MATH6103 Differential and Integral Calculus
MATH6500 Elementary Mathematics for Engineers

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1 Notes on Complex Numbers

1.1 The Basics

Suppose we want to solve the the quadratic equation $x^2 + 1 = 0$, inserting this into the equation to solve quadratics:

$$x = \frac{-0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{\pm\sqrt{-4}}{2} = \pm\sqrt{-1}$$

So we can't solve the equation. So we define the following quantity $i = \sqrt{-1}$, and we call i the imaginary number, if $i = \sqrt{-1}$, then $i^2 = -1$. All numbers can be built out of real numbers and imaginary numbers which we call complex numbers. A complex number is usually written as z, and we write $z = x + yi$, we add complex numbers in the following way, if $z_1 = a + bi$ and $z_2 = c + di$ then:

$$z_1 + z_2 = (a + c) + (b + d)i \tag{1}$$

The set of complex numbers is denoted by $\mathbb{C}$, complex numbers obey the the following sets.

1. order doesn't matter in addition $z_1 + z_2 = z_2 + z_1$
2. order doesn't matter in multiplication $z_1 z_2 = z_2 z_1$
3. Addition is associative $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. Multiplication is associative $(z_1 z_2)z_3 = z_1(z_2 z_3)$
5. There is an associative identity $z + 0 = z$

6. There is a multiplicative identity $1 \cdot z = z$

7. For every z, there is a $-z$ such that $z + (-z) = 0$

8. For every z, there is a number $1/z$ such that $z \cdot (1/z) = 1$

9. The distribution law holds $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

If $z = a + bi$, then we will write down the inverse.

$$\begin{aligned} \frac{1}{z} &= \frac{1}{a+bi} \\ &= \frac{1}{a+bi}\frac{a-bi}{a-bi} \\ &= \frac{a-bi}{(a+bi)(a-bi)} \\ &= \frac{a-bi}{a(a-bi)+bi(a-bi)} \\ &= \frac{a-bi}{a^2 - abi + abi - b^2 i^2} \\ &= \frac{a-bi}{a^2+b^2} \end{aligned}$$

So:

$$\frac{1}{z} = \frac{a-bi}{a^2+b^2} \tag{2}$$

Given a complex number $z = a + bi$, we define the complex conjugate, $\bar{z}$ by:

$$\bar{z} = a - bi \tag{3}$$

We can write the real part of a general complex number $z = a + bi$ as $Re(z) = a$ and the imaginary part of z as $Im(z) = b$, so the general complex number can be written $z = Re(z) + Im(z)i$. The modulus of the complex number is written as $|z|$, if $z = a + bi$, then:

$$|z|^2 = z\bar{z} = (a+bi)(a-bi) = a^2 + b^2 \tag{4}$$

We can now solve quadratics like $x^2 - 4x + 13 = 0$, inserting this into the equation for solving quadratics shows:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} \end{aligned}$$

$$\begin{aligned} &= \frac{4 \pm \sqrt{16-52}}{2} \\ &= \frac{4 \pm \sqrt{-36}}{2} \\ &= \frac{4 \pm 6\sqrt{-1}}{2} \\ &= 2 \pm 3\sqrt{-1} \\ &= 2 \pm 3i \end{aligned}$$

1.2 The Argand Diagram/Complex Plane

Complex numbers can be written in the form $z = (a, b)$ where $z = a + bi$ and this notation is suggestive of the usual plane which we're familiar with. We can associate the x co-ordinate with $Re(z)$ and the y co-ordinate with $Im(z)$, so a general complex number $z = a + bi$ as a point on the complex plane. The argand diagram suggest that it is possible for yet another representation of a complex number, the use of polar co-ordinates. The distance r is just the modulus, so for a complex number $z = a + bi$, $r = |z| = \sqrt{a^2 + b^2}$, the angle θ is called the argument and is written $Arg(z)$. The argument is calculated as follows:

$$Arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right) \tag{5}$$

From the definition of $\cos\theta$ and $\sin\theta$:

$$\sin\theta = \frac{b}{r}, \quad \cos\theta = \frac{a}{r}$$

So re-arranging:

$$a = r\cos\theta, \quad b = r\sin\theta$$

As $z = a + bi$, we can write it as:

$$z = r(\cos\theta + i\sin\theta) \tag{6}$$

This is called the polar form of a complex number.

Example. Compute the modulus and argument of $z = -\sqrt{3} + i$ and plot it on and argant diagram.
The modulus can be computed as

$$\begin{aligned}
|z| &= \sqrt{z\bar{z}} \\
&= \sqrt{(-\sqrt{3}+i)(-\sqrt{3}-i)} \\
&= \sqrt{(3+1+i\sqrt{3}-i\sqrt{3})} \\
&= \sqrt{4} \\
&= 2
\end{aligned}$$

To calculate the Argument, we compute:

$$\begin{aligned}
Arg(z) &= \theta \\
&= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) \\
&= \pi - \frac{1}{\sqrt{3}} \\
&= \pi - \frac{\pi}{6} \\
&= \frac{5\pi}{6}
\end{aligned}$$

So the polar form of the complex number is:

$$z = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$$

1.3 Other Identities Associate With Complex Numbers

1.3.1 Euler's Formula

Previously we computed a series for for e^x, there is a special equation called Eulers formula which deals with $e^{i\theta}$. Then:

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{\theta^3 i}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5 i}{5!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)i \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

The result

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{7}$$

Is known as Eulers formula. Setting $\theta = \pi$, shows that:

$$e^{i\pi} + 1 = 0 \tag{8}$$

which links all the most important numbers in maths. A general complex number can be written as $z = re^{i\theta}$

1.3.2 De Moivre's Theorem

We have spoken about the polar representation of a complex number $z = r(\cos\theta + i\sin\theta)$, let us examine z^2.

$$\begin{aligned}
z^2 &= (r(\cos\theta + i\sin\theta))^2 \\
&= r^2(\cos\theta + i\sin\theta)^2 \\
&= r^2(\cos^2\theta + (i\sin\theta)^2 + 2i\sin\theta\cos\theta) \\
&= r^2(\cos^2\theta + (i)^2(\sin\theta)^2 + 2i\sin\theta\cos\theta) \\
&= r^2(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta) \\
&= r^2(\cos 2\theta + i\sin 2\theta)
\end{aligned}$$

So we have shown something rather remarkable!

$$(\cos\theta + \sin\theta)^2 = \cos 2\theta + i\sin 2\theta \tag{9}$$

A natural question to ask is if this true for general powers, we can compute for $z^3 = (r(\cos\theta + i\sin\theta))^3$.

$$\begin{aligned}
z^3 &= z \cdot z^2 \\
&= (r(\cos\theta + i\sin\theta))r^2(\cos 2\theta + i\sin 2\theta) \\
&= r^3(\cos\theta\cos 2\theta + (i\sin\theta)(i\sin 2\theta) + i\sin\theta\cos 2\theta + i\sin 2\theta\cos\theta) \\
&= r^3(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta + (\sin\theta\cos 2\theta + \cos\theta\sin 2\theta)i) \\
&= r^3(\cos(\theta + 2\theta) + i\sin(\theta + 2\theta)) \\
&= r^3(\cos 3\theta + i\sin 3\theta)
\end{aligned}$$

We can do the same for any whole number n and it shows that:

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta) \tag{10}$$