3.3 Exponential growth and decay

Let $y = f(t)$ represent some physical quantity, such as the volume of a substance, the population of a certain species or the mass of a decaying radioactive substance. We want to measure the growth or decay of $f(t)$.

Definition 3.1. *We define the "relative rate of growth (or decay)" as*

$$\frac{\textit{actual rate of growth (or decay)}}{\textit{size of } f(t)} = \frac{f'(t)}{f(t)} = \frac{dy/dt}{y}. \tag{3.1}$$

In many applications, the growth (or decay) rate of the given physical quantity is constant, that is

$$\frac{dy/dt}{y} = \alpha, \quad \text{or } \frac{dy}{dt} = \alpha y, \quad \alpha = \text{constant}.$$

This is a *differential equation* whose solution is

$$y(t) = ce^{\alpha t},$$

where constant c is determined by an initial condition, say, $y(0) = y_0$ (given). Therefore we have

$$y(t) = y_0 e^{\alpha t}.$$

This means that if you start with y_0, after time t you have $y(t)$.

If $\alpha > 0$, the quantity is increasing (growth),

If $\alpha < 0$, the quantity is decreasing (decay).

3.3.1 Radioactive decay

Atoms of elements which have the same number of protons but differing numbers of neutrons are referred to as isotopes of each other. Radioisotopes are isotopes that decompose and in doing so emit harmful particles and/or radiation.

It has been found experimentally that the atomic nuclei of so-called radioactive elements spontaneously decay. They do it at a characteristic rate.

If we start with an amount M_0 of an element with decay rate λ (where $\lambda > 0$), then after time t, the amount remaining is

$$M = M_0 e^{-\lambda t}.$$

This is the radioactive decay equation. The proportion left after time t is

$$\frac{M}{M_0} = e^{-\lambda t},$$

and the proportion decayed is

$$1 - \frac{M}{M_0} = 1 - e^{-\lambda t},$$

3.3.2 Carbon dating

Carbon dating is a technique used by archeologists and others who want to estimate the age of certain artefacts and fossils they uncover. The technique is based on certain properties of the carbon atom.

In its natural state, the nucleus of the carbon atom C^{12} has 6 protons and 6 neutrons. The isotope carbon-14, C^{14}, has 6 protons and 8 neutrons and is radioactive. It decays by beta emission.

Living plants and animals do not distinguish between C^{12} and C^{14}, so at the time of death, the ratio C^{12} to C^{14} in an organism is the same as the ratio in the atmosphere. However, this ratio changes after death, since C^{14} is converted into C^{12} but no further C^{14} is taken in.

Example 3.8. Half-lives: how long before half of what you start with has decayed? When do we get $M = \frac{1}{2}M_0$? We need to solve

$$\frac{M}{M_0} = \frac{1}{2} = e^{-\lambda t},$$

taking logarithms of both sides gives

$$\ln\left(\frac{1}{2}\right) = -\lambda t \quad \Longrightarrow \quad t = \frac{\ln(2)}{\lambda}.$$

So, the half-life, $T_{1/2}$ is given by

$$T_{1/2} = \frac{\ln(2)}{\lambda}.$$

If λ is in "per year", then $T_{1/2}$ is in years.

Example 3.9. Carbon-14 (C^{14}) exists in plants and animals, and is used to estimate the age of certain fossils uncovered. It is also used to trade metabolic pathways. C^{14} is radioactive and has a decay rate of $\lambda = 0.000125$ (per year). So we can calculate its half-life as

$$T_{1/2} = \frac{\ln 2}{0.000125} \approx 5545 \text{ years}.$$

Example 3.10. A certain element has $T_{1/2}$ of 10^6 years

1. What is the decay rate?

$$\lambda = \frac{\ln 2}{T_{1/2}} \approx \frac{0.693}{10^3} \approx 7 \times 10^{-7} \text{ (per year)}.$$

2. How much of this will have decayed after 1000 years? The proportion remaining is

$$\frac{M}{M_0} = e^{-\lambda t} = e^{-7\times 10^{-7}\times 10^3} = e^{-7\times 10^{-4}} \approx 0.9993.$$

The proportion decayed is

$$1 - \frac{M}{M_0} \approx 1 - 0.9993 = 0.0007.$$

3. How long before 95% has decayed?

$$\frac{M}{M_0} = 1 - 0.95 = 0.05 = e^{-7\times 10^{-7}t},$$

taking logarithms of both sides we have

$$\ln(0.05) = -7 \times 10^{-7} t$$

which implies

$$t = \frac{\ln(0.05)}{-7 \times 10^{-7}} \approx \frac{-2.996}{-7 \times 10^{-7}} \approx 4.3 \times 10^6 \text{ (years)}.$$

WARNING: Half-life $T_{1/2}$ of a particular element does not mean that in $2 \times T_{1/2}$, the element will completely decay.[2]

[2] End Lecture 13.