Example 4.31. Using the trapezium method, estimate

$$\int_0^1 \frac{1}{1+x^4}\,dx.$$

We choose $n = 4$, then

$$h = \frac{1-0}{4} = \frac{1}{4}, \quad x_k = kh, \quad k = 0, 1, 2, 3, 4.$$

Also, note that

$$f(x) = \frac{1}{1+x^4}, \quad \text{i.e.} \quad f(x_k) = \frac{1}{1+x_k^4}.$$

Therefore, we have

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^4}\,dx &\approx \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3)) + f(x_4)\right] \\
&= \frac{1}{8}\left[f(0) + 2(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)) + f(1)\right] \\
&= \frac{1}{8}\left[1 + 2\left(\frac{256}{257} + \frac{16}{17} + \frac{256}{337}\right) + \frac{1}{2}\right] \\
&= 0.862.
\end{aligned}$$

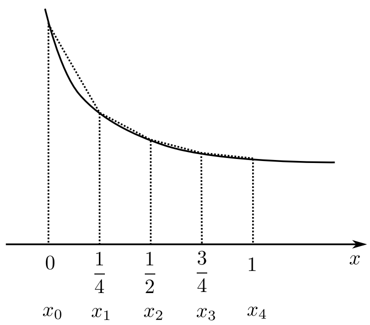


Figure 4.11: *Numerically integrating under $y = 1/(1+x^4)$. Dividing interval into 4 pieces of width $h = 1/4$.*

This is an over-estimate of the integral since $y = f(x)$ is convex (i.e. it curves up like a cup). If it were concave (i.e. curved down like a cap), then you would have an under-estimate.

Example 4.32. Estimate the following integral

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx, \quad \text{i.e.} \quad f(x) = \frac{1}{\sqrt{1-x^4}}.$$

However, notice that we can't calculate $f(1)$. This is because $y = f(x)$ has a vertical asymptote at $x = 1$.

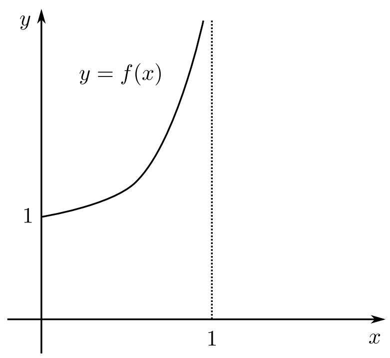


Figure 4.12: *Graph of* $y = 1/(\sqrt{1-x^4})$*, with an asymptote at* $x = 1$*. Estimating integral on interval* $[0, 1]$*.*

The problem here is that at $x \to 1$, $1 - x^4 \to 0$, rather like $1 - x$.

Since $1 - x^4 = (1 - x)(1 + x + x^2 + x^3)$, i.e. $1 - x^4$ contains a factor $1 - x$, which makes $f(x)$ become *singular* at $x = 1$. So we may try to get rid of it by a substitution.

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx = \int_0^1 \frac{1}{\sqrt{1-x}}\,\frac{1}{\sqrt{1+x+x^2+x^3}}\,dx.$$

Let us try $u = \sqrt{1-x}$, then $u^2 = 1 - x$ or $x = 1 - u^2$ and $dx = 2 - udu$. Now, at $x = 0 \to u = 1$ and $x = 1 \to u = 0$. Thus, the integral becomes

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx &= \int_1^0 \frac{1}{u}\,\frac{-2u}{\sqrt{1+1-u^2+(1-u^2)^2+(1-u^2)^3}}\,du \\
&= -2\int_1^0 \frac{1}{\sqrt{4-6u^2+4u^4-u^6}}\,du \\
&= +2\int_0^1 \frac{1}{\sqrt{4-6u^2+4u^4-u^6}}\,du,
\end{aligned}$$

where we have applied the rule

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

i.e. if you switch the limits, the integral changes sign.

Let us put

$$g(u) = \frac{1}{\sqrt{4-6u^2+4u^4-u^6}}.$$

Choosing $n = 4$, we have $h = \frac{1}{4}$, so

$$u_k = kh, \quad k = 0, 1, 2, 3, 4.$$

Then, we estimate

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx &= 2\int_0^1 g(u)\,du \\ &\approx 2\cdot\frac{h}{2}\left[g(0)+2\left(g\left(\frac{1}{4}\right)+g\left(\frac{1}{2}\right)+g\left(\frac{3}{4}\right)\right)+g(1)\right] \\ &= 1.32\ldots\end{aligned}$$

The exact result is $1.311\ldots$, so we have a close estimate given we only chose 4 divisions of the interval.

4.6 Application of the definite integral

4.6.1 Area bounded by curves

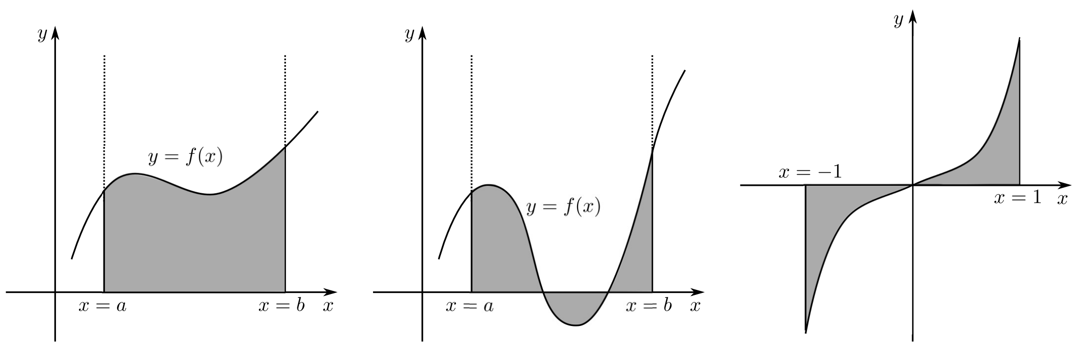
As we have discussed, integrating allows us to find the area bounded by the x-axis and a curve $y = f(x)$. We can extend this to find the area between different curves on the same axis.

