1.3.1 Exponents

We have seen the function $f(x) = x^2$. In general, a function of the form $f(x) = x^a$, where a is a constant, is called a power function. For example,

$$f(x) = x^5, \quad a = 5,$$
$$f(x) = \sqrt{x}, \quad a = 1/2, \quad x \geq 0.$$

What about $f(x) = a^x$? Can we define a function of this form? Yes, we can, but we need to explore the meaning of a^x, when x is not a positive integer.

REVISION: when we wish to multiply a number by itself several times, we make use of index or power notation. We have notation for powers:

$$a^2 = a \cdot a, \quad a^3 = a \cdot a \cdot a, \quad a^x = \overbrace{a \cdot a \ldots a \cdot a}^{x}, \quad a \in \mathbb{R}, \quad x \in \mathbb{N}.$$

Here, a is called the *base* and x is called the *index* or *power*. We also know the following properties (laws of exponents)

1. $\boxed{a^{x+y} = a^x \cdot a^y}$, e.g. $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^3 \cdot 2^2, \quad x = 3, \quad y = 2.$
2. $\boxed{(a^x)^y = a^{xy}}$, e.g. $3^6 = 3^{2\times 3} = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^2 \cdot 3^2 \cdot 3^2 = (3^2)^3.$
3. $\boxed{a^x \cdot b^x = (ab)^x}$, e.g. $2^3 \cdot 3^3 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = (2 \cdot 3)(2 \cdot 3)(2 \cdot 3) = (2 \cdot 3)^3.$

These laws hold for any $a \in \mathbb{R}$ and $x, y \in \mathbb{N}$.

Now we want to generalise the notation for the power to include whole numbers, fractions and irrational numbers, so that a^x makes sense for most values of a and all $x \in \mathbb{R}$. The idea is to make laws of exponents hold generally.

First, we choose $a^0 = 1$ $(a \neq 0)$ so that

$$a^2 = a^{2+0} = a^2 \cdot a^0 = a^2, \quad \text{i.e. } a^x = a^{x+0} = a^x \cdot a^0 = a^x.$$

Second, we choose

$$a^{-1} = \frac{1}{a}, \quad a^{-2} = \frac{1}{a^2}, \ldots, a^{-n} = \frac{1}{a^n}, \quad n \in \mathbb{N},$$

so that

$$a^2 \cdot a^{-2} = a^{2-2} = a^0 = 1, \quad \text{or } a^n \cdot a^{-n} = a^{n-n} = a^0 = 1, \quad \text{for all } n \in \mathbb{N}.$$

We choose

$$a^{1/2} = \sqrt{a} \quad (\text{the square root of } a,\ a \geq 0),$$

$a^{1/3} = \sqrt[3]{a}$ (the cube root of a, $a \in \mathbb{R}$),

$\vdots$

$a^{1/n} = \sqrt[n]{a}$ (the nth root of a. If n is even, $a \geq 0$; otherwise any $a \in \mathbb{R}$ is O.K.)

so that

$$a^{1/2} \cdot a^{1/2} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a,$$
$$a^{1/3} \cdot a^{1/3} \cdot a^{1/3} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a,$$
$$\vdots$$
$$\overbrace{a^{1/n} \cdot a^{1/n} \ldots a^{1/n}}^{n} = a^{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}} = a^{\frac{n}{n}} = a^1 = a.$$

If x is a rational number, then $x = p/q$, where p and q are integers and $q > 0$. Then

$$a^x = a^{p/q} = (a^{1/q})^p = (a^p)^{1/q} \quad \text{(order doesn't matter)}.$$

Therefore, a^x makes sense for any rational number x.

If x is an irrational number, then we can always find two rational numbers c and d which are sufficiently close to x and which satisfy $c < x < d$. So

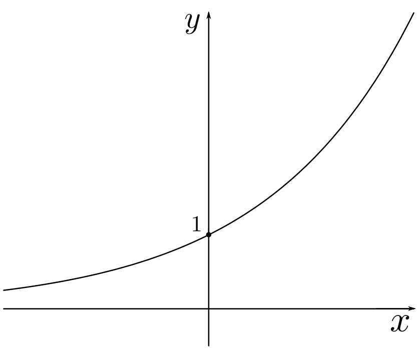
$$a^c < a^d \quad \text{if} \quad a \geq 1; \quad a^c > a^d \quad \text{if} \quad 0 < a < 1.$$

It can be shown that there is exactly one number between a^c and a^d. We define this number as a^x.

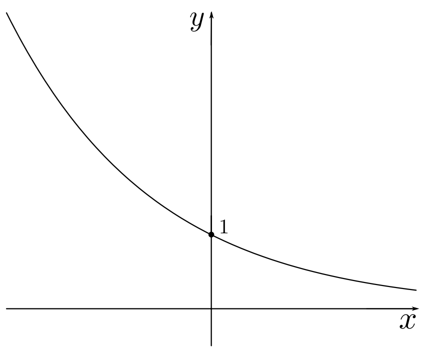
Finally, an exponential function can be defined by

$$f(x) = a^x, \quad x \in \mathbb{R},$$

where a is a positive constant. The domain of f is $\mathbb{R}$ and the range is $\mathbb{R}^+$. Graph:



(a) $y = a^x, \quad a > 1$.



(b) $y = a^x, \quad 0 < a < 1$.

Figure 1.4: *Comparison of exponent graphs for different values of a.*

Summary:

If a and b are positive numbers, x and y are any real numbers, then we have

1 $a^{x+y} = a^x \cdot a^y$,

2 $a^{x-y} = \frac{a^x}{a^y}$,

3 $(a^x)^y = a^{xy}$,

4 $a^x \cdot b^x = (ab)^x$.

There is also a special exponential function, $f = e^x$, we will investigate this further later in the course.[2]

[2] End Lecture 2.