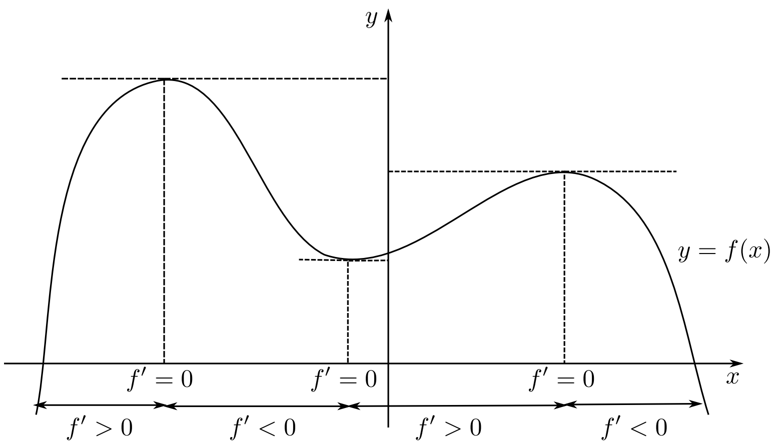
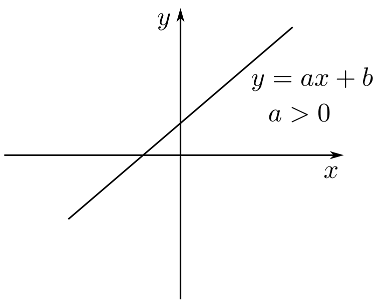
2.3 Maxima, minima and second derivatives

Consider the following question: given some function f, where does it achieve its maximum or minimum values?

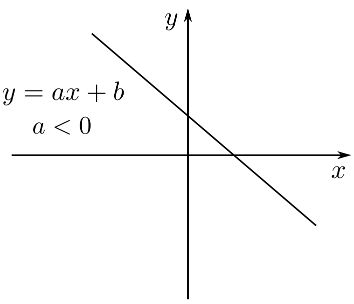
First let us examine in more detail what $f'(x)$ tells us about $f(x)$.



(a) *Maxima and minima of some graph, $y = f(x)$.*



(b) *Positive slope.*



(c) *Negative slope.*

Figure 2.16: *Displaying the change in first derivative before, at and after maxima and minima.*

We notice that:

If $f(x)$ is increasing, then $f'(x) > 0$,

If $f(x)$ is decreasing, then $f'(x) < 0$.

Therefore, troughs and humps occur at places through which f' changes sign, i.e. when $f' = 0$, where

$$\text{Trough} = \text{local minimum}, \quad \text{Hump} = \text{local maximum}.$$

The derivative gives us a way of finding troughs and humps, and so provides good places to look for maximum and minimum values of a function.

Example 2.21. Find the maximum and minimum values of the function

$$f(x) = x^3 - 3x, \quad \text{on the domain } -\frac{3}{2} \leq x \leq \frac{3}{2}.$$

Differentiating $f(x)$ we have

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1).$$

Hence, at $x = \pm 1$, we have $f'(x) = 0$. The value of the function at these points are

$$f(1) = -2, \quad f(-1) = 2.$$

It also maybe possible for the function f to achieve its maximum or minimum at the ends of the domain. Therefore we calculate

$$f\left(\frac{3}{2}\right) = -\frac{9}{8}, \quad f\left(-\frac{3}{2}\right) = \frac{9}{8}.$$

So comparing the four values of f, we know where $f(x)$ is biggest or smallest. So we finally conclude

$$f(1) = -2 \text{ (min)}, \quad f(-1) = 2 \text{ (max)},$$

i.e. we have a minimum at the point $(1, -2)$ and a maximum at $(-1, 2)$.

If we had been looking in the range $-3 \leq x \leq 3$, then at the ends

$$f(3) = 18 \text{ (max)}, \quad f(-3) = -18 \text{ (min)}.$$

To draw the graph of f, we need to find the values of f at some important points, such as when $x = 0$, when $f(x) = 0$ and $f'(x) = 0$. We already know the points at which $f'(x) = 0$.

When $x = 0$ we have $f(0) = 0$, i.e. the graph passes through the point $(0, 0)$.

