Hele-Shaw flow driven by an electric field

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Picture courtesy of Antony Hall, antonyhall.net.

Overview

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   - Boundary Integral Method
   - Example
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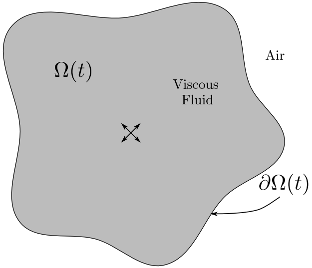
   - Check: numerical result
4. Concluding remarks

1.1 Free boundary problem

- Flow of viscous fluid ‘sandwiched’ between two parallel plates

- Fluid air interface: Hele-Shaw free boundary problem



$$\nabla^2 \phi = \sum_{j=1}^{N} Q_j \delta(x - x_j, y - y_j), \quad (x_j, y_j) \in \Omega(t),$$

$$\phi = \Psi(x, y), \quad (x, y) \in \partial\Omega(t),$$

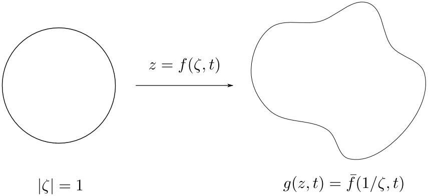
$$v_n = \frac{\partial \phi}{\partial n} = \nabla \phi \cdot \mathbf{n}, \quad (x, y) \in \partial\Omega(t).$$

see e.g. Entov & Etingof (EJAM, 2007)

- Q_j - strength hydrodynamic singularities, $\mathbf{n}$ - unit normal vector on $\partial\Omega(t)$, Ψ - external potential, ϕ - velocity potential ($\mathbf{u} = \nabla \phi$)

1.2 Schwarz function approach

- In complex variables $z = x + iy$, conformally map the unit ζ-disk to the domain $\Omega(t)$



- One can calculate the Schwarz function of the curve $\partial\Omega(t)$ by

$$g(z,t) := \bar{z} = \overline{f(\zeta,t)} = \bar{f}(1/\zeta,t), \quad z \in \partial\Omega(t) \quad \Longleftrightarrow \quad |\zeta|^2 = 1$$

- It has been shown (McDonald, EJAM, 2011) the following *generalised* Schwarz function equation holds on the entire domain $\Omega(t)$:

$$\frac{\partial F}{\partial z} = \frac{\partial \Psi}{\partial z} + \frac{1}{2}\frac{\partial g}{\partial t}$$

1.3 Schwarz function equation remarks

$$\frac{\partial F}{\partial z} = \frac{\partial \Psi}{\partial z} + \frac{1}{2}\frac{\partial g}{\partial t}$$

- The above equation is useful since it must hold in the limit where singularities of $F'(z)$ are approached
- So $\dot{g}$ must have the same structure of singularity as other terms $\Rightarrow$ find $g(z,t)$, i.e. the shape of the boundary
- Driving singularities of $F(z)$ and initial boundary shape, $g(z,0)$, complete initial description
- In the absence of external fields i.e. $\Psi \equiv 0$, equation reduces to that previously used, see e.g. Cummings *et al.* (EJAM, 1999), Abanov *et al.* (Physica D, 2007)

1.4 Well known example

- Slightly deformed circular fluid blob driven by a hydrodynamic sink of strength Q at $z = 0$ (here $\Psi \equiv 0$)

- Polynomial map: $z(\zeta, t) = a(t)\left(\zeta + \frac{b(t)}{n}\zeta^n\right)$

- Schwarz function: $g(z, t) = -\frac{a^{n+1}b}{n}\frac{1}{z^n} + a^2\left(1 + \frac{b^2}{n}\right)\frac{1}{z} + O(1)$

