Finding a particular solution to $y'' + ry' + sy = p(x)$

It depends on the function $p(x)$. We only consider three kinds of $p(x)$:

1. Polynomials,
2. trigonometric functions,
3. the exponential functions:
   i. $e^{\mu x}$, μ is not a root of of the homogeneous equation $y'' + ry' + sy = 0$,
   ii. $e^{\mu x}$, μ is a root of the homogenous equation.

Example 5.20. Find the general solution to the differential equation

$$y'' + 2y' + y = x^2.$$

Recall, the general solution takes the form $y = f(x) + g(x)$. First we find the general solution to the homogeneous equation

$$y'' + 2y' + y = 0,$$

i.e. we seek the complimentary function (C.F.) $y = g(x)$. The auxiliary equation is

$$\lambda^2 + 2\lambda + 1 \quad \Longleftrightarrow \quad (\lambda + 1)^2 = 0 \quad \Longrightarrow \quad \lambda_1 = -1,$$

i.e. it has only one root. So the C.F. is

$$g = C_1 e^{-x} + C_2 x e^{-x} = (C_1 + C_2 x)e^{-x}.$$

Second, we find the particular integral (P.I.), we try

$$f = ax^2 + bx + x, \quad \text{since } p(x) = x^2.$$

so we have

$$f' = 2ax + b, \quad f'' = 2a.$$

Substituting $y = f(x)$ into the differential equation gives

$$\begin{aligned} f'' + 2f' + f &= 2a + 2(2ax + b) + ax^2 + bx + c \\ &= ax^2 + (4a + b)x + 2a + 2b + c \\ &\equiv x^2. \end{aligned}$$

Comparing coefficients between the LHS and the RHS we have

$$\left.\begin{array}{l} a = 1 \\ 4a + b = 0 \\ 2a + 2b + c = 0 \end{array}\right\} \quad \Longrightarrow \quad \left.\begin{array}{l} a = 1 \\ b = -4 \\ c = 6 \end{array}\right\} \quad \Longrightarrow \quad f(x) = x^2 - 4x + 6,$$

Finally, we can write the general solution as

$$y(x) = x^2 - 4x + 6 + (C_1 + C_2 x)e^{-x}.$$

Example 5.21. Solve the following initial-value problem:

$$y'' - 2y' + y = \sin x, \quad y(0) = -2, \quad y'(0) = 2.$$

Note, we have two conditions here because second order differential equations have two constants of integration to be found. The auxiliary equation for this problem is

$$\lambda^2 - 2\lambda + 1 = 0 \quad \iff \quad (\lambda - 1)^2 = 0 \implies \lambda = 1,$$

i.e. we have a repeated root and hence, the C.F. is

$$g(x) = (C_1 + C_2 x)e^x.$$

To find the P.I. we try

$$f = a\sin x + b\cos x,$$

since $p(x) = \sin x$. Then we have

$$f' = a\cos x - b\sin x, \quad f'' = -a\sin x - b\cos x.$$

Substituting into the differential equation we have

$$\begin{aligned} f'' - 2f' + f &= -a\sin x - b\cos x - 2a\cos x + 2b\sin x + a\sin x + b\cos x \\ &= (-a + 2b + a)\sin x + (-b - 2a + b)\cos x \\ &= 2b\sin x - 2a\cos x \\ &\equiv \sin x. \end{aligned}$$

Comparing coefficients, we have

$$a = 0, \quad b = \frac{1}{2} \quad \implies \quad f = \frac{1}{2}\cos x.$$

Therefore the general solution to the initial-value problem is

$$y(x) = \frac{1}{2}\cos x + (C_1 + C_2 x)e^x.$$

In order to find the unknown constants C_1 and C_2 using the initial conditions, we need to find $y'(x)$, so we differentiate the above to give

$$y'(x) = -\frac{1}{2}\sin x + C_2 e^x + (C_1 + C_2 x)e^x = -\frac{1}{2}\sin x + (C_1 + C_2 + C_2 x)e^x.$$

