Chapter 3

Exponentials and Logarithms

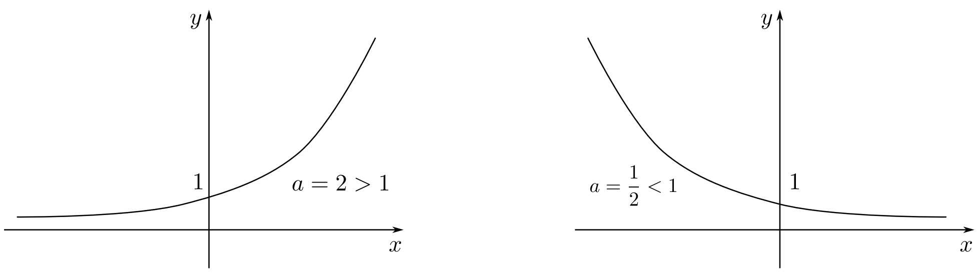
3.1 Exponentials

An exponential function is a function of the form

$$f(x) = a^x,$$

where a is a positive constant.

Example 3.1.



(a) $y = 2^x$. (b) $y = \left(\frac{1}{2}\right)^x$.

Figure 3.1: *The domain:* $-\infty < x < +\infty$*; the range:* $(0, +\infty)$.

For

$a > 1$, $f(x)$ increases as x increases.

$a < 1$, $f(x)$ decreases as x increases.

$a = 1$, $f(x) = 1$.

$a^0 = 1$ for each a, so the graph always passes through the point $(0, 1)$.

3.1.1 Slope of exponentials

First let us consider the slope at $x = 0$.

Example 3.2. Suppose we have $f(x) = 2^x$, then applying the definition of the derivative we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{2^h - 1}{h}.$$

h	$f'(0) \approx$
0.1	0.7177
0.01	0.6955
0.001	0.6933
0.0001	0.6932

So for $f(x) = 2^x$, $(a = 2)$, we have slope ≈ 0.693 at $x = 0$.

Similarly, for $f(x) = 3^x$, $(a = 3)$, we have slope ≈ 1.698 at $x = 0$.

Therefore, we expect that there is a number between 2 and 3 such that the slope at $x = 0$ is 1. This number is called e, where $e \approx 2.718281828459\ldots$. The number e is irrational.

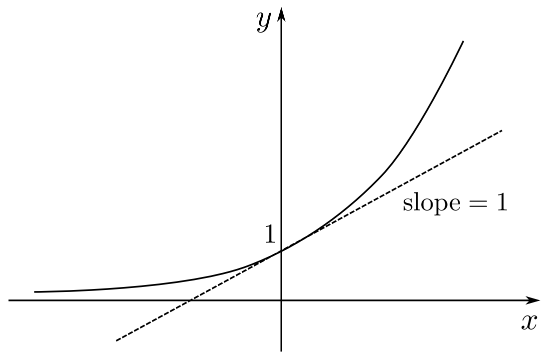


Figure 3.2: *Graph of $y = e^x$, which has tangent with slope of 1 at $x = 1$.*

The fact that the slope is 1 at $x = 0$ tells us that

$$\frac{e^h - e^0}{h} = \frac{e^h - 1}{h} \to 1, \quad ash \to 0.$$

Therefore, if h is small, then $e^h - 1 \approx h$, i.e.

$$e^h \approx 1 + h.$$

> We call $f(x) = e^x = \exp(x)$ the *exponential function*.

To find the slope at $x = c$, we need to look at

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0} \frac{e^{c+h} - e^c}{h} = \lim_{h\to 0} \frac{e^c(e^h - 1)}{h} = e^c \lim_{h\to 0} \frac{e^h - 1}{h} = e^c,$$

i.e. the derivative of e^x is itself,

$$\frac{d}{dx}(e^x) = e^x.$$

Example 3.3. Consider $f(x) = e^{\sqrt{1+x}}$. Here we will employ the chain rule. Choose $g(x) = \sqrt{1+x}$ and $f(u) = e^u$, so we have $g'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ and $f'(u) = e^u$.

$$\begin{aligned} \frac{d}{dx}\left(e^{\sqrt{1+x}}\right) &= f'(g(x))g'(x) \\ &= e^{\sqrt{1+x}}\frac{1}{2}(1+x)^{-\frac{1}{2}} \\ &= \frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}}. \end{aligned}$$

