4.3.2 Trigonometric substitution

Example 4.20. We know that

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C,$$

(see pg. 38), since

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Actually, we can work out this integral by a substitution like $x = \sin u$ because we know that

$$1 - \sin^2 u = \cos^2 u,$$

and

$$\frac{dx}{du} = \cos u \quad \text{or} \quad dx = \cos u\, du.$$

Thus, we calculate the integral as

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\, du = \int du = u + C = \sin^{-1} x + C.$$

since

$$\sin^{-1} x = \sin^{-1}(\sin u) = u.$$

Example 4.21.

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C, \quad \text{since} \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Let us try the following

$$x = \tan\theta, \quad \frac{dx}{d\theta} = \frac{1}{\cos^2\theta} = 1 + \tan^2\theta \quad \Longrightarrow \quad dx = (1+\tan^2\theta)d\theta.$$

So we calculate the integral as

$$\int \frac{1}{1+x^2}\, dx = \int \frac{1}{1+\tan^2\theta}(1+\tan^2\theta)\, d\theta = \int d\theta = \theta + C = \tan^{-1} x + C.$$

Since we have

$$x = \tan\theta \quad \Longrightarrow \quad \tan^{-1} x = \tan^{-1}(\tan\theta) = \theta.$$

Example 4.22. Consider the integrand $1/(1+2x^2)$. This is similar to the previous example. If we try

$$\sqrt{2}x = \tan\theta \quad \Longrightarrow \quad dx = \frac{1}{\sqrt{2}}(1+\tan^2\theta)d\theta, \quad \theta = \tan^{-1}(\sqrt{2}x).$$

So we calculate the integral as:

$$\begin{aligned}
\int \frac{1}{1+2x^2}\, dx &= \int \frac{1}{1+(\sqrt{2}x)^2}\, dx \\
&= \frac{1}{\sqrt{2}} \int \frac{1}{1+\tan^2\theta}(1+\tan^2\theta)\, d\theta \\
&= \frac{1}{\sqrt{2}}\theta + C \\
&= \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}x) + C.
\end{aligned}$$

Ceck:

$$\frac{d}{dx}\left[\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x)+C\right]=\frac{1}{\sqrt{2}}\frac{1}{1+(\sqrt{2}x)^2}\cdot\sqrt{2}=\frac{1}{1+2x^2},$$

which is correct.

Example 4.23. Consider the integrand $x/(1+2x^2)$, which is a variation of the above example. So let us try $\sqrt{2}x=\tan\theta$. Then, we can write

$$\begin{aligned}\int\frac{x}{1+2x^2}\,dx &= \int\frac{\frac{1}{\sqrt{2}}\tan\theta}{1+\tan^2\theta}\cdot\frac{1}{\sqrt{2}}(1+\tan^2\theta\,d\theta\\ &= \frac{1}{2}\int\tan\theta\,d\theta\\ &= \frac{1}{2}\int\frac{\sin\theta}{\cos\theta}\,d\theta\\ &= -\frac{1}{2}\int\frac{1}{\cos\theta}d(\cos\theta).\end{aligned}$$

We achieve the last line from knowing that

$$\frac{d(\cos\theta)}{d\theta}=-\sin\theta.$$

Now we use another substitution to tackle the integral we have so far, that is put $u=\cos\theta$, so

$$\begin{aligned}\int\frac{x}{1+2x^2}\,dx &= -\frac{1}{2}\int\frac{1}{u}\,du\\ &= -\frac{1}{2}\ln|u|+C\\ &= -\frac{1}{2}\ln|\cos\theta|+C\\ &= -\frac{1}{2}\ln\left(\frac{1}{1+\tan^2\theta}\right)^{\frac{1}{2}}+C\\ &= -\frac{1}{4}\ln\left(\frac{1}{1+2x^2}\right)+C\\ &= \frac{1}{4}\ln(1+2x^2)+C.\end{aligned}$$

We gained the final result by using the fact that $|\cos\theta|=\sqrt{\cos^2\theta}$, $\cos^2\theta=1/(1+\tan^2\theta)$ and two log rules.

