If $y_1(x)$ and $y_2(x)$ are independent and they are the solutions to (5.10) then the general solution to (5.10) can be written in the form

$$y(x) = C_1 y_1(x) + C_2 y_2(x), \tag{5.11}$$

where C_1 and C_2 are constants. In other words, if you have two independent solutions, then you can represent any other solution in terms of these two solutions.

Now, we verify that (5.11) is the solution to (5.10):

$$\begin{aligned} y'' + ry' + sy &= C_1 y_1'' + C_1 r y_1' + C_1 s y_1 + C_2 y_2'' + C_2 r y_2' + C_2 s y_2 \\ &= C_1 \underbrace{(y_1'' + r y_1' + s y_1)}_{=0} + C_2 \underbrace{(y_2'' + r y_2' + s y_2)}_{=0} \\ &= 0. \end{aligned}$$

Therefore we need to find two independent solutions to (5.10). But how do we find them? Let us assume that the solutions are of the form

$$y = e^{\lambda x}.$$

Then

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x},$$

and so substituting into (5.10) we have

$$y'' + ry' + sy = \lambda^2 e^{\lambda x} + r\lambda e^{\lambda x} + s e^{\lambda x} = e^{\lambda x}(\lambda^2 + r\lambda + s) \equiv 0,$$

which tells us that if λ is a root of the equation

$$\lambda^2 + r\lambda + s = 0, \tag{5.12}$$

then $y = e^{\lambda x}$ will be a solution to (5.10). Equation (5.12) is called the *auxiliary equation* or *characteristic equation* and is quadratic, so we expect two solutions for λ and thus two independent solutions to (5.10).

Example 5.17. Consider the second-order differential equation

$$y'' + 3y' + 2y = 0.$$

If we put $y = e^{\lambda x}$, we see that the auxiliary equation is

$$\lambda^2 + 3\lambda + 2 = 0 \iff (\lambda + 1)(\lambda + 2) = 0.$$

This has two roots:

$$\lambda_1 = -1, \quad \lambda_2 = -2.$$

Therefore we have two solutions e^{-x} and e^{-2x}, and they are independent. So the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{-2x}.$$

In general, we know that the roots of (5.12) are given by

$$\lambda = \frac{-r \pm \sqrt{\Delta}}{2}, \quad \Delta = r^2 - 4s.$$

There are three cases, depending on the value of Δ.

(a) $\Delta > 0$: two distinct real roots (see Fig. 5.1(a)),

$$\lambda_1 = \frac{-r+\sqrt{\Delta}}{2}, \quad \lambda_2 = \frac{-r-\sqrt{\Delta}}{2}.$$

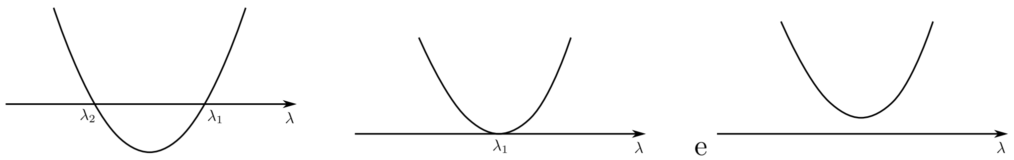
We have two independent solutions to the differential equation (5.10),

$$e^{\lambda_1 x} \quad \text{and} \quad e^{\lambda_2 x},$$

so

$$g(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x},$$

is the general solution to (5.10), i.e. the C.F. of (5.9).



(a) *Two roots.* (b) *One roots.* (c) *No roots (complex).*

Figure 5.1: *Different options for the curve* $y = \lambda^2 + r\lambda + s$ *when solving the equation* $\lambda^2 + r\lambda + s = 0$.

(b) $\Delta = 0$, (5.12) has one root which is real (see Fig. 5.1(b)), given by

$$\lambda_1 = \frac{r}{s}, \quad \Delta = 0 \quad \Longrightarrow \quad s = \frac{r^2}{4}.$$

So $e^{\lambda_1 x}$ is one solution to (5.10). We need to find another solution, which is independent of $e^{\lambda_1 x}$. So we try

$$y = xe^{\lambda_1 x},$$

then

$$y' = e^{\lambda_1 x} + \lambda_1 x e^{\lambda_1 x} \quad \text{and} \quad y'' = \lambda_1 e^{\lambda_1 x} + \lambda_1 e^{\lambda_1 x} + \lambda_1^2 x e^{\lambda_1 x} = \lambda_1^2 x e^{\lambda_1 x} + 2\lambda_1 e^{\lambda_1 x}.$$

