1.3.2 Polynomials

A polynomial is a function P with a general form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{1.4}$$

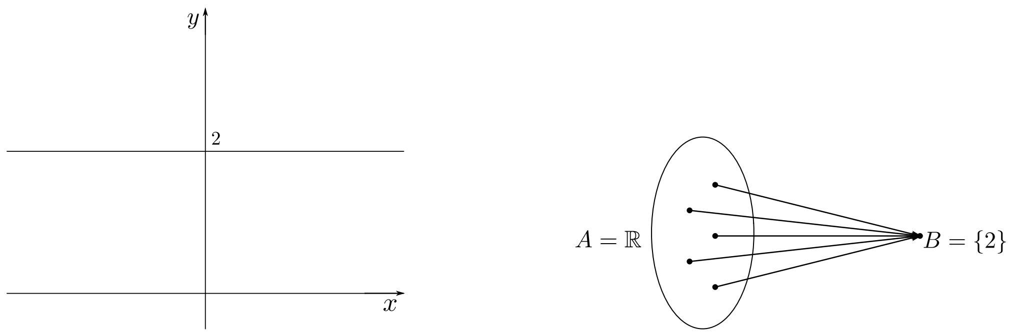
where the coefficients a_i $(i = 0, 1, \ldots, n)$ are numbers and n is a non-negative whole number. The highest power whose coefficient is not zero is called the *degree* of the polynomial P.

Example 1.5.

$P(x)$	2	$3x^2 + 4x + 2$	$\frac{1}{1+x}$	$\sqrt{x}$	$1 - 3x + \pi x^3$	$2t + 4$
Polynomial?	Yes	Yes	No	No	Yes	Yes
Order	0	2	N/A	N/A	3	1

Importance of polynomials: analytical & computational points of view

Degree 0: $P(x) = a_0 = a_0 x^0$, say $P(x) = 2$. This polynomial is simply a constant.



(a) $y = P(x) = 2$.

(b) *Entire domain mapped to one point.*

Figure 1.5: *A straight line parallel to the x-axis.*

Degre 1: $P(x) = ax + b$, $a \neq 0$. These are called *linear*, since the graph of $y = ax + b$ is a straight line. The linear equation $ax + b = 0$ has solution $x = -b/a$.

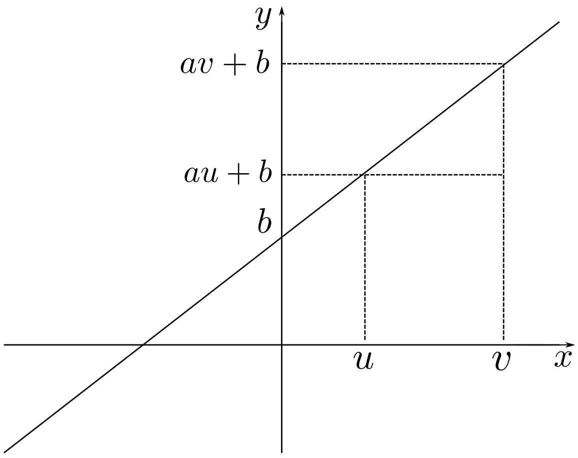


Figure 1.6: *Linear graph given by $y = ax + b$.*

The *slope* or *gradient* of $y = ax + b$ is a. It can be worked out as follows:

$$\begin{aligned} slope &= \frac{\text{change in height}}{\text{change in distance}} \\ &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{(av+b)-(au+b)}{v-u} \\ &= \frac{a(v-u)}{v-u} = a. \end{aligned} \tag{1.5}$$

Degree 2: $P(x) = ax^2 + bx + c$, $a \neq 0$. This is known as a quadratic polynomial. The quadratic equation $ax^2 + bx + c = 0$ has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.6}$$

Proof. Start by re-arranging the equation and dividing through by a so that

$$x^2 + \frac{b}{a}x = -\frac{c}{a},$$

next we add $b^2/(2a)^2$ to both sides, hence

$$x^2 + \frac{b}{a}x + \frac{b^2}{(2a)^2} = -\frac{c}{a} + \frac{b^2}{(2a)^2}.$$

Now we can search for a common denominator on the right hand side (RHS) and complete the square or factorise on the left hand side (LHS), i.e.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{2^2ac}{(2a)^2} + \frac{b^2}{(2a)^2}.$$

