Example 5.24. Find the general solution to the equation

$$y'' + 2y' + y = 2e^{-x}.$$

The general solution takes the form $y(x) = f(x) + g(x)$.

Find the C.F.: The auxiliary equation for the above differential equation is

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Longleftrightarrow \quad (\lambda + 1)^2 = 0 \quad \Longrightarrow \quad \lambda_1 = -1,$$

i.e. we have a repeated root so

$$g(x) = C_1 e^{-x} + C_2 x e^{-x}.$$

Here e^{-x} and xe^{-x} are two independent solutions to the homogeneous equation

$$y'' + 2y' + y = 0.$$

Find a P.I.: We have to try

$$f = ax^2 e^{-x},$$

since e^{-x} and xe^{-x} can't be the solution to the original differential equation as they satisfy the homogeneous equation. So we work out the derivatives

$$f' = 2axe^{-x} - ax^2 e^{-x},$$

$$f'' = 2ae^{-x} - 2axe^{-x} - 2axe^{-x} + ax^2 e^{-x} = axe^{-x} - 4axe^{-x} + 2ae^{-x}.$$

Substituting $y = f(x)$ into the differential equation, we have

$$\begin{aligned} f'' + 2f' + f &= ax^2 e^{-x} - 4axe^{-x} + 2ae^{-x} + 4axe^{-x} - 2ax^2 e^{-x} + ax^2 e^{-x} \\ &= 2ae^{-x} \\ &\equiv 2e^{-x}, \end{aligned}$$

therefore we have $a = 1$. So finally, we have the general solution

$$y(x) = (C_1 + C_2 x + x^2) e^{-x}.$$

5.2 Simple Harmonic Motion (SHM)

SHM is essentially standard trigonometric oscillation at a single frequency, for example a pendulum.

An ideal pendulum consists of a weightless rod of length l attached at one end to a frictionless hinge and supporting a body of mass m at the other end. We describe the motion in terms of angle θ, made by the rod and the vertical.

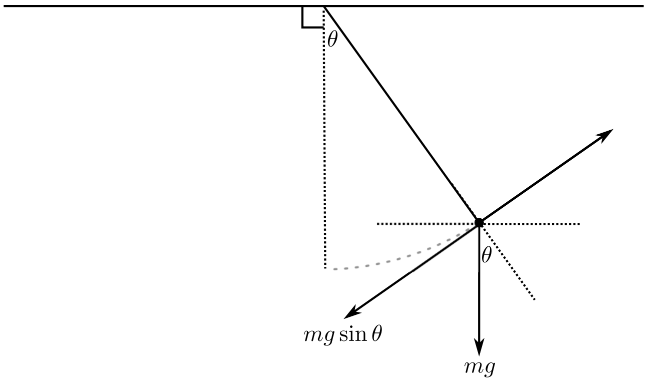

Figure 5.2: *Sketch of a pendulum of length l with a mass m, displaying the forces acting on the mass resolved in the tangential direction relative to the motion.*

Using Newton's second law of motion $F = ma$, we have the differential equation

$$-mg\sin\theta = ml\ddot{\theta},$$

which describes the motion of the mass m, where the RHS is the tangential acceleration and the LHS is the tangential component of gravitation force.

NOTATION: $\dot{\theta} = d\theta/dt$ and $\ddot{\theta} = d^2\theta/dt^2$.

We re-write the equation as

$$\ddot{\theta} + \omega^2\sin\theta = 0, \quad \omega^2 = \frac{g}{l}.$$

This is a nonlinear equation, and we can not solve it analytically.