When $f(x) = 0$, we need to solve the equation

$$x^3 - 3x = x(x^2 - 3x) = 0.$$

So either $x = 0$ (we already have this point), or

$$x^2 - 3 = 0 \quad \Longrightarrow \quad x = \pm\sqrt{3}.$$

So the graph also passes through the points $(0, \sqrt{3})$ and $(0, -\sqrt{3})$.

Now let us examine the derivative further, $f'(x) = 3x^2 - 3$.

$$\begin{array}{lll} f'(x) > 0 & \text{when} & x < -1, \\ f'(x) \leq 0 & \text{when} & -1 \leq x \leq 1, \\ f'(x) > 0 & \text{when} & x > 1. \end{array}$$

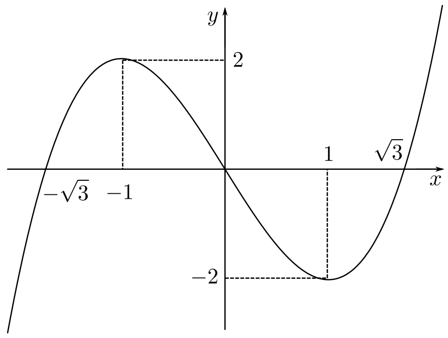


Figure 2.17: *Sketch of $f(x) = x^3 - 3x$, displaying turning points (maximum and minimum).*

Sketching graphs, things to remember:

1. Find $f(x)$ when $x = 0$, i.e. where the graph cuts the y-axis.
2. Find x when $f(x) = 0$, i.e. when the graphs cuts the x-axis.
3. Find x when $f'(x) = 0$, i.e. the *stationary* points of the graph. (Plug into f(x) to find y value).
4. Determine the sign of $f'(x)$ on either side of the stationary points to determine weather stationary points are minimum, maximum or points of inflection.

Basic principles:

Let $f : [a, b] \to \mathbb{R}$. If f achieves a local maximum or local minimum at x, then either:

(i) $f'(x) = 0$, or

(ii) $x = a$, or

(iii) $x = b$, or

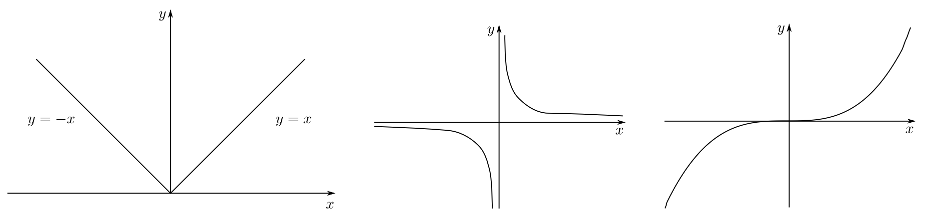
(iv) where $f'(x)$ doesn't exist.

So to find the local maximum/minimum of f, it suffices to list possibilities in (i)-(iv) and then check.

Example 2.22. Consider $f(x) = |x|$, on the domain $-1 \leq x \leq 1$. We know $f'(0)$ doesn't exist, but at $x = 0$, $f(x)$ achieves its minimum value of 0.

Example 2.23. Consider $f(x) = 1/x$ on the domain $-2 \leq x \leq 2$. There is no maximum or minimum on this range. $f'(x) = -1/x^2$ is not defined at $x = 0$ (it actually tends to $\pm\infty$ either side of $x = 0$).

Example 2.24. Consider $f(x) = x^3$ on the domain $-2 \leq x \leq 2$. $f'(x) = 3x^2$, which is zero if $x = 0$. So zero is a stationary point for x^3, but it is neither a hump or a trough. It's a point of inflection.



(a) $y = |x|$ (Ex. 2.22). (b) $y = \frac{1}{x}$ (Ex. 2.23). (c) $y = x^3$ (Ex. 2.24).

Figure 2.18: *Different options for when* $f'(x) = 0$.

2.3.1 Second derivative

To characterise troughs and humps (local maximums and minimums), we need the knowledge of second derivatives.

If we start with some function, say $f(x) = x^2$, we know the derivative $f'(x)$ is well defined for every x, and $f'(x) = 2x$. We can view $f'(x)$ itself as a function, so we may differentiate it again to have

$$\frac{d}{dx}(f'(x)) = \frac{d^2}{dx^2}(f(x)) = f''(x) = 2.$$

In this way, we can define f'', $f^{(3)}$, $f^{(4)}$ and so on.

