2^{nd} Order ODE example – BVP.

Q: Solve the following Boundary Value Problem:

$$y'' - 8y' + 16y = 2\cos x$$

with $y(0) = 0$, $y(\pi/2) = 0$.

A: (i) Solve Homogeneous Eq:

$$y'' - 8y' + 16y = 0$$

Auxilliary Eq: $\lambda^2 - 8\lambda + 16 = 0$

$$\therefore \quad (\lambda - 4)^2 = 0$$

i.e. $\lambda_1 = \lambda_2 = 4$. (repeated root)

$$\therefore \quad y_{CF} = Ae^{4x} + Bxe^{4x}$$ [linearly independant]

(ii) Solve non-Homogeneous Eq.

$P(x) = 2\cos x$.

Try $y_{PI} = \alpha\cos x + \beta\sin x$.

$$y'_{PI} = -\alpha\sin x + \beta\cos x.$$

$$y''_{PI} = -\alpha\cos x - \beta\sin x.$$

substitute and compare coefficients:

$$-\alpha\cos x - \beta\sin x + 8\alpha\sin x - 8\beta\cos x + 16\alpha\cos x + 16\beta\sin x = 2\cos x$$

$$(15\alpha - 8\beta)\cos x + (15\beta + 8\alpha)\sin x = 2\cos x$$

$$\therefore \quad 15\alpha - 8\beta = 2 \qquad (1)$$

$$15\beta + 8\alpha = 0 \qquad (2). \Rightarrow \beta = -\frac{8\alpha}{15}$$

$$\therefore (2) \Rightarrow \quad 15\alpha - 8\left(-\frac{8\alpha}{15}\right) = 2$$

$$\therefore \quad \frac{225\alpha + 64\alpha}{15} = 2.$$

$$\therefore \quad 289\alpha = 30 \Rightarrow \alpha = \frac{30}{289}$$

$$\Rightarrow \beta = \frac{-16}{289}$$

$$y_{PI} = \frac{1}{289}(30\cos x - 16\sin x).$$

(iii) General solution $y = y_{CF} + y_{PI}$.

$$\therefore \quad y = Ae^{4x} + Bxe^{4x} + \frac{1}{289}(30\cos x - 16\sin x).$$

Now apply $y(0) = 0$, $y(\pi/2) = 0$.

$$\therefore \quad 0 = A + \frac{1}{289}(-16) \Rightarrow A = \frac{16}{289}$$

$$0 = Ae^{2\pi} + \frac{B\pi}{2}e^{2\pi} + \frac{30}{289}$$

$$\therefore \quad 0 = \frac{16}{289}e^{2\pi} + \frac{B\pi}{2}e^{2\pi} + \frac{30}{289}$$

$$\therefore \quad -\frac{B\pi}{2}e^{2\pi} = \frac{16}{289}e^{2\pi} + \frac{30}{289}.$$

$$\Rightarrow \quad B = -\frac{2}{\pi}\cdot\frac{16}{289} + \frac{30}{289}e^{-2\pi}\cdot\left(-\frac{2}{\pi}\right)$$

$$B = \frac{-2}{289\pi}\left(16 + 30e^{-2\pi}\right)$$

Finally:

$$y = \frac{1}{289}\left[16e^{4x} - \frac{2}{\pi}\left(16+30e^{-2\pi}\right)xe^{4} + 30\cos x - 16\sin x\right]$$

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looks scary but after step (i) and (ii) all we have done is solve simultaneous equation and have done much re-arranging only!