Previously, we chose an antiderivative which is correct for the given integrand $1/x^2$. However, recall

$$\frac{d}{dx}\left(-\frac{1}{x}\right) \neq \frac{1}{x^2} \quad \text{if } x = 0.$$

That is $F'(x) = f(x)$ doesn't hold for $-1 \leq x \leq 1$. We have to be sure the function is well defined over the entire interval over which we integrate.

Example 4.8. Consider the function $f(x) = x^3$. Then the integral over the interval $[-1, 1]$ is

$$\int_{-1}^{1} x^3\, dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0.$$

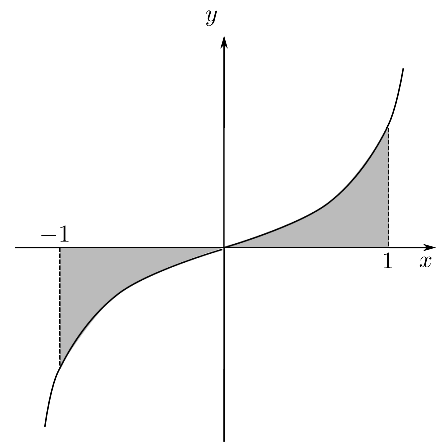

Figure 4.8: *Integrating to find the shaded area under the curve $y = x^3$ on the interval $[-1, 1]$. Have two shaded regions bounded by the curve and the x-axis.*

In this case, the area cancels out. The shaded area is actually given by

$$\int_{0}^{1} x^3\, dx + \left|\int_{-1}^{0} x^3\, dx\right| = \left[\frac{1}{4}x^4\right]_{0}^{1} + \left|\left[\frac{1}{4}x^4\right]_{-1}^{0}\right| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

4.3 Indefinite integrals

So far we know

$$\int f(x)\, dx = F(x) + C, \quad F'(x) = f(x).$$

What function F can we differentiate to get f?

<u>Powers of x:</u>

$$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C, \quad (n \neq -1),$$

since

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n.$$

For $n = -1$,

$$\int \frac{1}{x}\, dx = \ln|x| + C,$$

since for $x > 0$

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

and for $x < 0$,

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}.$$

Special rule:
Let us consider the derivative of the logarithm of some general function $f(x)$, i.e.

$$\frac{d}{dx}\left[\ln(f(x))\right] = \frac{1}{f(x)} \cdot \frac{d}{dx}\left[f(x)\right] = \frac{f'(x)}{f(x)}.$$

This implies that

$$\int \frac{f'(x)}{f(x)}\, dx = \ln(f(x)) + C,$$

where C is some arbitrary constant of integration. Hence, if we can calculate integrals by inspection if the integrand takes the form $f'(x)/f(x)$ by the above formula.

Example 4.9. Consider the the following integral:

$$I = \int \frac{2x+5}{x^2+5x+3}\, dx.$$

Now, if we choose $f(x) = x^2+5x+3$, then $f'(x) = 2x+5$. So, if we differentiate $\ln(f(x))$, in this case we have

$$\frac{d}{dx}\left[\ln(x^2+5x+3)\right] = \frac{2x+5}{x^2+5x+3},$$

by the chain rule. Thus, we know the integral must be

$$I = \ln(x^2+5x+3) + C,$$

where C is some arbitrary constant of integration.

Example 4.10. Consider the following integral:

$$I = \int \frac{3}{2x+2}\, dx.$$

Now, if we choose $f(x) = 2x+2$ then $f'(x) = 2$. However, the numerator of the integrand is 3. Not to worry, as we can simply re-write or manipulate the initial integral as follows:

$$I = \int \frac{3}{2x+2}\, dx = 3\int \frac{1}{2}\frac{2}{2x+2}\, dx = \frac{3}{2}\int \frac{2}{2x+2}\, dx.$$

Since $3/2$ is a constant, which we are able to take out of the integral sign, we need not worry about this and can proceed with the integration using what we have learnt above, giving

$$I = \frac{3}{2}\ln(2x+2) + C.$$

To check, we differentiate the above expression, so

$$\frac{dI}{dx} = \frac{d}{dx}\left[\frac{3}{2}\ln(2x+2)+C\right] = \frac{3}{2}\cdot\frac{1}{2x+2}\cdot 2,$$

which is correct!

This "special case" is an example of a method called *substitution*, and is not limited to integrals which give you logarithms. Nevertheless, it is a good sighter for what's to follow. Together with inspection, it can be extended for other function by choosing suitable substitutions (i.e. $f(x)$).

