**Example 5.2.** Consider the differential equation

$$y' = \lambda y.$$

We already know the solution to this equation. Now let us see how to derive it using separation of variables.

$$\frac{dy}{dt} = \lambda y,$$

taking all things relating to $y$ to the left, and for $t$ to the right, we have

$$\frac{1}{y} dy = \lambda dt.$$

Integrating both sides we have

$$\int \frac{1}{y}\, dy = \int \lambda\, dt,$$

hence, using what we have learnt in previous chapters we get

$$\ln y = \lambda t + C.$$

Finally, re-arranging for $y$, we have

$$y = e^{\lambda t + C} = Ae^{\lambda t}, \quad A = e^C.$$

**Example 5.3.** Consider the equation

$$\frac{dy}{dx} = xy,$$

following the procedure as in the previous example, we have

$$\frac{1}{y}\, dy = x\, dx.$$

Integrating both sides we have

$$\int \frac{1}{y}\, dy = \int x\, dx,$$
$$\implies \qquad \ln y = \frac{1}{2}x^2 + C.$$

Taking exponentials of both sides in order to re-arrange for $y$, we get

$$y = e^{\frac{1}{2}x^2 + C} = Ae^{\frac{1}{2}x^2}, \quad A = e^C.$$

We can check if this satisfies the original equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(Ae^{\frac{1}{2}x^2}\right) = Ae^{\frac{1}{2}x^2 +} \cdot \frac{1}{2} \cdot 2x = xy.$$

**Example 5.4.** Consider the differential equation

$$y^2 y' = x.$$

We first write it in the form $y' = f(x, y)$, i.e.

$$\frac{dy}{dx} = \frac{x}{y^2},$$

now we realises that we can apply separation of variable, so

$$y^2 dy = x\, dx,$$

$$\begin{aligned}
\implies \quad & \int y^2 dy = \int x\, dx \\
\implies \quad & \frac{1}{3}y^3 = \frac{1}{2}x^2 + C, \\
\implies \quad & y = \left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}},
\end{aligned}$$

where $C'$ is some constant (different to $C$, since we multiplied through by 3). Again, we check the solution satisfies the equation

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}\right] \\
&= \frac{1}{3}\left(\frac{3}{2}x^2 + C'\right)^{-\frac{2}{3}} \cdot \frac{3}{2} \cdot 2x \\
&= x\left(\frac{3}{2}x^2 + C'\right)^{-\frac{2}{3}} \\
&= x\left[\left(\frac{3}{2}x^2 + C'\right)^{\frac{1}{3}}\right]^{-2} \\
&= \frac{x}{y^2}.
\end{aligned}$$

**Example 5.5.** Consider the following initial-value problem:

$$\frac{dy}{dx} = y^2(1 + x^2), \quad y(0) = 1.$$

First, we find the general solution, note, we can use separation of variables in this example, so

$$\frac{1}{y^2}\, dy = (1 + x^2)\, dx$$

$$\begin{aligned}
\implies \quad & \int \frac{1}{y^2}\, dy = \int (1 + x^2)\, dx \\
\implies \quad & -\frac{1}{y} = x + \frac{1}{3}x^3 + C \\
\implies \quad & y = -\frac{1}{x + \frac{1}{3}x^3 + C}.
\end{aligned}$$

Now check that the general solution satisfies the original differential equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\frac{1}{x + \frac{1}{3}x^3 + C}\right) = \frac{1 + x^2}{\left(x + \frac{1}{3}x^3 + C\right)^2} = (1 + x^2)y^2.$$

Now it remains to find the constant $C$, by applying the condition $y(0) = 1$, i.e. we put $x = 0$.

$$y(0) = -\frac{1}{0 + \frac{1}{3} \cdot 0^3 + C} = -\frac{1}{C} = 1, \quad \implies \quad C = -1.$$

So the solution to the initial value problem is

$$y = \frac{1}{1 - x - \frac{1}{3}x^3}.$$

**Example 5.6.** Consider the initial value problem

$$e^y y' = 3x^2, \quad y(0) = 2.$$

First, find the general solution,

$$\begin{aligned}
& e^y y' = 3x^2 \\
\implies \quad & \frac{dy}{dx} = 3x^2 e^{-y} \\
\implies \quad & e^y \, dy = 3x^2 \, dx \\
\implies \quad & \int e^y \, dy = \int 3x^2 \, dx \\
\implies \quad & e^y = x^3 + C \\
\implies \quad & y = \ln(x^3 + C).
\end{aligned}$$

Check:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln(x^3 + C)\right) = \frac{3x^2}{x^3 + C} = 3x^2 \frac{1}{x^3 + C}.$$

Recall $e^{\ln(a)} = a$, using this, we can write

$$\frac{dy}{dx} = 3x^2 e^{\ln\left(\frac{1}{x^3+C}\right)} = 3x^2 e^{-\ln(x^3+C)} = 3x^2 e^{-y}.$$

Now we apply the initial condition,

$$y(0) = \ln(C) = 2 \qquad \implies \qquad C = e^2,$$

so we have the final solution

$$y(x) = \ln(x^3 + e^2).$$

### 5.1.2 Linear first-order differential equations

An $n$-th order differential equation is linear if it can be written in the form:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + a_{n-2}(x)y^{(n-2)} + \cdots + a_1(x)y' + a_0(x)y = f(x), \tag{5.4}$$

or

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{5.5}$$

where $a_i$ $(i = 0, 1, 2, \ldots, n)$ and $f(x)$ are known functions of $x$.

NOTE: here we have leading order coefficient of 1.

NOTATION: the $n$-th derivative of $y$ with respect to $x$ is written as

$$y^{(n)} \equiv \frac{d^n y}{dx^n}.$$

**Example 5.7.**

1. $$y' + 2y = e^x,$$
   is a first-order linear differential equation with constant coefficients, i.e.
   $$n = 1, \quad a_0(x) = 2, \quad f(x) = e^x.$$

2. $$y'' + 8y' + 16y = 0,$$
   is a second-order linear differential equation with constant coefficients, i.e.
   $$n = 2, \quad a_1(x) = 8, \quad a_0(x) = 16, \quad f(x) = 0.$$

3. $$y'' + yy' = x,$$
   is a second-order *non-linear* differential equation.

4. $$e^x y' + xy = \cos x \iff y' + xe^{-x}y = e^{-x}\cos x,$$
   is a first-order linear differential equation with
   $$n = 1, \quad a_0(x) = xe^{-x}, \quad f(x) = e^{-x}\cos x.$$

5. $$\frac{dx}{dt} = 3x + t^3 e^{3t},$$
   is a first-order linear differential equation with
   $$n = 1, \quad a_0(t) = 3, \quad f(t) = t^3 e^{3t}.$$

**Definition 5.1.** *If $f(x) \equiv 0$ (zero function), then the linear equation* (5.5) *is said to be homogeneous; otherwise, we say the equation is non-homogenous.*

**Example 5.8.**

$$\begin{aligned} y' &\ +2y = e^x && \text{- non-homogeneous,} \\ y' &\ +2y = 0 && \text{- homogeneous.} \end{aligned}$$

A first-order linear equation has the form

$$\frac{dy}{dx} + q(x)y = p(x). \tag{5.6}$$

If $p(x) \equiv 0$, then we can try to solve the differential equation by separation of variables.

In general, we solve as follows. Let

$$T(x) = \int q(x)\, dx,$$

then

$$e^{T(x)} = e^{\int q(x)\, dx},$$

is called the *integrating factor*. We will use this integrating factor to derive the general solution to the first-order linear differential equation.[2]

[2]End Lecture 22.