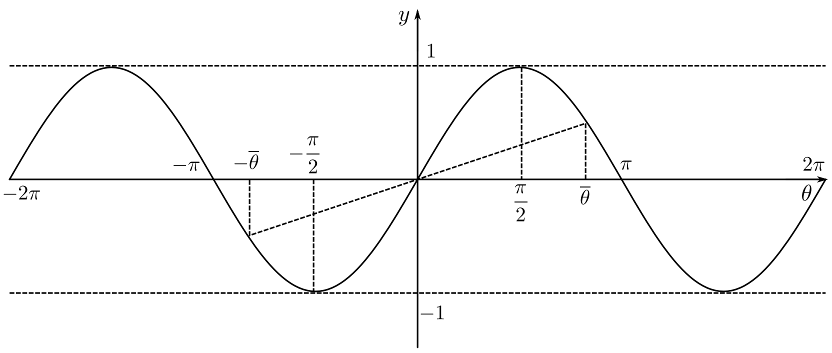


Figure 1.11: *Graph of* $\sin\theta$*.*

Note: $\cos : \mathbb{R} \to [-1,1]$ and $\sin : \mathbb{R} \to [-1,1]$.

3. cos is even, i.e. $\cos(-\theta) = \cos\theta$. sin is odd, i.e. $\sin(-\theta) = -\sin\theta$.

   In general, if $f(-x) = f(x)$ for all x, then we say f is even. If $f(-x) = -f(x)$ for all x, then we say f is odd.

4. Shift: cos and sin are the same shape but shifted by $\pi/2$, which means

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \quad \text{or} \quad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta,$$

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta, \quad \text{or} \quad \sin\left(\theta - \frac{\pi}{2}\right) = \cos\theta.$$

5. Addition formulae:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

6. Double-angle formulae:

$$\begin{aligned}\cos(2\theta) &= \cos(\theta+\theta)\\ &= \cos^2\theta - \sin^2\theta\\ &= \cos^2\theta - (1-\cos^2\theta)\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta.\end{aligned}$$

$$\begin{aligned}\sin 2\theta &= \sin(\theta+\theta)\\ &= 2\sin\theta\cos\theta.\end{aligned}$$

7. Half-angle formulae: Let $2\theta = \alpha$, then

$$\cos\alpha = 2\cos^2\left(\frac{\alpha}{2}\right) - 1 \implies \cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$$

$$\cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right) \implies \sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$$

Exercise 1.1. Try find $\cos(\theta-\phi)$ and $\sin(\theta-\phi)$ using properties (3) and (5).

We define tangent (tan) as follows:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cos\theta \neq 0.$$

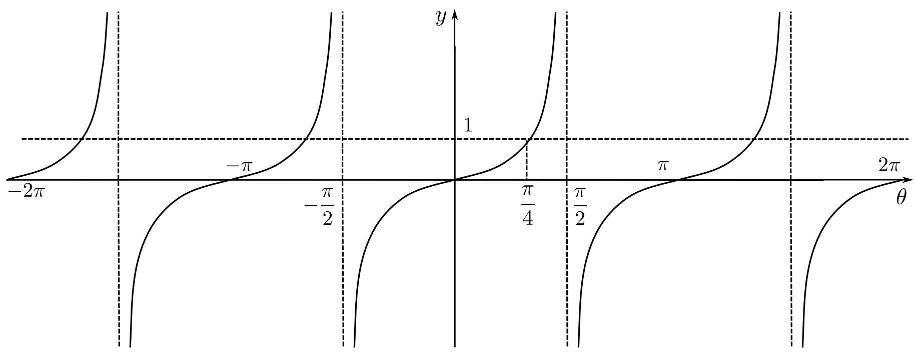


Figure 1.12: *Graph of* $\tan\theta$.

NOTE: $\tan : \mathbb{R} \to \mathbb{R}$. Also, as $\theta \to \pm\frac{\pi}{2}(2N-1)$, $\tan\theta \to \infty$ for $N \in \mathbb{N}$, this is the same as saying, as $\cos\theta \to 0$, $\tan\theta \to \infty$. We say the function has vertical *asymptotes* at these points. Also note that $\tan\theta$ has a periodicity of π!

The double angle formula for tan can be calculated using the definition as follows:

$$\begin{aligned}\tan(\theta+\phi) &= \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)}\\ &= \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi},\end{aligned}$$

dividing by $\cos\theta\cos\phi$ we get

$$\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}.$$

Also, if we divide property (1) on page 14 through by $\cos^2 x$, we have

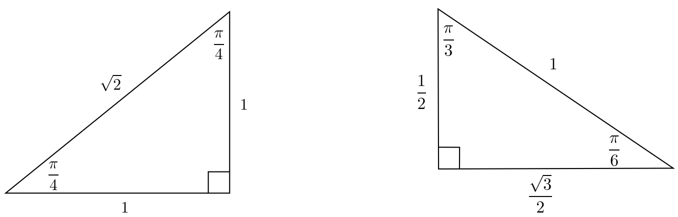
$$1+\frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x},$$

which can be written as

$$1+\tan^2 x = \sec^2 x.$$

The secant, cosecant and cotangent functions are defined as

$$\sec x = \frac{1}{\cos x}, \quad \mathrm{cosec} x = \frac{1}{\sin x}, \quad \cot x = \frac{1}{\tan x}.$$



(a) $\pi/4$ *triangle.* (b) $\pi/3$, $\pi/6$ *triangle.*

Figure 1.13: *Some well known results for particular angles can be derived by the above triangles for* sin, cos *and* tan.

Summary:

1 radian $= 180/\pi$, and 1 degree $= \pi/180$.

The ratio of arc length to radius gives the angle (in radians)

$\cos\theta$ and $\sin\theta$ have a periodicity of 2π, whilst $\tan\theta$ has a periodicity of π (convince yourself why!).

The range of $\cos\theta$ and $\sin\theta$ is $[-1,1]$, whilst the range of $\tan\theta$ is $(-\infty,\infty)$.

$\sin\theta$ and $\cos\theta$ can be obtained by shifting one another by $\pi/2$.

Take note of the various angle formulae![5]

[5]End Lecture 4.