Trigonometric functions:

$$\int \cos x\, dx = \sin x + C, \quad \text{since } \frac{d}{dx}(\sin x) = \cos x,$$
$$\int \sin x\, dx = -\cos x + C, \quad \text{since } \frac{d}{dx}(-\cos x) = \sin x.$$

Exponential function:

$$\int e^x\, dx = e^x + C, \quad \text{since } \frac{d}{dx}(e^x) = e^x.$$

As with differentiation, we also have a sum rule for integration, that is

$$\int (f(x)+g(x))\, dx = \int (fx)\, dx + \int g(x)\, dx. \tag{4.4}$$

In other words, the antiderivative of a sum is the sum of the antiderivatives.

Also,

$$\int Kf(x)\, dx = K\int f(x)\, dx, \tag{4.5}$$

where K is a constant, i.e. a constant can be taken outside of the integration sign.

Example 4.11.

$$\begin{aligned}\int (3x^4+2x+5)\, dx &= \int 3x^4\, dx + \int 2x\, dx + \int 5\, dx \\ &= 3\int x^4\, dx + 2\int x\, dx + 5\int dx \\ &= \frac{3}{4}x^5 + \frac{2}{3}x^3 + 5x + C.\end{aligned}$$

NOTE: Do not forget the constant C when the integral is indefinite!

Example 4.12.

$$\begin{aligned}\int (x^2-1)(x^4+2)\, dx &= \int x^6 - x^4 + 2x^2 - 2\, dx \\ &= \int x^6\, dx - \int x^4\, dx + 2\int x^2\, dx - \int 2\, dx \\ &= \frac{1}{7}x^7 - \frac{1}{5}x^5 + \frac{2}{3}x^3 - 2x + C.\end{aligned}$$

Example 4.13.

$$\begin{aligned}\int \frac{x^4+1}{x^2}\,dx &= \int \left(\frac{x^4}{x^2}+\frac{1}{x^2}\right)dx\\ &= \int \left(x^2+\frac{1}{x^2}\right)dx\\ &= \int x^2\,dx + \int \frac{1}{x^2}\,dx\\ &= \frac{1}{3}x^3 - \frac{1}{x} + C.\end{aligned}$$

4.3.1 Substitution

We can use substitution to convert a complicated integral into a simple one.

Example 4.14. Consider the indefinite integral with integrand $(2x+3)^{100}$. We make the substitution

$$u = 2x+3 \quad \Longrightarrow \quad \frac{du}{dx} = 2 \quad \text{i.e.} \quad dx = \frac{1}{2}du.$$

So we calculate the integral as follows:

$$\begin{aligned}\int (2x+3)^{100}\,dx &= \int u^{100}\cdot\frac{1}{2}\,du\\ &= \frac{1}{2}\int u^{100}\,du\\ &= \frac{1}{202}(2x+3)^{101} + C.\end{aligned}$$

We can check the result by performing the following differentiation:

$$\frac{d}{dx}\left[\frac{1}{202}(2x+3)^{101}+C\right] = \frac{101}{202}(2x+3)^{100}\cdot 2 = (2x+3)^{100},$$

which is correct.

Example 4.15. Suppose we have the integrand $x(x+1)^{50}$. Let us try the substitution

$$u = x+1, \quad \text{so } x = u-1 \quad \Longrightarrow \quad \frac{du}{dx} = 1 \quad \text{i.e.} \quad dx = du.$$

So we have

$$\begin{aligned}\int x(x+1)^{50}\,dx &= \int (u-1)u^{50}\,du\\ &= \int u^{51}\,du - \int u^{50}\,du\\ &= \frac{1}{52}u^{52} - \frac{1}{51}u^{51} + C\\ &= \frac{1}{52}(x+1)^{52} - \frac{1}{51}(x+1)^{51} + C\end{aligned}$$

Check:

$$\frac{d}{dx}\left[\frac{1}{52}(x+1)^{52} - \frac{1}{51}(x+1)^{51} + C\right] = (x+1)^{51} - (x+1)^{50} = (x+1)^{50}\cdot x,$$

which is correct.

Example 4.16. Consider the integrand $1/(x \ln x)$. Let us try

$$u = \ln x \quad \Longrightarrow \quad \frac{du}{dx} = \frac{1}{x} \quad \text{i.e.} \quad dx = x du.$$

Thus, we calculate the integral as

$$\begin{aligned} \int \frac{1}{x \ln x} \, dx &= \int \frac{1}{xu} \cdot x \, du \\ &= \int \frac{1}{u} \, du \\ &= \ln |u| + C \\ &= \ln |\ln x| + C. \end{aligned}$$

Check:

$$\frac{d}{dx} (\ln |\ln x|) = \frac{1}{\ln x} \cdot \frac{1}{x},$$

which is correct.