Example 2.25.

$f(x) = x^n$, n is a positive whole number.

$f'(x) = nx^{n-1}$, $f''(x) = n(n-1)x^{n-2}$,

$f^{(3)}(x) = n(n-2)(n-2)x^{n-3}$, ...

$f^{(n)}(x) = n(n-1)(n-2)\dots 2 \cdot 1 = n!$, recall ! $\equiv$ Factorial.

$f^{(m)}(x) = 0$, $m > n$.

In physics, if $f(t)$ represents distance as a function of time, then $f'(t)$ represents speed (i.e. the rate of change in distance), and $f''(t)$ represents acceleration, i.e. the rate of change of speed. This is the key to understanding how things move, in particular under gravity.

The geometric interpretation of f'':

1. If $f'' > 0$, then the slope of the tangent line is increasing in value (from left to right), so possible shapes for $f(x)$ are like:

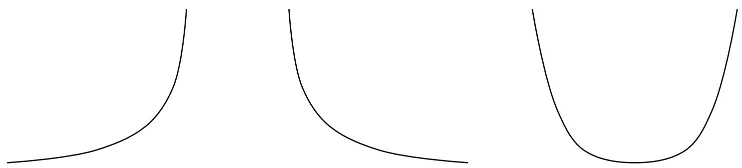


Figure 2.19: *Slope of tangent increases in value (possibly negative to positive).*

   Therefore if $f'(c) = 0$ and $f''(c) > 0$, then around c, $f(x)$ is a trough and we can expect a local minimum value of f at $x = c$.

   Example 2.26.

$$\begin{aligned} f(x) &= x^3 - 3x, \\ f'(x) &= 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1), \\ f''(x) &= 6x. \end{aligned}$$

   At $x = 1$, $f'(x) = 0$, $f''(x) = 6 > 0$, so we have a trough at $x = 1$.

2. If $f'' < 0$, then the slope of the tangent line is decreasing (from left to right), so possible shapes for $f(x)$ are like:

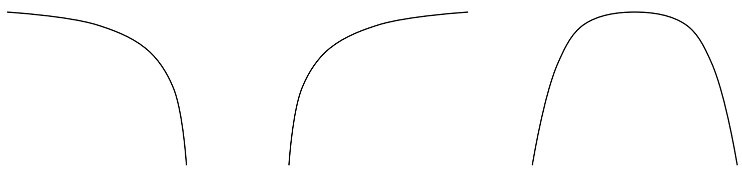


Figure 2.20: *Slope of tangent decreases in value (possibly positive to negative)* .

   Therefore if $f'(c) = 0$ and $f''(c) < 0$, then around c, $f(x)$ is a hump, and we expect a local maximum value of f at c.

   Example 2.27. Continuing with $f(x) = x^3 - 3x$, at $x = -1$ we have $f'(-1) = 0$, $f''(-1) = -6 < 0$.

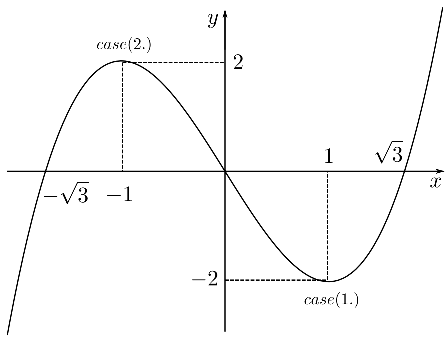


Figure 2.21: *Characterising the turning points using second derivatives of* $y = x^3 - 3x$.

