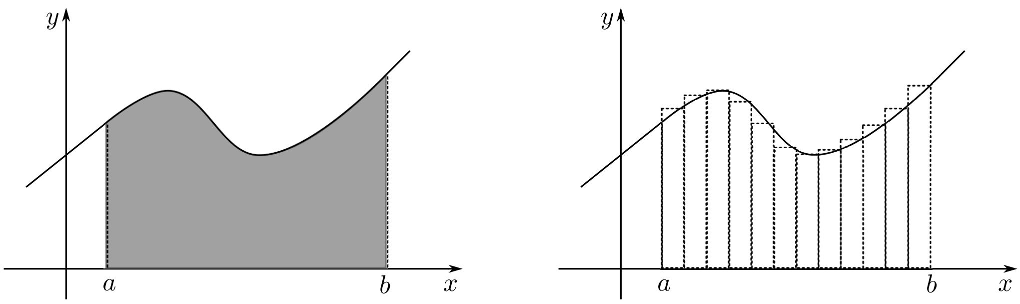
Chapter 4

Integration

4.1 The basic idea

We are interested in calculating areas under curves.

Example 4.1.



(a) *Shaded area under $y = f(x)$.*

(b) *Rectangular strips under $y = f(x)$.*

Figure 4.1: *Integration calculates the shaded area under the curve $y = f(x)$, we could do this by dividing the area into rectangles and summing the area of all rectangles.*

How do we do it?

1. We divide the interval $a \leq x \leq b$ into pieces (say equal length).
2. We build a rectangle on each piece, where the top touches the curve.
3. We calculate the total area of the rectangles.

We say, if the division is very fine, we will get a good measure of the area we want. We watch what happens as we make the division of the "strips" finer and finer.

Example 4.2. Consider the function $f(x) = x$ on the interval $0 \le x \le 1$.

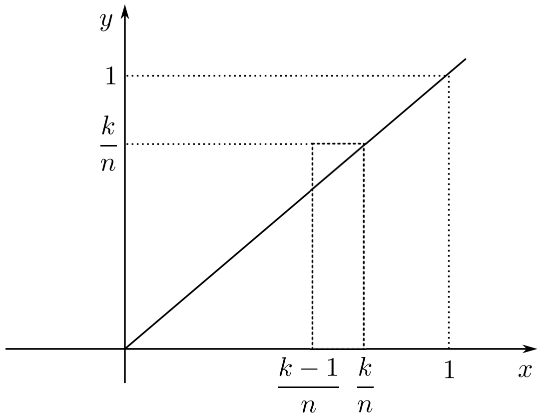


Figure 4.2: *Integrating under the curve $y = x$, from $x = 0$ to $x = 1$.*

We divide $[0, 1]$ into n equal pieces. The divisions occur at

$$0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{k-1}{n}, \frac{k}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1.$$

We have $n + 1$ points and we put a rectangle on each point. The rectangle between $\frac{k-1}{n}$ and $\frac{k}{n}$ will have height $f(\frac{k}{n}) = \frac{k}{n}$ (see Fig. 4.2), and the area of this rectangle is

$$\underbrace{\frac{k}{n}}_{\text{height}} \cdot \underbrace{\frac{1}{n}}_{\text{width}} = \frac{k}{n^2}.$$

The sum of the area of all rectangles on the interval is

$$\begin{aligned}
\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{k}{n^2} + \cdots + \frac{n}{n^2} &= \frac{1}{n^2}(1 + 2 + \cdots + k + \cdots + n) \\
&= \frac{1}{n^2}\frac{n(n+1)}{2} \\
&= \frac{1}{2}\left(\frac{n+1}{n}\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{n}\right),
\end{aligned}$$

and

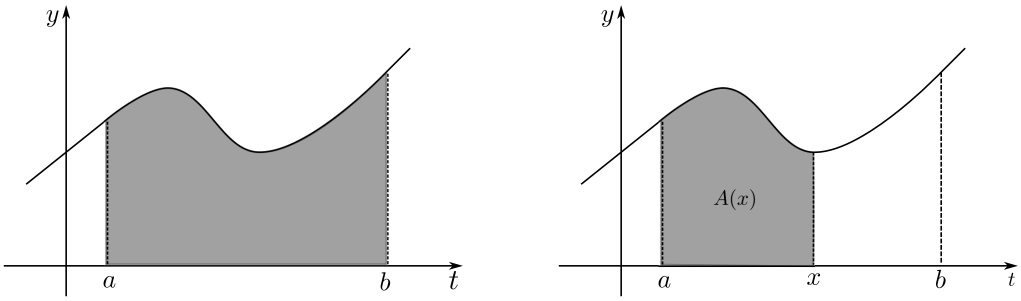
$$\frac{1}{2}\left(1 + \frac{1}{n}\right) \to \frac{1}{2}, \quad \text{as } n \to \infty.$$

That is the sum is approaching the actual area $\frac{1}{2}$.

Therefore, as we increase n so as to get finer divisions, the area approaches the exact area under the curve. You could do this for all functions, however, there is a much quicker way.

Idea: we want to find the area under the curve, let us call it $y = f(t)$.

Example 4.3.



(a) *Shaded area under $y = f(t)$.* (b) *Integrating $y = f(t)$ up to point x.*

Figure 4.3: *Writing the shaded area under the curve $y = f(t)$ as a function of x, where x is point within the desired interval $[a, b]$.*

We think of the area as a function of x, say $A(x)$. If we know $A(x)$, then we know the area, i.e. $A(b) - A(a)$. So we want to find this function $A(x)$.

We don't know A, but we can say something about it. Think about how it is related to f. What we know is that

$$A'(x) = f(x). \tag{4.1}$$

Why can we say this? We need to understand what happens to

$$\frac{A(x+h) - A(x)}{h} \quad \text{as } h \to 0. \tag{4.2}$$

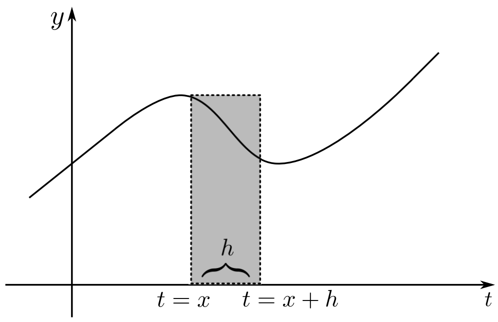


Figure 4.4: *Consider one "strip" under the curve $y = f(t)$, to find the limit* (4.2).

The difference $A(x+h) - A(x)$ is the area between $t = x$ and $t = x + h$. So the area is roughly rectangular (if h is small) with height $f(x)$ and base h. So the area is approximately $f(x) \cdot h$. Therefore

$$A(x+h) - A(x) \approx f(x) \cdot h \quad \Longrightarrow \quad \frac{A(x+h) - A(x)}{h} \approx f(x),$$

and

$$\frac{A(x+h)-A(x)}{h} \to f(x) \quad \text{as } h \to 0.$$

Thus, by the definition of the derivative, we have $A'(x) = f(x)$. We defined $A(x)$ as the *antiderivative* of $f(x)$.[1]

[1] End Lecture 15.