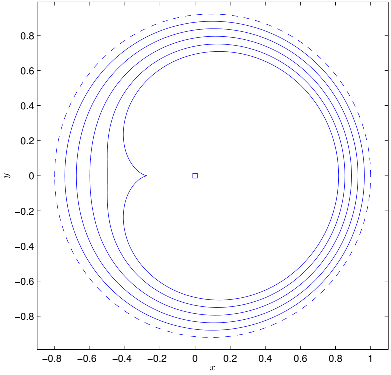
- As $z \to 0$, $F(z) = \frac{Q}{2\pi}\log(z) \Rightarrow F'(z) = \frac{Q}{2\pi z}$

- Compare terms of $\mathcal{O}(z^{-1})$, $\mathcal{O}(z^{-n})$ in Schwarz function equation:

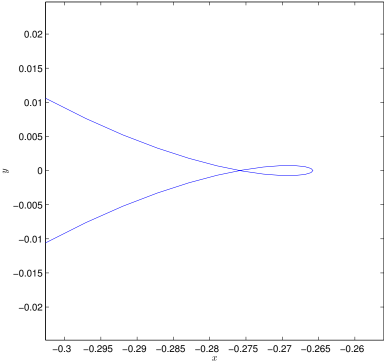
$$\frac{d}{dt}\left[a(t)^2\left(1 + \frac{b(t)^2}{n}\right)\right] = \frac{Q}{\pi}, \qquad \frac{d}{dt}\left[a(t)^{n+1}b(t)\right] = 0$$

1.5 Well known example (cont.)

For $n = 2,\ Q = -1,\ a(0) = 0.9,\ b(0) = 0.1$



$t = 0$ to $t = t^* \approx 1.22$.



beyond $t = t^*$.

2.1 Formulation Boundary Integral Method

- Suppose you are given initial boundary $\partial\Omega(0)$ and its velocity, i.e. $x_j^1 = x(0)$, $y_j^1 = y(0)$, $u_j^1 = u(0)$ and $v_j^1 = v(0)$, for $j = 1, \ldots, N$

- Step in time by advection

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v \tag{1}$$

i.e.

$$x_j^{k+1} = x_j^k + u_j^k \Delta t, \quad y_j^{k+1} = y_j^k + v_j^k \Delta t$$

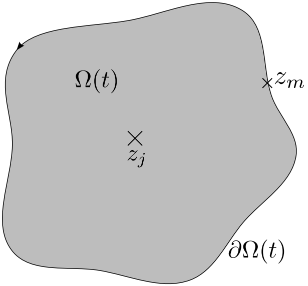
- Require 2 equations for the 2 unknowns u_j^{k+1} and v_j^{k+1}

2.2 Formulation Boundary Integral Method (cont.)

- (i) Consider simple smooth curve $\partial\Omega(t)$ of fluid blob driven by hydrodynamic singularity

- Suppose we choose a point $z_m = x_m + iy_m$ on the $\partial\Omega(t)$ and consider the following integral (boundary integral equation)

$$I = \frac{1}{2\pi i}\int_{\partial\Omega(t)} \frac{u(z) - iv(z)}{z - z_m}\,dz = \frac{1}{2}\left\{u(z_m) - iv(z_m)\right\} + \sum_j \left[\mathrm{Res}\left(\frac{u(z) - iv(z)}{z - z_m}; z_j\right)\right] \quad (2)$$



2.3 Formulation Boundary Integral Method (cont.)

- (ii) Along $\partial\Omega(t)$ given $\phi = \Psi$ and since $u = \phi_x$ and $v = \phi_y$ then (dynamic condition)

$$u\frac{dx}{ds} + v\frac{dy}{ds} = \Psi_x\frac{dx}{ds} + \Psi_y\frac{dy}{ds} \tag{3}$$

  s - arclength parameter

- Since x_j^{k+1}, y_j^{k+1} (and their derivatives) are known, discretise and solve (2) and (3) for u_j^{k+1} and v_j^{k+1}.

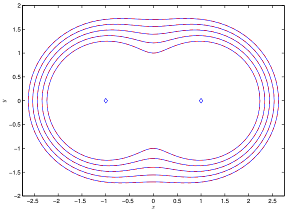
- Obtain N equations from (2) by placing N midpoints, z_m, on equispaced mesh around $\partial\Omega(t)$ and N equations from (3) at mesh points $\Rightarrow$ solve linear system of $2N$ equations at each time step.