Put $x = 0$, so

$$y(0) = \frac{1}{2}\cos 0 + e^0(C_1 + C_2 \cdot 0) = \frac{1}{2} + C_1 = -2,$$

$$y'(0) = -\frac{1}{2}\sin 0 + e^0(C_1 + C_2 + C_2 \cdot 0) = C_1 + C_2 = 2,$$

thus, we have the constants

$$C_1 = -\frac{5}{2}, \quad C_2 = 2 - C_1 = 2 + 52 = \frac{9}{2}.$$

Finally, the solution to the initial value problem is

$$y(x) = \frac{1}{2}\cos x + \frac{1}{2}e^x(9x - 5).$$

Example 5.22. Find the general solution of the following differential equation,

$$y'' + 4y' + 3y = 5e^{4x}.$$

The auxiliary equation is

$$\lambda^2 + 4\lambda + 3 = 0 \quad \Longleftrightarrow \quad (\lambda + 1)(\lambda + 3) = 0.$$

Hence, this has two distinct real roots, namely $\lambda_1 = -1$, $\lambda_2 = -3$. So the C.F. (from the homogeneous equation) is given by

$$g(x) = C_1 e^{-x} + C_2 e^{-3x}.$$

To find the P.I. we try

$$f = ae^{4x},$$

since $p(x) = 5e^4 x$. Differentiating, we have

$$f' = 4a^{4x}, \quad f'' = 16a^{4x}.$$

Substituting into the differential equation we see that

$$f'' + 4f' + 3f = 16ae^{4x} + 16ae^{4x} + 3ae^{4x} = 35ae^{4x} \equiv 5e^{4x}.$$

Therefore we have $a = 1/7$ and so the P.I. is

$$f = \frac{1}{7}a^{4x}.$$

Finally, the general solution is

$$y(x) = f(x) + g(x) = \frac{1}{7}e^{4x} + C_1 e^{-x} + C_2 e^{-3x}.$$

Example 5.23. Solve the initial-value problem given by

$$y'' + 4y' + 3y = e^{-x}, \quad y(0) = 0, \quad y'(0) = 0.$$

We know from example 5.22 that the C.F. to the homogeneous equation $y'' + 4y' + 3y = 0$ is

$$g = C_1 e^{-x} + C_2 e^{-3x},$$

i.e. $\lambda_1 = -1$ and $\lambda_2 = -3$.

Now let us find the P.I., if we try $f = ae^{-x}$, we know that it wouldn't work since ae^{-x} is actually a solution to the homogeneous equation $y'' + 4y' + 3y = 0$. Therefore, we try

$$f = axe^{-x},$$

thus

$$f' = ae^{-x} - axe^{-x}, \quad \text{and} \quad f'' = -2ae^{-x} + axe^{-x}.$$

Substituting into the differential equation we have

$$\begin{aligned} f'' + 4f' + 3f &= -2ae^{-x} + axe^{-x} + 4ae^{-x} - 4axe^{-x} + 3axe^{-x} \\ &= 2ae^{-x} \\ &\equiv e^{-x}. \end{aligned}$$

Therefore we must have $a = 1/2$ and so the general solution to the differential equations is

$$y(x) = \frac{1}{2}xe^{-x} + C_1e^{-x} + C_2e^{-3x}.$$

Differentiating the general solution we have

$$y'(x) = \frac{1}{2}e^{-x} - \frac{1}{2}xe^{-x} - C_1e^{-x} - 3C_2e^{-3x}.$$

Putting $x = 0$, we have (from the initial conditions)

$$\left.\begin{array}{l} y(0) = C_1 + C_2 = 0 \\ y'(0) = \frac{1}{2} - C_1 - 3C_2 = 0 \end{array}\right\} \quad \Longrightarrow \quad \begin{array}{l} C_1 = -\frac{1}{4} \\ C_2 = \frac{1}{4} \end{array},$$

thus

$$y(x) = \frac{1}{2}xe^{-x} - \frac{1}{4}e^{-x} + \frac{1}{4}e^{-3x},$$

is the solution to the initial-value problem.[6]

[6] End Lecture 26.