We will consider three different kinds of options for $p(x)$:

1. Polynomial,
2. trigonometric functions sin or cos,
3. exponential function:
   (a) p has the form e^{rx} where $r \neq -\lambda$,
   (b) p has the form e^{rx} where $r = -\lambda$.

Example 5.12. Consider the differential equation

$$y' + y = x,$$

so we have

$$\lambda = 1, \quad p(x) = x, \quad \text{solution: } y = f(x) + Ce^{-x},$$

i.e. we see that the C.F. from the homogeneous equation is $g = Ce^{-x}$. Now, here f should be a polynomial with a degree of one, since $p(x) = x$. So we try the most general first order polynomial, $f(x) = ax + b$, and so $f'(x) = a$. Substituting $y = f(x)$ into the differential equation we have that

$$f' + f = a + ax + b = ax + (a + b) \equiv x,$$

Thus, comparing coefficients from the LHS and RHS we must have that

$$\left.\begin{array}{l} a = 1 \\ a + b = 0 \end{array}\right\} \implies \left.\begin{array}{l} a = 1 \\ b = -1 \end{array}\right\} \implies f(x) = x - 1,$$

so $f(x) = x - 1$ is a solution. Therefore, the general solution to the original equation is

$$y(x) = x - 1 + Ce^{-x}.$$

Example 5.13. Consider the differential equation

$$y' + 2y = x^2, \quad \lambda = 2, \quad p(x) = x^2.$$

We can easily work out the C.F., the general solution will take the form

$$y(x) = f(x) + Ce^{-2x}.$$

For the particular integral, we try $f(x) = ax^2 + bx + c$, since $p(x) = x^2$. So, we have $f'(x) = 2ax + b$. Substituting into the differential equation, we have

$$f' + 2f = 2ax + b + 2ax^2 + 2bx + 2c = 2ax^2 + (2a + 2b)x + (b + 2c) \equiv ex^2.$$

Comparing coefficients between the LHS and RHS we have

$$\left.\begin{array}{l} 2a = 1 \\ a + b = 0 \\ b + 2c = 0 \end{array}\right\} \implies \left.\begin{array}{l} a = \frac{1}{2} \\ b = -\frac{1}{2} \\ c = \frac{1}{4} \end{array}\right\} \implies f(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4},$$

and the general solution is

$$y(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}.$$

Example 5.14. Consider the differential equation

$$y' + y = \sin 2x, \quad \lambda = 1, \quad p(x) = \sin 2x.$$

The general solution will have the form

$$y(x) = f(x) + Ce^{-x}.$$

To find the particular integral, we try $f(x) = a\sin 2x + b\cos 2x$, since $p(x) = \sin 2x$. Therefore $f'(x) = 2a\cos 2x - 2b\sin 2x$. Substituting into the differential equation tells us

$$f' + f = 2a\cos 2x - 2b\sin 2x + a\sin 2x + b\cos 2x = (a - 2b)\sin 2x + (2a + b)\cos 2x \equiv \sin 2x.$$

Comparing coefficient from LHS and RHS then

$$\left.\begin{array}{l} a - 2b = 1 \\ 2a + b = 0 \end{array}\right\} \implies \left.\begin{array}{l} a = \frac{1}{5} \\ b = -\frac{2}{5} \end{array}\right\} \implies f(x) = \frac{1}{5}\sin 2x - \frac{2}{5}\cos 2x.$$

So the general solution is

$$y(x) = \frac{1}{5}\sin 2x - \frac{2}{5}\cos 2x + Ce^{-x}.$$

Example 5.15. Consider the differential equation

$$y' + y = e^{2x}, \quad \lambda = 1, \quad p(x) = e^{2x}, \quad r = 2 \neq -\lambda = -1.$$

The general solution takes the form

$$y(x) = f(x) + Ce^{-x}.$$

For the particular integral, we shall try $f(x) = ae^{2x}$, since $p(x) = e^{2x}$ and $f'(x) = 2ae^{2x}$. Substituting into the differential equation gives

$$f' + f = 2ae^{2x} + ae^{2x} = 3ae^{2x} \equiv e^{2x}.$$

Comparing coefficients gives $a = \frac{1}{3}$, thus the general solution is

$$y(x) = \frac{1}{3}e^{2x} + Ce^{-x}.$$

Example 5.16. Consider the differential equation

$$y' + y = e^{-x}, \quad \lambda = 1, \quad p(x) = e^{-x}.$$

The general solution has the form

$$y(x) = f(x) + Ce^{-x}.$$

Now, if we try $f(x) = ae^{-x}$, then

$$f' + f = -ae^{-x} + ae^{-x} = 0 \neq e^{-x},$$

so it doesn't work since $r = -1 = -\lambda$. But we may try

$$f(x) = axe^{-x}, \quad \text{then } f'(x) = ae^{-x} - axe^{-x}.$$

Substituting into the differential equation gives

$$f'(x) + f(x) = -axe^{-x} + ae^{-x} + axe^{-x} = ae^{-x} \equiv e^{-x},$$

therefore, comparing coefficients, we have $a = 1$, so the general solution is

$$y(x) = xe^{-x} + Ce^{-x} = e^{-x}(x + C).$$

In general, for

$$y' + \lambda y = e^{-\lambda x},$$

the general solution is of the form

$$y(x) = f(x) + Ce^{-\lambda x}.$$

For the particular integral we try $f(x) = axe^{-\lambda x}$, then

$$f' + \lambda f = ae^{-\lambda x} - \lambda axe^{-\lambda x} + \lambda axe^{-\lambda x} = ae^{-\lambda x} \equiv e^{-\lambda x},$$

therefore, comparing coefficient, we have $a = 1$. So the general solution is

$$y(x) = xe^{-\lambda x} + Ce^{-\lambda x} = e^{-\lambda x}(x + C).$$

This can be easily generalised for the case

$$y' + \lambda y = be^{-\lambda x}, \quad \lambda \text{ and } b \text{ are constants.}$$

5.1.3 Second-order linear differential equations with constant coefficients

$$y'' + ry' + sy = p(x), \tag{5.9}$$

where r and s are constant. The general solution to (5.9) has the form

$$y(x) = f(x) + g(x),$$

where $y = f(x)$ is a *particular integral* from the solution to (5.9) and $y = g(x)$ is the *complementary function* from the solution to the homogeneous equation

$$y'' + ry' + sy = 0. \tag{5.10}$$

Solving (5.10) to build $y = g(x)$

Two functions (say, two solutions to (5.10)) are said to be independent if one is not a *constant* multiple of the other; otherwise, they are said to be dependent. For example

- $y_1(x) = 1$, $y_2(x) = x$, here y_1 and y_2 are independent.
- $y_1(x) = e^{ax}$, $y_2(x) = e^{bx}$, if $a \neq b$, then y_1 and y_2 are independent since

  $$y_1 = e^{ax} = e^{bx+(a-b)x} = e^{(b-a)x}y_2,$$

  i.e. if $a \neq b$, then the multiple is not a constant.
- $y_1(x) = e^{ax}$, $y_2(x) = e^{ax+b}$, a and b are constant, then y_1 and y_2 are dependent, since $y_2(x) = e^b y_1(x)$, e^b is constant.[4]

[4] End Lecture 24.