Approximation: if θ is small, then $\sin\theta \approx \theta$, and in this situation we have an approximate equation given by

$$\ddot{\theta} + \omega^2\theta = 0.$$

We solve the equation for $\theta(t)$. Here we have $r = 0$, $s = \omega^2$ and $\Delta = -4\omega^2 < 0$. The auxiliary equation is

$$\lambda^2 + \omega^2 = 0 \quad \Longleftrightarrow \quad \lambda^2 = -\omega^2 \quad \Longrightarrow \quad \lambda_1 = i\omega, \quad \lambda_2 = -i\omega,$$

where $i = \sqrt{-1}$. So

$$\alpha = -\frac{r}{2} = 0 \quad \text{and} \quad \beta = \frac{1}{2}\sqrt{-1} = \omega,$$

therefore we have $e^{\alpha t}\cos\beta t = \cos\omega t$ and $e^{\alpha t}\sin\beta t = \sin\omega t$. Hence, the general solution is

$$\theta(t) = A\cos\omega t + B\sin\omega t.$$

Differentiating we have

$$\begin{aligned}
\dot{\theta}(t) &= -A\omega\sin\omega t + B\omega\cos\omega t \\
\ddot{\theta}(t) &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t.
\end{aligned}$$

We can verify $\theta(t)$ satisfies the original differential equation as

$$\ddot{\theta}(t) = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t = -\omega^2 \underbrace{(A\cos\omega t + B\sin\omega t.)}_{=\theta(t)} = -\omega^2\theta(t).$$

The solution $\theta(t)$ can be written as

$$\begin{aligned}
\theta(t) &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\omega t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega t\right) \\
&= \sqrt{A^2+B^2}(\sin\phi\cos\omega t + \cos\phi\sin\omega t) \\
&= R\sin(\omega t + \phi).
\end{aligned}$$

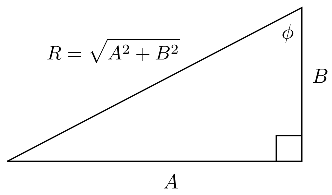

Figure 5.3: *Using Pythagarus' theorem to write the constants A and B in terms of the phase angle ϕ.*

R - amplitude of the motion.

ϕ - phase angle, i.e. the amount of shift.

$\omega = \sqrt{g/l}$ - the natural frequency, i.e. the number of complete oscillations per unit time.

$T = 2\pi/\omega$ - period, the time taken for a complete cycle (two complete swings). T depends on the length of the pendulum, but doesn't depend on the mass and initial conditions.

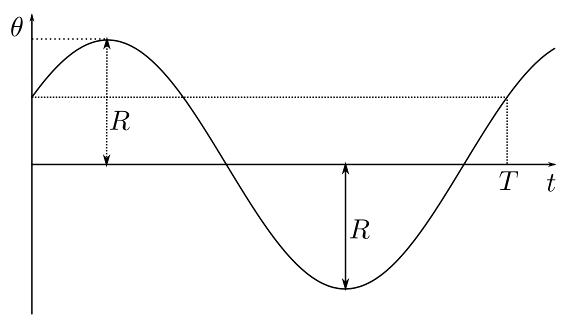

Figure 5.4: *Graph showing the change in θ over time t, displaying oscillations of period T.*

If the pendulum is initially at rest, i.e. $\theta(0) = 0$, $\dot{\theta}(0) = 0$, then

$$\theta(0) = A = 0, \quad \dot{\theta}(0) = B\omega = 0 \quad \Longrightarrow \quad B = 0 \quad \Longrightarrow \quad \theta(t) = 0,$$

i.e. the pendulum will remain at test for all time t.

If the pendulum is displaced by an angle θ_0 and released, then $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, so

$$\theta(0) = A = \theta_0, \quad \dot{\theta}(0) = B\omega = 0 \quad \Longrightarrow \quad B = 0,$$

therefore

$$\theta(t) = \theta_0 \cos \omega t \quad \Longrightarrow \quad |\theta(t)| \leq \theta_0.$$

That is, if the displaced angle θ_0 at initial time is small, then the small angle approximation makes sense.

The solution tells us the oscillation, once started, goes forever. But in reality, the frictional force and air resistance would eventually bring the pendulum to a rest.[7]

[7]End Lecture 27.