3.2 The natural logarithm

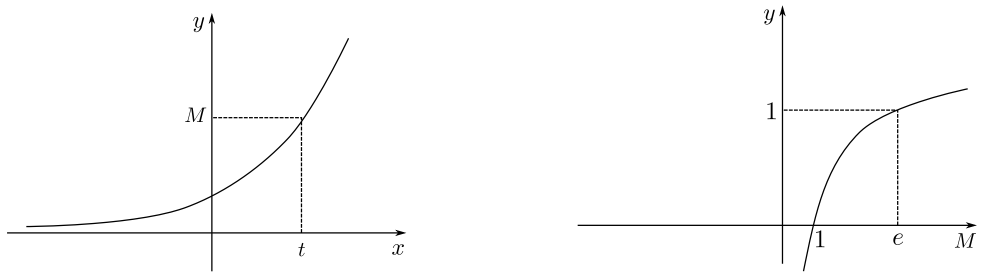
Consider the inverse function of $f(x) = e^x$. In $f(x)$, every positive number occurs as the exponential of something, i.e. $M = e^t$ for an appropriate t. The number t is called $\ln(M)$: the natural logarithm of M. In other words

"$\ln M$ is the number whose exponential is M": $e^{\ln M} = M$.

In this way, we define the function of the natural logarithm

$$g(M) = \ln M.$$

Example 3.4.



(a) $y = e^x$. (b) $y = \ln x$.

Figure 3.3: *For* $\ln x$ *the domain:* $(0, +\infty)$*; the range:* $(-\infty, +\infty)$*. The graph* $y = e^x$ *has a horizontal asymptote at* $y = 0$*, while* $y = \ln x$ *has a vertical asymptote at* $x = 0$*.*

By definition,

$$e^{\ln M} = M, \quad \ln(e^x) = x,$$

which means that if you perform ln(exp) or take the exp(ln), then we get back to where we started.

3.2.1 Characteristic properties

1. $\ln(MN) = \ln M + \ln N$.
2. $\ln(M^p) = p \ln M$.

Logarithms are used among other things to solve "exponential equations".

Example 3.5. Find x, given $3^x = 7$. Taking the logarithm of both sides we have

$$\ln(3^x) = \ln 7 \qquad \Longrightarrow \qquad x \ln 3 = \ln 7.$$

Rearranging we have

$$x = \frac{\ln 3}{\ln 7} \approx \frac{1.95}{1.10} \approx 1.77.$$

Exercise 3.1. Show that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Hint: put $y = \ln x$.

Example 3.6. Consider the function $f(x) = \ln(\cos x)$. Choose $g(x) = \cos x$ and $f(u) = \ln u$, so we have $g'(x) = -\sin x$ and $f'(u) = 1/u$. Thus

$$\begin{aligned} \frac{d}{dx}(\ln(\cos x)) &= f'(g(x))g'(x) \\ &= \frac{1}{\cos x} \cdot (-\sin x) \\ &= -\tan x. \end{aligned}$$

Similarly we have

$$\begin{aligned} \frac{d}{dx}(\sin(\ln x)) &= \cos(\ln x) \cdot \frac{1}{x} \\ &= \frac{\cos(\ln x)}{x}. \end{aligned}$$

Differentiation of other exponentials

In order to differentiate for example 3^x, we must express it in firms of e^x :

$$3 = e^{\ln 3} \qquad \Longrightarrow \qquad 3^x = (e^{\ln 3})^x = e^{x \ln 3}$$

Therefore we calculate the derivative of 3^x as follows:

$$\begin{aligned} \frac{d}{dx}(3^x) &= \frac{d}{dx}(e^{x \ln 3}) \\ &= f'(g(x))g'(x) \\ &= e^{x \ln 3} \cdot \ln 3 \\ &= 3^x \cdot \ln 3. \end{aligned}$$

Here we chose $g(x) = x \ln 3$ and $f(u) = e^u$ so that $g'(x) = \ln 3$ and $f'(u) = e^u$.

In general, for any positive constant a

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

NOTATION: $\log_e x = \ln x$, is the "proper" way of writing the natural logarithm.

3.2.2 Logarithms base a

We can also define $\log_a(x)$ to be the number m, i.e. $\log_a(x) = m$ is such that $a^m = x$. In this way we can think of logarithms as a different form of writing powers.

Example 3.7. $\log_{10}(1000) = 3$.

The derivative of $\log_a(x)$ is

$$\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x \ln(a)}.$$

Exercise 3.2. Try to show the above statement is true. Hint: use the chain rule.[1]

[1] End Lecture 12.