3.3.3 Population growth

Example 3.11. Suppose a certain bacterium divides each hour. Each hour the population doubles:

Hours	1	2	3	4	5	...
Population	2	4	8	16	32	...

After t hours you have 2^t times more bacteria than what you started with. In general, we write it as an exponential form.

If a population, initially P_0 grows exponentially with growth rate λ (where $\lambda > 0$), then at time t, the population is

$$P(t) = P_0 e^{\lambda t}.$$

Example 3.12. Bacterium divides every hour.

1. What is the growth rate?

   We know that when $t = 1$ hour, we are supposed to have

   $$P = 2P_0,$$

   so

   $$2P_0 = P_0 e^{\lambda \cdot 1} \quad \Longrightarrow \quad 2 = e^{\lambda} \quad \Longrightarrow \quad \lambda = \ln 2 \approx 0.693.$$

2. How long for 1 bacterium to become 1 billion?

   $$P_0 = 1, \quad \lambda = 0.693, \quad P = 10^9,$$

   therefore we may write

   $$P = P_0 e^{\lambda t} \quad \Longrightarrow \quad 10^9 = e^{0.693t},$$

   taking logarithms of both sides and re-arranging for t, we have

   $$t = \frac{9 \ln 10}{0.693} \approx 30 \text{ hours}.$$

3.3.4 Interest rate

An annual interest rate of 5% tells you that £100 investment at the start of the year grows to £105. Each subsequent year you leave your investment, it will be multiplied by the factor 1.05.

In general, if you initially invest M_0 (amount) with an annual interest rate r (given as percentage/100), then after t years you have

$$A = M_0(1 + r)^t,$$

where A is the future value. We could write this as an exponential as follows:

$$A(t) = M_0 e^{\lambda t} = M_0(1 + r)^t.$$

Taking logarithms we have

$$\lambda t = t \ln(1+r),$$

so we may write

$$A(t) = M_0 e^{\ln(1+r)t}.$$

ASIDE: In fact, for small r, $\ln(1+r) \approx r$. [3]

[3] End Lecture 14.