If $f(x)$ is a non-negative function on $a \le x \le b$, then $\int_a^b f(x)\,dx$ is the area between the curves $y = f(x)$ and $y = 0$ (i.e. the x-axis) from a to b. That is, the region is bounded by

$$y = f(x), \quad y = 0, \quad x = a, \quad x = b.$$

In general, the area between the curves $y = f(x)$ and x-axis from a to b is

$$\int_a^b |f(x)|\,dx.$$



(a) *Use $y = f(x)$.* (b) *Use $y = |f(x)|$.* (c) *Use $y = |x^3|$.*

Figure 4.13: *Examples of area bounded between curves $y = f(x)$ and $y = 0$, on the interval $[a, b]$.*

Recall:

$$\int_{-1}^{1} x^3\,dx = \left.\frac{x^4}{4}\right|_{-1}^{1},$$

but the shaded area is given by

$$A = \int_{-1}^{1} |x^3|\,dx = \int_{-1}^{0} (-x^3)\,dx + \int_{0}^{1} x^3\,dx = \frac{1}{2}.$$

Suppose the region is bounded above and below by two curves $y = f_1(x)$ (top) and $y = f_2(x)$ (bottom) from a to b, then the area of the region is

$$\int_a^b |f_1(x) - f_2(x)|\,dx. \tag{4.7}$$

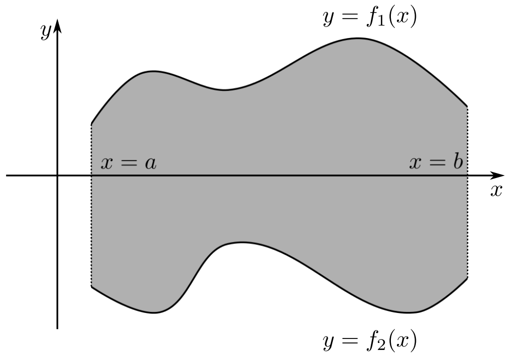


Figure 4.14: *Area bounded between two curves on the interval* $[a, b]$.

Example 4.33. Find the area of the region between $y = x + 1$ and $y = 7 - x$ from $x = 2$ to $x = 5$.

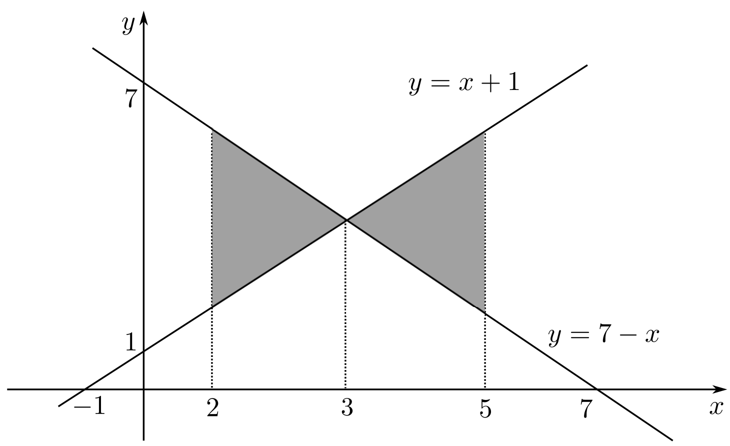


Figure 4.15: *Shaded area between* $y = x + 1$ *and* $y = 7 - x$ *over the interval* $[2, 5]$.

First, we need to find the point where the two curves intersect. The point should satisfy both equations, therefore we solve

$$\left.\begin{array}{l} y = x+1 \\ y = 7x-1 \end{array}\right\} \quad \Longrightarrow \quad x+1 = 7-x \quad \Longrightarrow \quad \left\{\begin{array}{l} x = 3 \\ y = 4 \end{array}\right.$$

We need to know the point of intersection because from the graph, it is easy to see that on the left of the point of intersection, $y = 7 - x$ is above $y = x + 1$, whilst on the right $y = x + 1$ is above $y = 7 - x$. So we have to be careful when employing the formula (4.7).

Therefore, we can finally calculate the area as follows:

$$\begin{aligned} A &= \int_2^3 [(7-x)-(x+1)]\ dx + \int_3^5 [(x+1)-(7-x)]\ dx \\ &= \int_2^3 (6-2x)\,dx + \int_3^5 (2x-6)\,dx \\ &= \left[6x - x^2\right]_2^3 + \left[-6x + x^2\right]_3^5 \\ &= 5. \end{aligned}$$

Exercise 4.1. Find the area bounded by the curves $y = x^2$ and $y = 1 - x^2$, on the interval $[-1, 1]$. Hint: is there more than one point of intersection?[6]

[6] End Lecture 20.