### 4.6.2 Finding a distance by the integral of velocity

If you know the velocity $v(t)$, then the distance $s$ as a function of time, i.e. $s = s(t)$, is

$$s(t) = \int v(t)\,dt, \quad (\text{since } s'(t) = v(t)).$$

Similarly, if you know the acceleration $a(t)$, then the velocity can be found by the integral of $a(t)$,

$$v(t) = \int a(t)\,dt \quad (\text{since } v'(t) = a(t))$$

**Example 4.34.** A ball is thrown down from a tall building with an initial velocity of 100ft/sec. Then its velocity after $t$ seconds is given by $v(t) = 32t + 100$. How far does the ball fall between 1 and 3 seconds of elapsed time?

First let us write the distance as the integral of the velocity, that is

$$s(t) = \int v(t) \ = \int 32t + 100\,dt = 16t^2 + 100t + C.$$

Then the distance fallen, $\bar{s}$ say, is given by

$$\bar{s} = s(3) - s(1) = (16t^2 + 100t + C)\big|_{t=3} - (16t^2 + 100t + C)\big|_{t=1} = 328 \text{ ft}.$$

Notice that

$$\begin{aligned}(16t^2 + 100t + C)\big|_{t=3} - (16t^2 + 100t + C)\big|_{t=1} &= (16t^2 + 100t + C)\big|_1^3 \\ &= \int_1^3 (32t + 100)\,dt \\ &= \int_1^3 v(t)\,dt.\end{aligned}$$

In general, if $v(t)$ is a velocity function, then the change in distance between $t_1$ and $t_2$ is

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\,dt,$$

and the total distance is

$$s(t) = \int_0^t v(\tau)\,d\tau.$$

NOTE: here we call $\tau$ a *dummy variable.*

# Chapter 5

# Differential Equations

Differential equations are equations involving a certain unknown function and its derivative (equations for functions). For example

$$y' = x^2 \quad \text{or} \quad \frac{dy}{dx} = x^2,$$

where $y$ is a function of $x$, i.e. $y = y(x)$. We know that in this particular case,

$$y(x) = \frac{1}{3}x^3 + C,$$

where $C$ is an arbitrary constant.

Recall exponential growth and decay: we define the relative rate of growth as $y'/y$. If we assume that the rate is constant, say $\lambda$, then we have

$$y' = \lambda y, \quad (y = y(t)). \tag{5.1}$$

This has solution $y = Ae^{\lambda t}$, since

$$y'(t) = \lambda A e^{\lambda t} = \lambda y(t),$$

where $A$ is an unknown constant.

The solution $Ae^{\lambda t}$ is called the *general solution* to the differential equation in (5.1), implying that we have found all possible solutions.

To determine the constant, we need an extra piece of information. For instance, if we know $y$ at the initial time, say $y(0) = y_0$ (given), then

$$y(0) = Ae^{\lambda \cdot 0} = A = y_0,$$

so $y(t) = y_0 e^{\lambda t}$.

The problem $y' = \lambda y$, $y(0) = y_0$ is call an *initial value problem* (IVP), which has a unique solution $y = y_0 e^{\lambda t}$.

The *order* of a differential equation is the order of the highest derivative appearing in the equation.

**Example 5.1.**

$$
\begin{aligned}
&y' + 2xy = e^x \text{ - first order,}\\
&y'' + 3y' + 4y = 0 \text{ - second order,}\\
&(y')^2 + 2\ln y + 4e^x = x^3 \text{ - first order,}\\
&y^{(3)} + 3y'' + 2y' + 4y = 10 \text{ - third order.}
\end{aligned}
$$

## 5.1 First order differential equations

Here we will consider different techniques to solve first order ordinary differential equations (1st order ODEs).

### 5.1.1 Separation of variables

Let us consider the simplest case of a first order differential equation, that is

$$\frac{dy}{dx} = f(x, y), \tag{5.2}$$

for example

$$
\begin{aligned}
&y' + 2xy = e^x \quad \Longrightarrow \quad y' = -2xy + e^x = f(x, y),\\
&(y')^2 + 2\ln y + 4e^x = x^3 \quad \Longrightarrow \quad y' = \pm\sqrt{x^3 - 2\ln y + 4e^x} = f(x, y),\\
&y^2 y' - x = 0 \quad \Longrightarrow \quad y' = x/y^2 = f(x, y).
\end{aligned}
$$

Suppose that $f(x, y)$ is separable, i.e. it can be written as

$$f(x, y) = g(x)h(y), \tag{5.3}$$

for example

$$f(x, y) = \frac{x}{y^2} = x \cdot \frac{1}{y^2} \quad \Longrightarrow \quad g(x) = x, \quad h(y) = \frac{1}{y^2}.$$

Then we have

$$\frac{1}{h(y)}\, dy = g(x)\, dx,$$

where the LHS only depends on $y$ and the RHS only depends on $x$. Integrating both sides we have

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx.$$

Therefore, if we can work out the integrals, then we can obtain the general solution to the equation. Also, since we expect a constant of integration, we can merge the constants from both sides into one constant, say $C$, since we are integrating the equation once only.[1]

[1] End Lecture 21.