2.4 Flow near a wall, equal dumbbell

Non-trivial example:

Fluid Blob Movie

Exact = dashed red, Numerical = solid blue, $Q = \pi$ at $z = \pm 1$



Time evolution, $t = \frac{3\pi}{2}$ to $t = \frac{3\pi+6}{2}$.

Numerical model relates to problem as:

$$\nabla^2\phi = \sum_{j=1}^{N} Q_j \delta(x - x_j, y - y_j), \quad (x_j, y_j) \in \Omega(t)$$

$$\phi = \Psi(x, y), \quad (x, y) \in \partial\Omega(t)$$

$$v_n = \frac{\partial\phi}{\partial n} = \nabla\phi \cdot \mathbf{n}, \quad (x, y) \in \partial\Omega(t)$$

✓ (1) Advection, step boundary in time

✓ (2) Boundary integral equation

✓ (3) Dynamic boundary (pressure) condition

3.1 Flow due to an electric point charge (i)

- Consider a circular fluid blob of conducting fluid centred at $z = 0$ with radius R, subject to an electric point charge at $z = 0$, where

$$\Psi = \frac{E}{4\pi} \log(zg) \tag{1}$$

- Here $g(z,t) = R^2/z$ is the Schwarz function of $\partial\Omega(t)$ and $E \in \mathbb{R}$
- The Schwarz function equation becomes

$$\frac{\partial F}{\partial z} = \frac{1}{2}\frac{\partial g}{\partial t} + \frac{E}{4\pi}\left(\frac{1}{z} + \frac{g_z}{g}\right) \tag{2}$$

- As $z \to 0$, considering singularities of $\mathcal{O}(z^{-1})$ above gives $\dot{R} = 0$
- We have a steady solution in which the fluid blob remains circular with constant radius $R_0 = R(0)$
- Turns out the flow is stable for $E < 0$.

3.2 Flow due to an electric point charge (ii)

- Consider a circular fluid blob centred at $z = \epsilon \in \mathbb{R}$ subject to an electric point charge at $z = 0$, then for small ϵ

$$g(z,t) = \epsilon + \frac{R^2}{z-\epsilon} \tag{3}$$

- Considering structure of the singularities of $\mathcal{O}(z^{-1})$ and $\mathcal{O}(z^{-2})$ on both sides of the Schwarz function equation yields:

$$R\dot{R} = 0, \tag{4a}$$

$$\frac{R^2\dot{\epsilon}}{2} + \epsilon R\dot{R} - \frac{E}{4\pi}\epsilon = 0, \quad E < 0 \tag{4b}$$

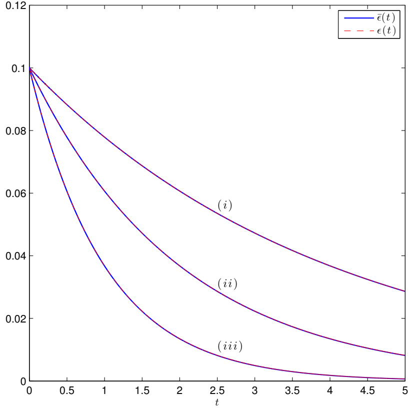
- Hence, $\dot{R} = 0$, so $R(t) = R_0$, where R_0 is the radius of the initial blob $\implies$ consistent with conservation of area, and (4b) gives

$$\epsilon(t) = \epsilon_0 \exp\left(\frac{E}{2R_0^2\pi}t\right). \tag{5}$$

- For small ϵ, the fluid blob remains circular throughout the motion and its centre emigrates towards the position of the point charge

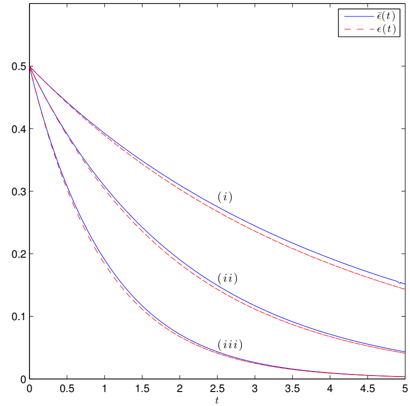
3.3 Checking analytical result

Circular blob with point charge strengths: (i)$E = -\frac{\pi}{2}$, (ii) $E = -\pi$, (iii) $E = -2\pi$



$\epsilon(0) = 0.1,\ R_0 = 1$

Excellent agreement!