Example 4.17. Consider the integrand $1/(1+\sqrt{x})$. Let us try the substitution

$$u = 1 + \sqrt{x} \quad \Longrightarrow \quad \sqrt{x} = u - 1.$$

Differentiating we have

$$\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} \quad \Longrightarrow \quad dx = 2x^{\frac{1}{2}} du = 2(u-1)du.$$

Therefore, we calculate the integral as

$$\begin{aligned} \int \frac{1}{1+\sqrt{x}} \, dx &= \int \frac{1}{u} \cdot 2(u-1) \, du \\ &= 2 \int \left(1 - \frac{1}{u}\right) \, du \\ &= 2 \int du - 2 \int \frac{1}{u} \, du \\ &= 2u - 2 \ln |u| + C \\ &= 2(1+\sqrt{x}) - 2 \ln |1+\sqrt{x}| + C. \end{aligned}$$

Check:

$$\frac{d}{dx} \left[2(1+\sqrt{x}) - 2 \ln |1+\sqrt{x}| + C \right] = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}(1+\sqrt{x})} = \frac{1+\sqrt{x}-1}{\sqrt{x}(1+\sqrt{x})} = \frac{1}{1+\sqrt{x}}.$$

Example 4.18. Consider the integrand $\sin(3x+1)$. Let us try the substitution

$$u = 3x + 1 \quad \Longrightarrow \quad \frac{du}{dx} = 3 \quad \text{i.e.} \quad dx = \frac{1}{3} du.$$

So integrate as follows:

$$\begin{aligned} \int \sin 3x + 1 \, dx &= \frac{1}{3} \int \sin u \, du \\ &= -\frac{1}{3} \cos u + C \\ &= -\frac{1}{3} \cos(3x+1) + C. \end{aligned}$$

Check:

$$\frac{d}{dx} \left[-\frac{1}{3} \cos(3x+1) + C \right] = +\frac{1}{3} \cdot 3 \cdot \sin(3x+1) = \sin(3x+1).$$

Example 4.19. Consider the integrand $\sin\left(\frac{1}{x}\right)/x^2$. Let us try the substitution

$$u = \frac{1}{x} \quad \Longrightarrow \quad \frac{du}{dx} = -\frac{1}{x^2} \quad \text{i.e.} \quad dx = -x^2 du.$$

So we calculate the integral as follows:

$$\begin{aligned}
\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\, dx &= \int \frac{\sin u}{x^2} \cdot (-x^2)\, du \\
&= -\int \sin u\, du \\
&= \cos u + C \\
&= \cos\left(\frac{1}{x}\right) + C.
\end{aligned}$$

Check:

$$\frac{d}{dx}\left[\cos\left(\frac{1}{x}\right) + C\right] = -\sin\left(\frac{1}{x}\right) \cdot \left(-x^{-2}\right) = \frac{\sin\left(\frac{1}{x}\right)}{x^2},$$

which is correct.

How do we find a suitable substitution? Usually by observation and using our knowledge of differentiation. Or we put

$$u = f(x), \quad \text{then} \quad \frac{du}{dx} = f'(x) \quad \Longrightarrow \quad du = f'(x)dx.$$

If we have

$$\frac{d}{dx}[f(x)] = f'(x),$$

then we can write

$$d[f(x)] = f'(x)dx.$$

For instance, in the case

$$\int \frac{1}{x \ln x}\, dx,$$

we know that

$$\frac{d(\ln x)}{dx} = \frac{1}{x} \quad \text{i.e.} \quad d(\ln x) = \frac{1}{x}dx.$$

So we write the integral as

$$\int \frac{1}{x\ln x} dx = \int \frac{1}{x \ln x} \cdot x d(\ln x) = \int \frac{1}{\ln x} d(\ln x).$$

Therefore we know $u = \ln x$ will work!

Similarly we know

$$\frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2}, \quad \text{i.e.} \quad \frac{1}{x^2}dx = -d\left(\frac{1}{x}\right),$$

so the integral from example 4.19 can be written as

$$\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\, dx = -\int \sin\left(\frac{1}{x}\right)\, d\left(\frac{1}{x}\right) \quad \Longrightarrow u = \frac{1}{x}.$$

Also, from example 4.14, we have

$$\int (2x+3)^{100}\,dx = \frac{1}{2}\int (2x+3)^{100}\,d(2x+3),$$

since

$$\frac{d(2x+3)}{dx} = 2 \quad\Longrightarrow\quad dx = \frac{1}{2}d(2x+3) \quad\Longrightarrow\quad \text{let } u = 2x+3,$$

which is the same as what we tried earlier.[3]

[3]End Lecture 17.