Example 4.26. Suppose we have the integrand xe^x. Let us choose

$$u = x, \quad v' = e^x \qquad \Longrightarrow \qquad u' = 1, \quad v = e^x.$$

Therefore, we can calculate the integral as follows:

$$\begin{aligned} \int xe^x\,dx &= \int uv'\,dx \\ &= uv - \int u'v\,dx \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C. \end{aligned}$$

Check:

$$\frac{d}{dx}\left[xe^x - e^x + C\right] = e^x + xe^x - e^x = xe^x.$$

Example 4.27. Suppose we want to integrate $\ln x = 1 \cdot \ln x$. We choose

$$u = \ln x, \quad v' = 1 \qquad \Longrightarrow \qquad u' = \frac{1}{x}, \quad v = x.$$

So we calculate the integral as

$$\begin{aligned} \int \ln x\,dx &= \int 1 \cdot \ln x\,dx \\ &= \int uv'\,dx \\ &= uv - \int u'v\,dx \\ &= x\ln x - \int \frac{1}{x} \cdot x\,dx \\ &= x\ln x - x + C. \end{aligned}$$

Check:

$$\frac{d}{dx}\left[x\ln x - x + C\right] = \ln x + x \cdot \frac{1}{x} - 1 = \ln x.$$

Example 4.28. Suppose we want to integrate $e^x \cos x$. First let us choose

$$u = \cos x, \quad v' = e^x \qquad \Longrightarrow \qquad u' = -\sin x, \quad v = e^x.$$

So we write our integral as

$$\begin{aligned} \int e^x \cos x\,dx &= \int uv'\,dx \\ &= uv - \int u'v\,dx \\ &= e^x \cos x + \int e^x \sin x\,dx. \end{aligned}$$

Now, we have an integral similar to what we started with, so let us integrate this by parts too, choosing

$$\bar{u} = \sin x, \quad \bar{v}' = e^x \qquad \Longrightarrow \qquad \bar{u}' = \cos x, \quad \bar{v} = e^x.$$

So our original integral becomes

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \cos x + \int \bar{u}\bar{v}'\,dx \\ &= e^x \cos x + \bar{u}\bar{v} - \int \bar{u}'\bar{v}\,dx \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x\,dx.\end{aligned}$$

Note, now on the RHS we have the same integral we started with. Rearranging this, we can make the integral the subject, i.e.

$$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx,$$

$$\therefore \quad 2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x.$$

So finally, we can write

$$\int e^x \cos x\,dx = \frac{1}{2}\left[e^x\left(\cos x + \sin x\right)\right] + C,$$

remembering the constant of integration! Check:

$$\frac{d}{dx}\left[\frac{1}{2}\left[e^x\left(\cos x + \sin x\right)\right] + C\right] = \frac{1}{2}\left\{e^x(\cos x + \sin x) + e^x(-\sin x + \cos x)\right\} = e^x \cos x,$$

which is correct.

4.4 Definite integrals

Remark: all the techniques acquired can be applied to definite integrals.

Example 4.29. Consider the integral

$$I = \int_0^1 (x+1)^3\,dx.$$

Let us choose the following substitution:

$$u = x + 1 \qquad \Longrightarrow \qquad dx = du,$$

then the limits of the integral become

$$x = 0 \to u = 1 \quad \text{and} \quad x = 1 \to u = 2.$$

Therefore, we calculate the integral as

$$I = \int_1^2 u^3\,du = \frac{1}{4}u^4\bigg|_1^2 = \frac{1}{4}\left[2^4 - 1^4\right] = \frac{15}{4}.$$

This is the same as finding the indefinite integral first,

$$\int (x+1)^3\,dx = \frac{1}{4}(x+1)^4 + C,$$

then imposing the limits, so

$$I = \int_1^2 u^3\,du = \left[\frac{1}{4}(x+1)^4 + C\right]\bigg|_{x=1} - \left[\frac{1}{4}(x+1)^4 + C\right]\bigg|_{x=0} = \frac{15}{4}.$$

Example 4.30. Consider the integral

$$I = \int_0^{\frac{\pi}{2}} \cos^3 x \, dx.$$

Let us choose

$$u = \cos^2 x, \quad v' = \cos x \quad \Longrightarrow \quad u' = -2\cos x \sin x, \quad v = \sin x.$$

Then the integral is calculated as

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \cos^2 x \cos x \, dx \\ &= \cos^2 x \sin x \,\Big|_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx \\ &= 2\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx. \end{aligned}$$

Now there are two ways to finish the integration.