Taking the square root of both sides we have

$$x + \frac{b}{2a} = \pm\sqrt{-\frac{2^2ac}{(2a)^2} + \frac{b^2}{(2a)^2}} = \frac{\pm\sqrt{-2^2ac + b^2}}{2a},$$

and finally re-arranging the above equation (or minus $b/2a$ from both sides of the equation) we get[3]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$\square$

Example 1.6. Consider the quadratic polynomial

$$P(x) = x^2 - 3x + 2. \tag{1.7}$$

We can represent $P(x)$ in a different way by factorising it, i.e.

$$P(x) = (x - 2)(x - 1). \tag{1.8}$$

Again, we can represent $P(x)$ in a different way, this time by completing the square. When completing the square, we do this based on the value of b as follows. We add and subtract $\left(\frac{b}{2}\right)^2$ to $P(x)$ such that $P(x) = x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2$ (i.e. we don't really change the equation). Now it is easy to see $x^2 - 3x + \left(\frac{3}{2}\right)^2$ is the same as $\left(x - \frac{3}{2}\right)^2$ i.e. it can be factorised. Thus we can finally write

$$P(x) = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}. \tag{1.9}$$

Now, (1.7), (1.8) and (1.9) are all equivalent and they are each able to give an insight on what the graph of the quadratic function $P(x)$ looks like. That is

(i) Equation (1.7) tells us the graph of $P(x)$ is a "cup" rather than a "cap" since the coefficient of x^2 is positive. Also we can easily see $P(0) = 2$.

(ii) Equation (1.8) tells us $P(x) = 0$ at $x = 1$ and $x = 2$

(iii) Equation (1.9) tells us $P(x)$ is minimal at $x = \frac{3}{2}$ and $P\left(\frac{3}{2}\right) = -\frac{1}{4}$. Note,

$$\left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4},$$

since anything squared is always positive!

Now that we have some suitable information regarding $P(x)$, we are able to produce an informed sketch:

[3] Aside: $\square \equiv Q.E.D$, where $Q.E.D =$ *"quod erat demonstratum"* which means *"which was to be demonstrated"* in Latin.

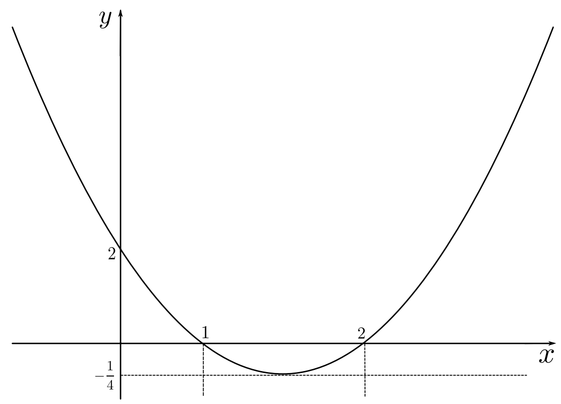


Figure 1.7: *Quadratic graph given by* $y = x^2 - 3x + 2$.

Degree ≥ 3: Things get a bit more complicated!

In general, we have the algebraic equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0, \tag{1.10}$$

which has n roots, including real and complex (imaginary numbers, $z = \alpha + i\beta$) roots.

$n = 2$	we have formulae for roots	(quadratics)
$n = 3$	we have formulae for roots	(cubic)
$n = 4$	we have formulae for roots	(quartics)
$n > 4$	No general formulae exist	(proven by Évariste Galois)

But in any case, we may try factorisation to find the roots. If you factorise a polynomial, say

$$P(x) = (x-1)(x+3)(x+4),$$

then you can easily solve $P(x) = 0$, in this case $x_1 = 1$, $x_2 = -3$, $x_3 = -4$.

