# Chapter 1

# Functions

## 1.1 What is a function?

**Example 1.1.** The area $A$ of a circle depends on the radius $r$ of the circle. The rule that connects $r$ and $A$ is given by the following equation,

$$A = \pi r^2. \tag{1.1}$$

With each positive number $r$, there is associated one value of $A$ and we say that $A$ is a function of $r$, denoted by $A = A(r)$.

**Example 1.2.** The human population of the world $P$ depends on the time $t$. The table below gives estimates of the world population at time $t$, for certain years:

| $t$ (years) | 1900 | 1910 | 1950 | 1960 | 1990 | 2000 |
|---|---|---|---|---|---|---|
| $P$ (millions) | 1650 | 1750 | 2650 | 3040 | 5280 | 6080 |

For instance, $P(1950) \approx 2,560,000,000$ and $P(2000) \approx 6,080,000,000$.

For each value of time $t$, there is a corresponding value of $P$, and we say that $P$ is a function of $t$.

However, in this case we don't know the rule that connects $t$ and $P$ at the moment. In fact, it is one of our tasks to find and understand the rule for a function.

## 1.2 The definition of a function

Here we will look at more precise definition of a function. First let us define a set.

**Definition 1.1.** *A* **set** *is a collection of objects, 'things' and states. A 'piece' or object of a set is called an element of the set.*

**Example 1.3.** The following are examples of sets:

(i) $\mathbb{Z} = \{\text{all whole numbers}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$,

(ii) $\mathbb{N} = \{\text{all positive whole numbers}\} = \{1, 2, 3, \dots\}$,

(iii) $\mathbb{R} = \{\text{all real numbers}\}$.

If we have two sets $A$ and $B$, then a function $f : A \to B$ is a rule that sends each element $x$ in $A$ to exactly one element called $f(x)$ in $B$.

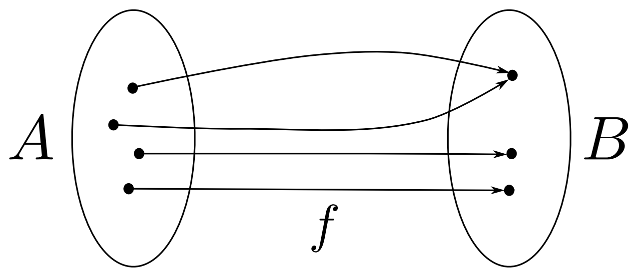


Figure 1.1: *Function $f$ 'maps' elements from set $A$ on to elements in set $B$. Many-to-one relationship.*

Here we call the set $A$ the *domain* of $f$ and $B$ is known as the *range* of $f$. Suppose $x$ is some element of $A$ which is denoted by $x \in A$, then $x$ is called an *independent variable* and $f(x)$ is called the dependent variable where $f(x) \in B$, i.e. $f(x)$ is an element of $B$.

NOTE: $f$ is not a function if it is multi-valued, i.e. if one element from $A$ maps to two distinct elements in $B$. Such a mapping is sometimes called multi-valued function, however this is just terminology, strictly by the definition for our purposes $f$ would not be a function in this case.

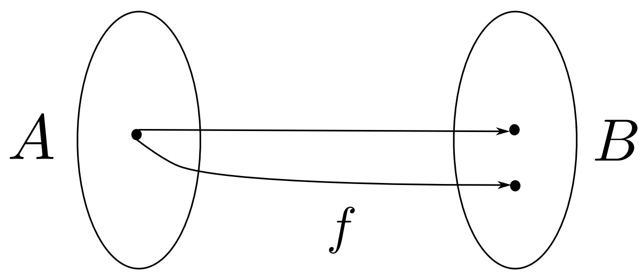


Figure 1.2: *The above 'map' $f$ is not a function, i.e. when it is one-to-many or multi-valued.*

## 1.3 Representing a function

**Example 1.4.** Let $S$ be the function from the real numbers to itself, that is $S : \mathbb{R} \to \mathbb{R}$ given by

$$S(x) = x^2. \tag{1.2}$$

Here we have defined the function S algebraically using an explicit formula.

That is, $S$ is the function that squares things (real numbers), i.e. a function performs some sort of 'action'. Here we have described the function verbally!

We can also represent a function numerically (by table of values):

| $x$ | 1 | 2 | 3 | ... | 7 | ... |
|---|---|---|---|---|---|---|
| $S(x)$ | 1 | 4 | 9 | ... | 49 | ... |

Finally, a function can also be represented visually, for example by a graph.

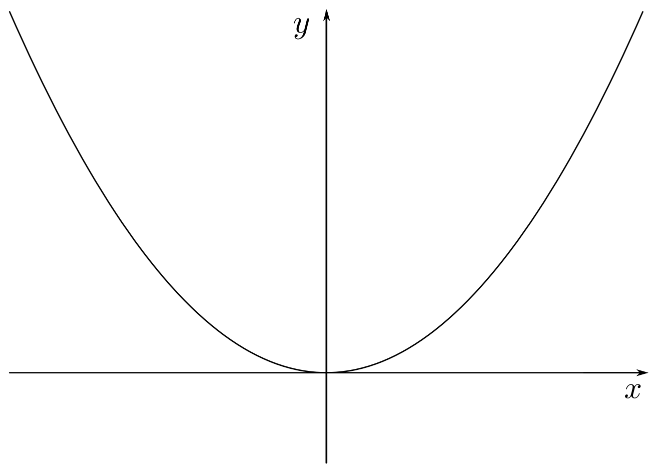


Figure 1.3: *Graph displaying the function $y = S(x) = x^2$.*

If the domain of $S$ is $\mathbb{R}$, then the range of $S$ is $\mathbb{R}^+ = \{\text{all positive real numbers}\}$.

Summary:

A function is a rule, it takes some independent variable (in the domain) for which there is some dependent variable (in the range).

There are four ways to represent function.

In the following, we take a look at some specific functions.[1]

[1] End Lecture 1.