Notice that if $A(x)$ is an antiderivative of $f(x)$, then $A(x)+C$, where C is any constant, is also an antiderivative of $f(x)$.

We employ the following notation:

$$\int f(x)\,dx = F(x) + C, \quad F'(x) = f(x).$$

$\int f(x)\,dx$ is referred to as an *indefinite integral*, where $f(x)$ is called the *integrand*.

4.2 Fundemental Theorem of Calculus

Example 4.4. Consider $f(t) = t^2$. We want to calculate the area under $y = t^2$ between 0 and 1.

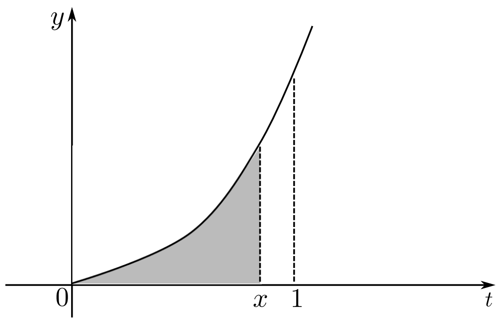

Figure 4.5: *Integrating to find the shaded area under the curve $y = t^2$.*

The area up to x is represented by

$$A(x) = \int_0^x t^2\,dt.$$

We know $A'(x) = x^2 = f(x)$. We need to find $A(x)$. Can we think of a function whose derivative is x^2? Yes! Consider $\frac{1}{3}x^3$. In fact, for any constant C,

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + C\right) = x^2.$$

Our function $A(x)$ is one of the family of functions $\frac{1}{3}x^3 + C$, but which one?

We need to fix the constant C. We have $A(0) = 0$, so if $A(x) = \frac{1}{3}x^3 + C$ then we must have $C = 0$. So finially $A(x) = \frac{1}{3}x^3$. Hence, $A(1) = \frac{1}{3}$, which is the answer to the original question.

We can interpret the above result as

$$\int_0^1 t^2\,dt = A(1) - A(0), \quad A'(x) = x^2.$$

This is actually the fundamental theorem of calculus, which says: If $F'(x) = f(x)$ between a and b, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{4.3}$$

The integral $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ from a (the lower limit) to b (the upper limit).

$\int_a^b f(x)\,dx$ is the limiting value of the sequences derived from the method of divisions when calculating the area under the curve, described previously. Thus, the integral may represent areas under curves.

Example 4.5. Suppose we want to integrate the function x^4 over the interval $[2, 3]$. That is, we want to calculate

$$\int_2^3 x^4\,dx.$$

Think of a function whose derivative is x^4. The answer is $\frac{1}{5}x^5$. So, we write

$$\int_2^3 x^4\,dx = \left[\frac{1}{5}x^5\right]_2^3 = \left[\frac{1}{5}3^5\right] - \left[\frac{1}{5}2^5\right] = \frac{211}{5}.$$

Example 4.6. Suppose we want to integrate the function $1/x^2$ over the interval $[1, 2]$. That is, we want to calculate

$$\int_1^2 \frac{1}{x^2}\,dx.$$

If we put $F(x) = -1/x$, then

$$F'(x) = \frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}.$$

So we can write

$$\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = \left[-\frac{1}{2}\right] - \left[-\frac{1}{1}\right] = \frac{1}{2}.$$

The integral $\int_1^2 \frac{1}{x^2}\,dx$ represents the area under the curve $y = \frac{1}{x^2}$ between 1 and 2, therefore we understand that this integral makes some geometrical sense.

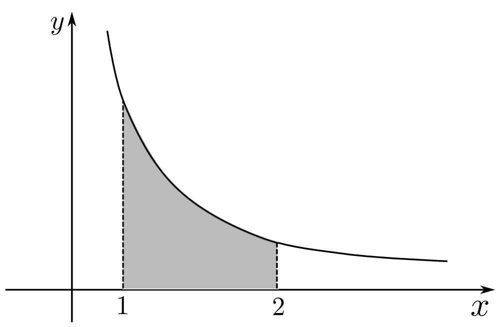

Figure 4.6: *Integrating to find the shaded area under the curve $y = \frac{1}{x^2}$ on the interval* $[1, 2]$.

Example 4.7. Consider the integral of $1/x^2$, but this time on the interval $[-1, 1]$.

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_{-1}^{1} = \left[-\frac{1}{1}\right] - \left[-\frac{1}{-1}\right] = -2.$$

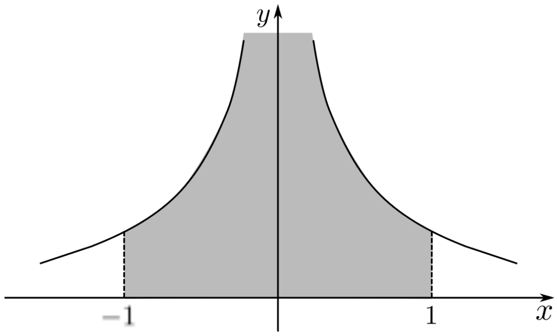

Figure 4.7: *Integrating to find the shaded area under the curve* $y = \frac{1}{x^2}$ *on the interval* $[-1, 1]$*. However, the curve has a vertical asymptote at* $x = 0$*.*

However, the area under the curve in this interval is not -2! What is wrong here? Think about this for next time.[2]

[2]End Lecture 16.