$\epsilon(0) = 0.5,\ R_0 = 1$

What's happening?

N.B. Calculate $\bar{\epsilon}(t)$ as the centre of mass of numerical solution

3.4 Numerical results as $\epsilon(0) \approx R_0$

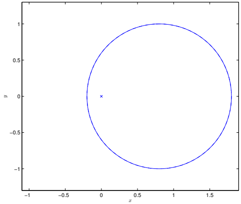
Placing electric charge close to the initial boundary e.g. $\epsilon(0) = 0.8$, $R_0 = 1$

Fluid Blob Movie

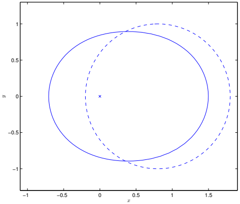
Eventual symmeterising about the location of the point charge, here $E = -2\pi$ at $z = 0$

3.5 Numerical results as $\epsilon(0) \approx R_0$ (cont.)

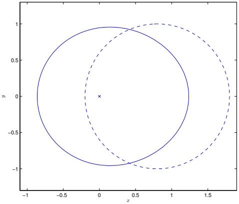
Placing electric charge close to the initial boundary e.g. $\epsilon(0) = 0.8, R_0 = 1$



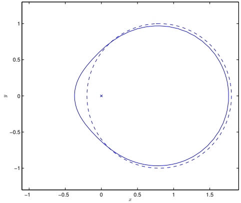
$t = 0$



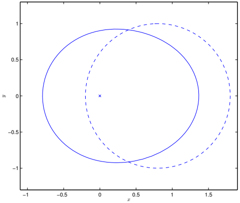
$t = 0.5$



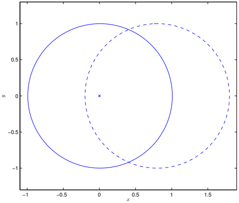
$t = 1.05$



$t = 0.06$



$t = 0.75$

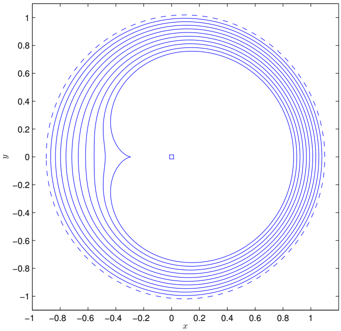


$t = 3$

Eventual symmeterising about the location of the point charge, here $E = -2\pi$ at $z = 0$

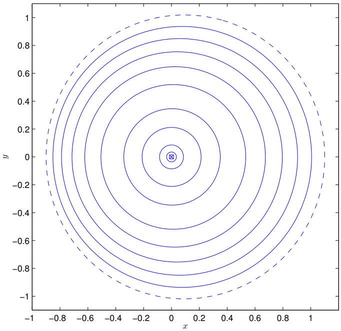
4.1 Remarks

- Superimposing sink & electric charge, extracting all fluid from a deformed circular blob?



Cusp formation at time $t^* \approx 1.22$

$Q = -1$



Exatracting more fluid, final time $t = 3$

$Q = -1,\ E = -2\pi$

4.1 Remarks

- Superimposing sink & electric charge, extracting all fluid from a deformed circular blob?

- Flow stable for point charge $E < 0$, symmeterising effect about its location

- $\partial\Omega(t)$ remains smooth for flows driven by solely by electric point charge

- Possible applications in (i) theory of fluid flows in microfluidic devices - manipulation of fluid blobs via electric fields, (ii) fluid extraction problems - prolonging cusps (contamination)

References

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