2nd order ODE example – Mixed RHS i.e. p(x).

Q: Find General Solution to

$$y'' - 2y' = e^{3x} + 4\sin 5x.$$

A: (i) Solve Homogenous Eq: $y'' - 2y' = 0$

Auxillary Eq: $\lambda^2 - 2\lambda = 0$

$\therefore \lambda(\lambda - 2) = 0$.

so $\lambda_1 = 0$, $\lambda_2 = 2$.

so $y_{CF} = Ae^0 + Be^{2x} = A + Be^{2x}$.

(ii) Solve Non-Homogenous. $p(x) = e^{3x} + 4\sin 5x$.

so try $y_{PI} = \alpha e^{3x} + \beta\sin 5x + \gamma\cos 5x$.

$y'_{PI} = 3\alpha e^{3x} + 5\beta\cos 5x - 5\gamma\sin 5x$

$y''_{PI} = 9\alpha e^{3x} - 25\beta\sin 5x - 25\gamma\cos 5x$.

Substitute into $y'' - 2y' = e^{3x} + 4\sin 5x$

and compare coefficients:

$$3\alpha e^{3x} + (-25\beta + 10\gamma)\sin 5x + (-25\gamma - 10\beta)\cos 5x = e^{3x} + 4\sin 5x.$$

$\Rightarrow 3\alpha = 1 \Rightarrow \alpha = 1/3$

$$-25\beta+10\gamma = 4$$

$$-25\gamma-10\beta = 0 \Rightarrow \beta = -\frac{5}{2}\gamma$$

$$\Rightarrow -25\left(-\frac{5}{2}\right)\gamma+10\gamma = 4 \Rightarrow \frac{145}{2}\gamma = 4 \Rightarrow \gamma = \frac{8}{145}$$

So $\beta = \frac{-4}{29}$

So $$y_{PI} = \frac{1}{3}e^{3x} - \frac{4}{29}\sin 5x + \frac{8}{145}\cos 5x.$$

(iii) General solution $y = y_{CF} + y_{PI}$

$$\therefore y(x) = A + Be^{2x} + \frac{1}{3}e^{3x} - \frac{4}{29}\sin 5x + \frac{8}{145}\cos 5x.$$