NOTE: this depends on the property of 0 on the RHS. You can't easily solve

$$(x-1)(x+3)(x+4) = 1.$$

Conversely, if you know that $P(\alpha) = 0$, then you may factorise $P(x)$ as

$$P(x) = (x - \alpha) q(x),$$

where $q(x)$ is some polynomial of one degree less than that of $P(x)$.

Example 1.7. Consider $P(x) = x^3 - 8x^2 + 19x - 12$. We know that $x = 1$ is a solution to $P(x) = 0$, then it can be shown that

$$P(x) = (x-1)q(x) = (x-1)(x^2 - 7x + 12).$$

Here $P(x)$ is a cubic and thus $q(x)$ is a quadratic.

Example 1.8. Consider $P(x) = x^3 - x^2 - 3x - 1$. By observation, we know

$$P(-1) = (-1)^3 - (-1)^2 - 3(-1) - 1 = 0.$$

So $x_1 = -1$ is a root. Let us write

$$P(x) = (x+1)(x^2 + ax + b),$$

then multiplying the brackets we have

$$P(x) = x^3 + (1+a)x^2 + (a+b)x + b,$$

which should be equivalent to $x^3 - x^2 - 3x - 1$. Thus, comparing the corresponding coefficients we have

$$\begin{aligned} 1+a &= -1, \\ a+b &= -3, \\ b &= -1. \end{aligned}$$

This set of simultaneous equations has the solution

$$a = -2, \quad b = -1.$$

So we can write

$$P(x) = (x+1)(x^2 - 2x - 1)$$

To find the other two solutions of $P(x) = 0$, we must set $(x^2 - 2x - 1) = 0$ which has solution $x_{2,3} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$, together with $x_1 = -1$ we have a complete set of solutions for $P(x) = 0$.

Example 1.9. Consider $P(x) = x^3 + 3x^2 - 2x - 2$. An obvious solution to $P(x) = 0$ is $x_1 = 1$. So we put

$$\begin{aligned} P(x) &= (x-1)(x^2 + ax + b) \\ &= x^3 + (a-1)x^2 + (b-a)x - b. \end{aligned}$$

Comparing corresponding coefficients with our original form of $P(x)$ we gain the following simultaneous equations:

$$\begin{aligned} a-1 &= 3, \\ b-a &= -2, \\ -b &= -2. \end{aligned}$$

These have solution $b = 2$, $a = 4$ and so we have

$$x^3 + 3x^2 - 2x - 2 = (x-1)(x^2 + 4x + 2).$$

The solutions to $(x^2 + 4x + 2) = 0$ are $x_{2,3} = -2 \pm \sqrt{2}$, completing the set solutions to $P(x) = 0$.

NOTE: In the above examples, the leading coefficient of $P(x)$ i.e. the coefficient of x^3 is equal to one!

As with many areas of mathematics, there are many ways to tackle a problem. Another way to find $q(x)$ given you know some factor of $P(x)$, is called *polynomial devision*.

Example 1.10. Consider $P(x) = x^3 - x^2 - 3x - 1$, we know $P(-1) = 0$. The idea is that we "divide" $P(x)$ by the factor $(x + 1)$, like so:

$$
\begin{array}{r|rrrr}
 & & x^2 & -2x & -1 \\
\hline
x+1 & x^3 & -x^2 & -3x & -1 \\
 & -x^3 & -x^2 & & \\
\hline
 & & -2x^2 & -3x & \\
 & & 2x^2 & +2x & \\
\hline
 & & & -x & -1 \\
 & & & x & +1 \\
\hline
 & & & & 0
\end{array}
$$

Hence, multiplying the *quotient* by the *divisor* we have $(x+1)(x^2-2x-1) = x^3-x^2-3x-1$.

Summary:

The highest power of the polynomial is known as the order of the polynomial.

The roots of a quadratic equation $ax^2 + bx + c$, can be found using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

No general formula for $n > 4$. However roots can be found of higher order polynomials by factorising first. This can be done by observation, the method of comparing coefficients or by long polynomial devision.

Later we will see that polynomials are functions which are easy to differentiate and integrate.

We can approximate most "nice" functions by polynomials, at least locally i.e. using power series.[4]

[4] End Lecture 3.