1.

$$\begin{aligned} I &= 2\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx \\ &= 2\int_0^{\frac{\pi}{2}} \cos x(1-\cos^2 x) \, dx \\ &= 2\int_0^{\frac{\pi}{2}} \cos x \, dx - 2\int_0^{\frac{\pi}{2}} \cos^3 x \, dx \\ &= 2\sin x\Big|_0^{\frac{\pi}{2}} - 2I \\ &= 2 - 2I. \end{aligned}$$

Finally we can write

$$I = 2 - 2I \quad \Longrightarrow \quad I = \frac{2}{3}.$$

2.

$$\begin{aligned} I &= 2\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx \\ &= 2\int_0^{\frac{\pi}{2}} \sin^2 x d(\sin x). \end{aligned}$$

Using the substitution $\bar{u} = \sin x$, then $x = 0 \to u = 0$ and $x = \frac{\pi}{2} \to u = 1$. So, we have

$$I = 2\int_0^1 u^2 \, du = \frac{2}{3}u^3\Big|_0^1 = \frac{2}{3}.$$

4.5 Numerical integration

Consider evaluating the definite integral

$$\int_a^b f(x)\,dx.$$

In practice, we may only know $f(x)$ at some discrete points, and even if we know $f(x)$, its antiderivative may not be expressed in terms of the functions we know, for example

$$\int \sqrt{1+x^3}\,dx \quad \text{or} \quad \int e^{x^2}\,dx.$$

Since most integrals can not be done analytically, we do them numerically. We do this using a "geometric idea".

4.5.1 Trapezium method

We want to estimate the integral of $f(x)$ on the interval $[a,b]$, which represents the area under the curve $y = f(x)$ from a to b.

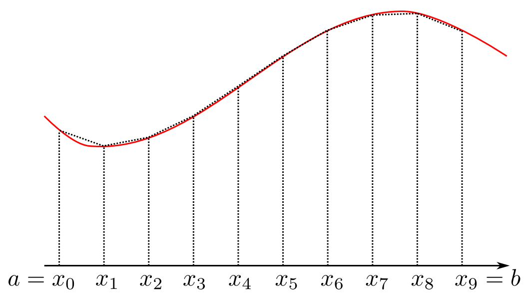

Figure 4.9: *Forming trapeziums with height of the sides dictated by the curve $y = f(x)$ over the interval $[a,b]$.*

We choose n number of pieces. Divide the interval $a \le x \le b$ into n (equal) pieces with points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

On each piece of the interval, we build a trapezium by joining points on the curve by a straight line. We calculate the total area by summing all the area of the trapezia. This is our estimate of the integral.

To start with, let h be the width of one piece of the interval, i.e.

$$h = \frac{b-a}{n},$$

then we have

$$x_k = x_0 + kh, \quad k = 0, 1, 2, \ldots, n. \quad x_0 = a, \quad x_n = b.$$

Let us consider the trapezium based on the piece $[x_{k-1}, x_k]$, whose width is h. The height of the sides of the trapezium are $f(x_{k-1})$ and $f(x_k)$. So the area is

$$h\frac{f(x_{k-1}) + f(x_k)}{2}.$$

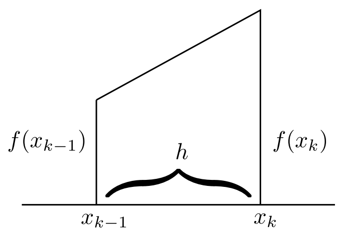

Figure 4.10: *Trapezium constructed over each piece of the interval, where each piece has width h.*

Then the total area under the curve over $[a, b]$ is the sum:

$$\begin{aligned}
Area &= h\frac{f(x_0) + f(1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2} \\
&= \frac{h}{2}\left[(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))\right] \\
&= \frac{h}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + \cdots + f(x_{n-1})) + f(x_n)\right] \\
&= \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right].
\end{aligned}$$

We can think of the sum as follows, we have the two outer sides of the first and last trapezium, then every trapezium in-between shares its sides with its neighbour, therefore we require two lots of the interior sides.[5]

[5] End Lecture 19.