5.3 Solving initial-value problems numerically: Euler's method

Most differential equations can not be solved analytically, so we try to solve them numerically.

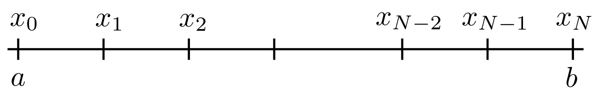
Suppose we have an initial-value problem:

$$\frac{dy}{dx} = f(x, y), \quad y(a) = y_0.$$

We want to find the solution $y(x)$ numerically on the interval $[a, b]$.

First we divide $[a, b]$ into N subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_k < \cdots < x_N = b.$$



If the subintervals have equal length, say h, then

$$x_k = a + kh, \quad h = \frac{b-a}{N} \quad \text{(step size)}, \quad k = 0, 1, \ldots, N.$$

Assume that $y(x)$ is the solution we want, then

$$\frac{d}{dx} y(x) = f(x, y(x)), \quad y(a) = y_0.$$

Integrating both sides on the subinterval $[x_k, x_{k+1}]$, we have

$$\int_{x_k}^{x_{k+1}} y'(x)\, dx = \int_{x_k}^{x_{k+1}} f(x, y(x))\, dx.$$

Considering the LHS, we have

$$\text{LHS} = y(x)|_{x_k}^{x_{k+1}} = y(x_{k+1}) - y(x_k),$$

thus

$$y(x_{k+1}) - y(x_k) = \int_{x_k}^{x_{k+1}} f(x, y(x))\, dx.$$

Let $g(x) = f(x, y(x))$, then

$$\text{RHS} = \int_{x_k}^{x_{k+1}} g(x)\, dx,$$

representing the area under the curve $y = g(x)$, between x_k and x_{k+1}.

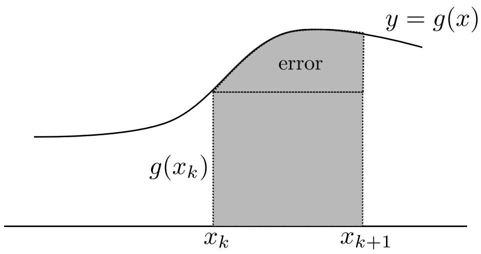


Figure 5.5: *The integral which represents the area under the curve $y = g(x)$ is approximated using rectangles. Error depends on the width of the rectangles, i.e. the number of sub-intervals.*

If we use the area of the rectangle

$$(x_{k+1} - x_k)g(x_k) = hg(x_k) = hf(x_k, y(x_k)),$$

to approximate the area, we have

$$y(x_{k+1}) - y(x_k) \approx hf(x_k, y(x_k)).$$

$k = 0$: $y(x_1) - y_0 \approx h(f(x_0, y_0) \implies y(x_1) \approx y_0 + hf(x_0, y_1) \triangleq y_1$, an approximation to $y(x_1)$.

$k = 1$: $y(x_2) - y(x_1) \approx h(f(x_1, y(x_1)) \implies y(x_2) \approx y(x_1) + hf(x_1, y(x_1)) \approx y_1 + hf(x_1, y_1) \triangleq y_2$, an approximation to $y(x_2)$.

$\vdots$

In general, we have

$$y_{k+1} = y_k + hf(x_k, y_k), \quad k = 0, 1, \ldots N - 1,$$

where y_k is an approximation to $y(x_k)$. This is a *difference equation* and we can solve it iteratively. This method is based on the above formula, and is called *Euler's method*.

Example 5.25. Estimate $y(1)$, where $y(x)$ satisfies the initial-value problem:

$$\frac{dy}{dx} = y, \quad y(0) = 1.$$

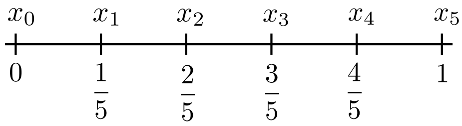
We know the exact solution is

$$y(x) = e^x, \quad \implies \quad y(1) = e \approx 2.71828.$$

Now we apply Euler's method to the problem. We have

$$f(x, y) = y.$$

First, we take $N = 5$, then $h = (1 - 0)/5 = 0.2$.



$$
\begin{aligned}
y_0 &= y(0) = 1 \\
y_1 &= y_0 + hf(x_0, y_0) = 1 + 0.2 \times 1 = 1.2 \\
y_2 &= y_1 + hf(x_1, y_1) = 1.2 + 0.2 \times 1.2 = (1.2)^2 \\
y_3 &= y_2 + hf(x_2, y_2) = y_2 + hy_2 = y_2(1+h) = (1.2)^2 \times 1.2 = (1.2)^3 \\
y_4 &= (1.2)^4 \\
y_5 &= (1.2)^5 \approx 2.48832.
\end{aligned}
$$

For $N = 5$, we have

$$\text{error} = e - y_5 = 2.71828 - 2.48832 = 0.22996.$$

Now, we double the number of subintervals: $N = 10$, $h = 0.1$ then we need 10 steps to reach $x_{10} = 1$.

$$y_{10} = (1.1)^{10} \approx 2.59374,$$

then we have

$$\text{error} = 2.71828 - 2.59374 = 0.12454.$$

For $N = 20$, $h = 0.05$ and so

$$y_{20} = (1.05)^{20} \approx 2.65330, \quad \text{error} = 0.0650.$$

For $N = 40$, $h = 0.025$ and so

$$y_{20} = (1.025)^{40} \approx 2.68506, \quad \text{error} = 0.0332.$$

Euler's method is first order, i.e. the error behaves like $O(h)$.

In general, if $h = 1/N$, then

$$
\begin{aligned}
y_1 &= y_0 + hy_0 = (1+h)y_0 = 1 + h = 1 + \frac{1}{N} \\
y_2 &= y_1 + hy_1 = (1+h)y_1 = \left(1 + \frac{1}{N}\right)^2 \\
&\vdots \\
y_N &= \left(1 + \frac{1}{N}\right)^N.
\end{aligned}
$$

Thus

$$y(1) \approx \left(1 + \frac{1}{N}\right)^N.$$

Actually,

$$\lim_{N \to \infty} \left\{ \left(1 + \frac{1}{N}\right)^N \right\} = e.$$

The source of the errors when approximating the function come from

1. discretisation error,
2. round-off error.[8]

[8] End Lecture 28. End of course.