Consider the general case for a first-order differential equation given by (5.6). First let us multiply both sides of the equation by $e^{T(x)}$,

$$e^{T(x)}\frac{dy}{dx} + e^{T(x)}q(x)y = e^{T(x)}p(x). \tag{5.7}$$

Now let us consider the derivative of $e^{T(x)}y$,

$$\begin{aligned}
\frac{d}{dx}(e^{T(x)}y) &= e^{T(x)}\frac{dy}{dx} + y\frac{d}{dx}(e^{T(x)}) \\
&= e^{T(x)}\frac{dy}{dx} + ye^{T(x)}\frac{d}{dx}(T(x)) \\
&= e^{T(x)}\frac{dy}{dx} + ye^{T(x)}q(x) \\
&= e^{T(x)}p(x).
\end{aligned}$$

Here we have put $T'(x) = q(x)$ and applied (5.7). So we have

$$\frac{d}{dx}(e^{T(x)}y) = e^{T(x)}p(x).$$

Therefore, integrating both sides we have

$$e^{T(x)}y = \int e^{T(x)}p(x)\,dx,$$

or

$$y = e^{-T(x)}\int e^{T(x)}p(x)\,dx.$$

we have shown that if $y' + q(x)y = p(x)$ and $T(x) = \int q(x)\,dx$, then

$$y = e^{-T(x)}\int e^{T(x)}p(x)\,dx. \tag{5.8}$$

Note, we expect a constant when we complete the integration above.

Example 5.9. Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x,$$

note that $f(x, y) = x - (y/x)$ can't be separated. So we put

$$q(x) = \frac{1}{x}, \quad p(x) = x \quad \Longrightarrow \quad T(x) = \int q(x)\,dx = \int \frac{1}{x}\,dx = \ln x.$$

Then we have the solution

$$y = e^{-\ln x}\int e^{\ln x}x\,dx = e^{\ln\frac{1}{x}}\int x^2\,dx = \frac{1}{x}\left[\frac{1}{3}x^3 + C\right] = \frac{1}{3}x^2 + \frac{C}{x}.$$

Example 5.10. Consider

$$\frac{dy}{dx} + xy = x.$$

So we put

$$q(x) = x, \quad p(x) = x \quad \Longrightarrow \quad T(x) = \int q(x)\,dx = \int x\,dx = \frac{1}{2}x^2.$$

Then the solution is

$$y = e^{-\frac{1}{2}x^2} \int e^{\frac{1}{2}x^2} x\,dx = e^{-\frac{1}{2}x^2} \left[\int e^{\frac{1}{2}x^2}\, d\left(\frac{1}{2}x^2\right) \right] = e^{-\frac{1}{2}x^2} \left[e^{\frac{1}{2}x^2} + C \right],$$

i.e.

$$y = 1 + Ce^{-\frac{1}{2}x^2}.$$

NOTE: this example could have been done using separation of variables.

Example 5.11. Solve the initial-value problem:

$$y' = y + x^2, \quad y(0) = 1,$$

so we put

$$q(x) = -1, \quad p(x) = x^2, \quad T(x) = \int -1\,dx = -x.$$

Therefore, the solution is

$$y = e^x \int e^{-x} x^2\,dx.$$

We calculate the integral using integration by parts,

$$\begin{aligned} \int e^{-x} x^2\,dx &= -x^2 e^{-x} + \int 2xe^{-x}\,dx \\ &= -x^2 e^{-x} - 2xe^{-x} + \int 2e^{-x}\,dx \\ &= e^{-x}\left[-x^2 - 2x\right] - 2\int e^{-x} \\ &= -e^{-x}\left(x^2 + 2x + 2\right) + C. \end{aligned}$$

Hence, the solution to the differential equation is

$$y = -(x^2 + 2x + 2) + Ce^x.$$

It remains to use the initial condition to find C, i.e.

$$y(0) = -2 + Ce^0 = 1 \quad \Longrightarrow \quad C = 3,$$

so the final solution is

$$y = -(x^2 + 2x + 2) + 3e^x.$$

In the previous three examples, we gained the following results:

Ex. 5.9.

$$y' + \frac{1}{x}y = x, \quad y = \frac{1}{3}x^2 + C \cdot .\frac{1}{x},$$

Ex. 5.10.

$$y' + xy = x, \quad y = 1 + C \cdot e^{-\frac{1}{2}x^2},$$

Ex. 5.11.

$$y' - y = x^2, \quad y = -(x^2 + 2x + 2) + Ce^x.$$

These examples have something very important in common, that is the solutions have the following form

$$y = f(x) + Cg(x),$$

with explicit functions f and g. Here $y = f(x)$ is a particular solution (take $C = 0$) of the non-homogeneous equation, and $y = g(x)$ is a solution of the corresponding homogeneous equation. For example

Ex. 5.9.

$$\begin{aligned} &\text{if} \quad y = \frac{1}{3}x^2, \quad \text{then } y' + \frac{1}{x}y = \frac{2}{3}x + \frac{1}{x}\cdot\frac{1}{3}x^2 = x, \\ &\text{if} \quad y = \frac{1}{x}, \quad \text{then } y' + \frac{1}{x}y = -\frac{1}{x^2} + \frac{1}{x}\cdot\frac{1}{x} = 0. \end{aligned}$$

Ex. 5.10.

$$\begin{aligned} &\text{if} \quad y = 1, \quad \text{then } y' + xy = 0 + x = x, \\ &\text{if} \quad y = e^{-\frac{1}{2}x^2}, \quad \text{then } y' + xy = e^{-\frac{1}{2}x^2}\left(\frac{1}{2}\cdot 2x\right) + xe^{-\frac{1}{2}x^2} = 0. \end{aligned}$$

These examples reveal an intrinsic structure of the general solution of a linear differential equation. We solve the equation by finding the solution of its homogeneous equation, and a particular solution to the non-homogeneous equation.

Now we will understand how to use this method to solve a first-order linear differential equation with constant coefficients:

$$y' + \lambda y = p(x), \quad \lambda \text{ is constant.}$$

We know that the general solution is

$$y(x) = \underbrace{f(x)}_{\text{particular integral (P.I.)}} + \underbrace{Cg(x)}_{\text{complementary function (C.F.)}},$$

and

$$f' + \lambda f = p(x), \quad g' + \lambda g = 0,$$

where C is the constant of integration to be found. We start by building g. So we need to solve

$$g' = \lambda g = 0,$$

solution:

$$g = Ce^{-\lambda x}.$$

Therefore, the general to $y' + \lambda y = p(x)$ is

$$y = f(x) + Ce^{-\lambda x},$$

where f is a particular solution (depending on $p(x)$). In what follows, we shall find $f(x)$ for certain kinds of function $p(x)$.[3]

[3] End Lecture 23.