This result can be gained in a quicker way, that is

$$\begin{aligned}\int\frac{x}{1+2x^2}\,dx &= \frac{1}{4}\int\frac{1}{1+2x^2}\,d(1+2x^2)\\ &= \frac{1}{4}\int\frac{1}{u}\,du\\ &= \frac{1}{4}\ln|u|+C\\ &= \frac{1}{4}\ln(1+2x^2)+C,\end{aligned}$$

where we used the substitution $u=1+2x^2$.

4.3.3 Partial fractions

Example 4.24. Consider the integrand $1/(x^2-1)$. We know that $x^2-1=(x+1)(x-1)$. So we may write

$$\frac{1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1},$$

where A and B are numbers to be found. Re-writing the RHS of the above, we have

$$\frac{A}{x-1}+\frac{B}{x+1}=\frac{(A+B)x+A-B}{(x-1)(x+1)}\equiv\frac{1}{x^2-1}.$$

Therefore, we require

$$\begin{aligned} A+B &= 0, \\ A-B &= 1, \end{aligned}$$

which has solution $A=\frac{1}{2}$ and $B=-\frac{1}{2}$. Therefore the integral becomes

$$\begin{aligned} \int\frac{1}{x^2-1}\,dx &= \frac{1}{2}\int\frac{1}{x-1}\,dx-\frac{1}{2}\int\frac{1}{x+1}\,dx \\ &= \frac{1}{2}\int\frac{1}{x-1}d(x-1)-\frac{1}{2}\int\frac{1}{x+1}d(x+1) \\ &= \frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C. \end{aligned}$$

Example 4.25. Consider the integral

$$\int\frac{x^2+6x+1}{3x^2+5x-2}\,dx.$$

We first factories the denominator so that $3x^2+5x-2=(3x-1)(x+2)$. Also, it would be easier if the polynomial in the numerator was of one degree less than the denominator. So we manipulate the integrand as follows:

$$\begin{aligned} \frac{x^2+6x+1}{3x^2+5x-2} &= \frac{\frac{1}{3}(3x^2+18x+3)}{3x^2+5x-2} \\ &= \frac{\frac{1}{3}(3x^2+5x-2+13x+5)}{3x^2+5x-2} \\ &= \frac{1}{3}\left[1+\frac{13x+5}{3x^2+5x-2}\right]. \end{aligned}$$

Now we proceed by changing the fraction in the square bracket using partial fractions, that is

$$\frac{13x+5}{3x^2+5x-2}=\frac{A}{3x-1}+\frac{B}{x+2}=\frac{(A+3B)x+2A-B}{(3x-1)(x+2)}.$$

Matching the coefficients on the numerator we must have

$$\begin{aligned} A+3B=13 \\ 2A-B=5 \end{aligned}$$

Solving simultaeneously we have the solution $A = 4$ and $B = 3$. So the integral becomes

$$\begin{aligned}
\int \frac{x^2+6x+1}{3x^2+5x-2}\,dx &= \frac{1}{2}\int 1+\frac{13x+5}{3x^2+5x-2}\,dx \\
&= \frac{1}{3}\int dx + \frac{1}{3}\int \frac{4}{3x-1}\,dx + \frac{1}{3}\int \frac{3}{x+2}\,dx \\
&= \frac{1}{3}\int dx + \frac{4}{9}\int \frac{1}{3x-1}\,d(3x-1) + \frac{3}{3}\int \frac{1}{x+2}\,d(x+2) \\
&= \frac{1}{3}x + \frac{4}{9}\ln|3x-1| + \ln|x+2| + C.
\end{aligned}$$

4.3.4 Integration by parts

This is equivalent to the product rule for integration. Suppose we have two function $u(x)$ and $v(x)$. Then the product rule states

$$\frac{d}{dx}(uv) = u'v + uv'.$$

Rearranging the above gives

$$uv' = \frac{d}{dx}(uv) - u'v.$$

Integrating both sides we get

$$\begin{aligned}
\int uv'\,dx &= \int \frac{d}{dx}(uv)dx - \int u'v\,dx \\
&= uv - \int u'v\,dx.
\end{aligned}$$

So we write the rule for integration by parts as:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx. \tag{4.6}$$

Given an integral whose integrand is the product of two functions, we choose one to be u and the other to be v', from which we can calculate u' and v. Plugging into the above equation hopefully leads to an easier integration.[4]

[4]End Lecture 18.