Free boundary problems in a Hele-Shaw cell

by

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Disclaimer

I, Ali Haseeb Khalid, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

The motion of a free boundary separating two immiscible fluids in an unbounded Hele-Shaw cell is considered. In the one-phase problem, a viscous fluid is separated from an inviscid fluid by a simple closed boundary. Preliminaries for a complex variable technique are presented by which the one-phase problem can be solved explicitly via conformal mappings. The Schwarz function of the boundary plays a major role giving rise to the so called Schwarz function equation which governs the evolution of exact solutions. The Schwarz function approach is used to study the stability of a translating elliptical bubble due to a uniform background flow, and the stability of a blob (or bubble) subject to an external electric field.

The one-phase problem of a translating free boundary and of a free boundary subject to an external field are studied numerically. A boundary integral method is formulated in the complex plane by considering the Cauchy integral formula and the complex velocity of a fluid particle on the free boundary. In the case of a free boundary subject to an external electric field due to a point charge, it is demonstrated that a stable steady state is achieved for appropriate charge strength. The method is also employed to study breakup of a single translating bubble in which the Schwarz function singularities (shown to be stationary) of the initial boundary play an important role.

The two-phase problem is also considered, where the free boundary now separates two viscous fluids, and the construction of exact solutions is studied. The one-phase numerical model is enhanced, where a boundary integral method is formulated to accommodate the variable pressure in both viscous phases. Some numerical experiments are presented with a comparison to analytical results, in particular for the case where the free boundary is driven by a uniform background flow.

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To the world of fluid mechanics. A subject that has afforded me the opportunity to conduct research in an area of applied mathematics I thoroughly enjoy studying.

Simplicity is the ultimate sophistication

—Leonardo da Vinci

Contents