Hence

$$\begin{aligned} y'' + ry' + sy &= \lambda_1^2 x e^{\lambda_1 x} + 2\lambda_1 e^{\lambda_1 x} + r e^{\lambda_1 x} + r\lambda_1 x e^{\lambda_1 x} + s x e^{\lambda_1 x} \\ &= \underbrace{(\lambda_1^2 + r\lambda_1 + s)}_{=0} x e^{\lambda_1 x} + \underbrace{(2\lambda_1 + r)}_{=0} e^{\lambda_1 x} \\ &= 0. \end{aligned}$$

That is, $xe^{\lambda_1 x}$ is a solution to (5.10). Also, $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$ are independent. Therefore, in this case the general solution of (5.10) is

$$g(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}.$$

Example 5.18. Consider the differential equation

$$y'' + 2y' + y = 0.$$

The auxiliary equation is

$$\lambda^2 + 2\lambda + 1 = 0 \iff (\lambda + 1)^2 = 0 \implies \lambda_1 = -1,$$

where λ_1 is the only root. But $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$ are two independent solutions. So the general solution is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} = e^{-x}(C_1 + C_2 x).$$

(c) $\Delta < 0$, there is no real solution to (5.12) (see Fig. 5.1(c)). But it is still possible to find two independent solutions, these will be complex roots given by

$$\lambda = \frac{-r \pm \sqrt{\Delta}}{2} = \frac{-r \pm \sqrt{(-1)(-\Delta)}}{2} = -\frac{r}{2} \pm \frac{1}{2}\sqrt{-1}\sqrt{-\Delta} = \alpha + i\beta,$$

where $-\Delta > 0$ and

$$i = \sqrt{-1}, \quad \alpha = -\frac{r}{2}, \quad \beta = \frac{1}{2}\sqrt{-\Delta} = \frac{1}{2}\sqrt{4s - r^2}.$$

In this case, the solutions are $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$, so we have the general solution

$$g(x) = \tilde{C}_1 e^{(\alpha+i\beta)x} + \tilde{C}_2 e^{(\alpha-i\beta)x} = e^{\alpha x}\left(\tilde{C}_1 e^{i\beta x} + \tilde{C}_2 e^{-i\beta x}\right).$$

But recall Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, which means we can write the general solution as

$$g(x) = e^{\alpha x}\left((\tilde{C}_1 + \tilde{C}_2)\cos\beta x + (\tilde{C}_1 - \tilde{C}_2)\sin\beta x\right) = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right),$$

where $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are two independent solutions.

Let us verify

$$y = e^{\alpha x}\cos\beta x$$

is a solution of (5.10). We have

$$y' = \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x,$$

$$\begin{aligned} y'' &= \alpha^2 e^{\alpha x}\cos\beta x - \alpha\beta e^{\alpha x}\sin\beta x - \alpha\beta e^{\alpha x}\sin\beta x - \beta^2 e^{\alpha x}\cos\beta x \\ &= (\alpha^2 - \beta^2)e^{\alpha x}\cos\beta x - 2\alpha\beta e^{\alpha x}\sin\beta x. \end{aligned}$$

Plugging these into equation (5.10), we have

$$\begin{aligned} y'' + ry' + sy = (\alpha^2 - \beta^2)e^{\alpha x}\cos\beta x - 2\alpha\beta e^{\alpha x}\sin\beta x + \\ + r\alpha e^{\alpha x}\cos\beta x - r\beta e^{\alpha x}\sin\beta x + s e^{\alpha x}\cos\beta x, \end{aligned}$$

which can be written as

$$y'' + ry' + sy = \underbrace{(\alpha^2 - \beta^2 + \alpha r + s)}_{\frac{r^2}{4} - \frac{1}{4}(4s - r^2) - \frac{r^2}{2} + s = 0} e^{\alpha x}\cos\beta x - \underbrace{(2\alpha + r)}_{=0}\beta e^{\alpha x}\sin\beta x = 0.$$

Similarly we can show $y = e^{\alpha x}\sin\beta x$ also satisfies (5.10), thus $g(x)$ above is a general solution to (5.10).

Example 5.19. Consider the differential equation

$$y'' - 6y' + 13y = 0.$$

The auxiliary equation is

$$\lambda^2 - 6\lambda + 13 = 0,$$

which has

$$r = -6, \quad s = 13, \quad \Delta = r^2 - 4s = 36 - 52 = -16 < 0,$$

i.e. it has complex roots. So we have

$$\alpha = -\frac{r}{2} = 3, \quad \beta = \frac{1}{2}\sqrt{-(-16)} = 2,$$

therefore $e^{3x}\cos 2x$ and $e^{3x}\sin 2x$ are two independent solutions, so

$$y = e^{3x}(C_1 \cos 2x + C_2 \sin 2x),$$

is the general solution.[5]

[5]End Lecture 25.