3. If $f'(c) = 0$ and $f''(c) = 0$, then the slope often doesn't change sign, i.e. it goes from positive slope to zero to positive slope (decreasing to zero then increasing), or negative to zero to negative (increasing to zero then decreasing). These points are points of inflection and possible shapes are like:

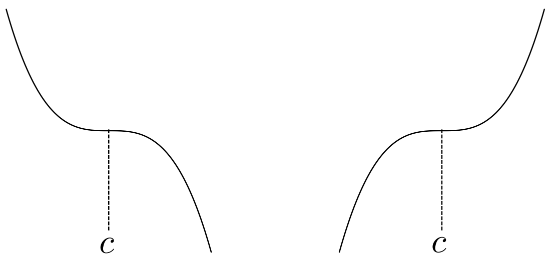


Figure 2.22: *Points of inflection at c, slope goes from positive to positive or negative to negative.*

Example 2.28. $f(x) = x^3$, so $f'(x) = 3x^2$ and $f''(x) = 6x$. At $x = 0$ we have $f'(0) = f''(0) = 0$.

WARNING: However, $f(x)$ doesn't necessarily need to be a point of inflection when the second derivative is zero.

Example 2.29. Consider $f(x) = x^4$. Now we have $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Notice that $f'(0) = f''(0) = 0$, however when we sketch $f(x)$, we realise it has a local minimum at $x = 0$.

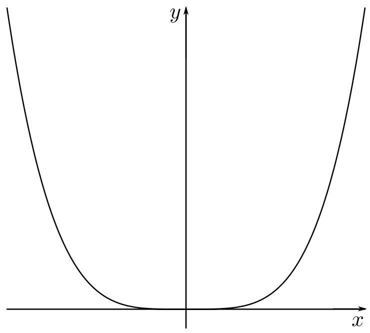


Figure 2.23: *Graph of $y = x^4$, clearly doesn't have an inflection point at $x = 0$, yet $f''(0) = 0$.*

Therefore we say that if f has a point of inflection at $x = c$, then $f'(c) = f''(c) = 0$ is a necessary condition.

Example 2.30. (A slightly more advanced example). Consider the function

$$f(x) = \frac{x^2 - 4}{x^2 - 1} = \frac{(x-2)(x+2)}{(x-1)(x+1)}.$$

So $f(x) = 0$ when the numerator of $f(x)$ is equal to zero. That is

$$(x-2)(x+2) = 0 \qquad \Longrightarrow x = 2,\ x = -2.$$

Something strange happens at the points $x = 1$ and $x = -1$. We get zero in the denominator and the numerator is negative in both cases so $f(\pm 1) \to -\infty$.

The graph cuts the y-axis when $x = 0$, that is

$$f(0) = \frac{-4}{-1} = 4.$$

Now let us calculate the derivative, it is a rational function so we may use the quotient rule (or apply the product rule), so we have

$$f'(x) = \frac{2x(x^2-1) - 2x(x^2-4)}{(x^2-1)^2} = \frac{6x}{(x^2-1)^2}.$$

So $f'(x) = 0$ when the numerator is zero i.e. when $6x = 0$, thus there is only one turning point at $x = 0$.

Now let us differentiate $f'(x)$, again using the quotient rule we have

$$f''(x) = \frac{6(x^2-1)^2 - 6x(2 \cdot (x^2-1) \cdot 2x)}{(x^2-1)^4} = \frac{6(x^2-1)^2 - 24x^2 \cdot (x^2-1)}{(x^2-1)^4}.$$

It remains to check the value of f'' at the turning point $x = 0$. Therefore

$$f''(0) = \frac{6(-1)^2 - 0}{(-1)^4} = 6 > 0,$$

and we have a local minimum at this point.

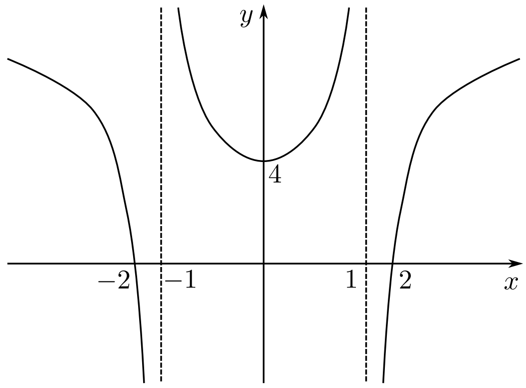


Figure 2.24: *Graph of* $y = (x^2 - 4)/(x^2 - 1)$*, i.e. a rational function with only one turning point (minimum) at* $x = 0$*.*

At $x = \pm 1$, we have what is known as *asymptotes*, lines the graph never touches, but gets very close to as we approach plus or minus infinity.[6]

[6]End Lecture 10.