Upstream Influence Leading to Discontinuous Solutions of the Boundary-Layer Equations

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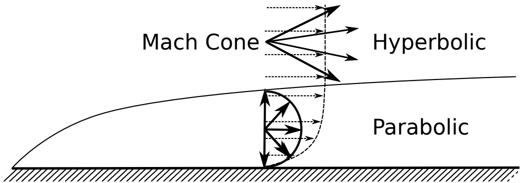
Relevance and Motivation

- Aerodynamics

- Personal interest - asymptotic methods, boundary-layer theory.

Laminar Boundary-Layers

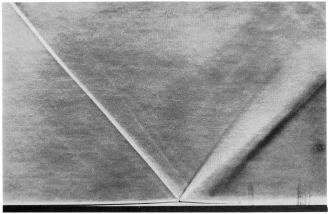
- Boundary-Layer theory/equations first formulated by Ludwig Prandtl in 1904.
- The solution is based on a hierarchal approach.
  - Outer flow is considered where the boundary-layer neglected.
  - Then turn to the boundary layer, solve boundary-layer equations.



- Perturbations unable to propagate upstream?

Experimentalists

- Experiments regarding boundary-layer/shock wave interaction conducted during the late 1930's to early 1950's.
- Findings were contrary to what a theoretician may have thought at the time.

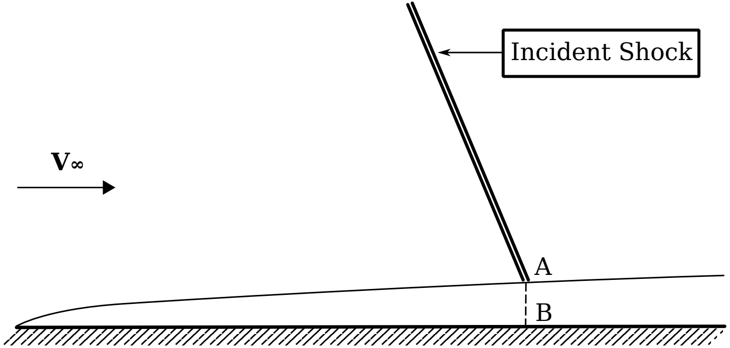


Liepmann *et al.* (1949)

- Upstream influence

Flow Layout

We consider the following simple flow layout:



Shock wave impinging upon the boundary-layer on a flat plate.

Governing Equations

- Navier-Stokes Equations for compressible, viscous flow[1]

x-momentum:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left\{\frac{\partial}{\partial x}\left[\mu\left(\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\},$$

y-momentum:

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left\{\frac{\partial}{\partial x}\left[\mu\left(\frac{4}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial u}{\partial x}\right)\right]+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right\},$$

continuity equation:

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0,$$

energy equation:

$$\rho\left(u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}\right)=u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+\frac{1}{Re}\left\{\frac{1}{Pr}\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial h}{\partial y}\right)\right]\right. +$$

$$\left.+\mu\left(\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial x}+\mu\left(\frac{4}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial y}+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right\},$$

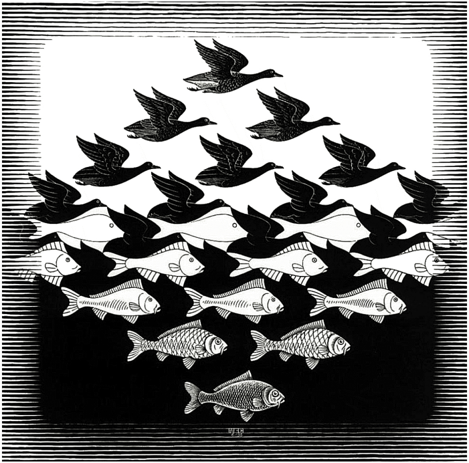
equation of state:

$$h=\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\frac{1}{M_\infty^2(\gamma-1)}\frac{1}{\rho}.$$

[1] see book for example by Rogers, Laminar Flow Analysis (1992).

Matched Asymptotic Expansions

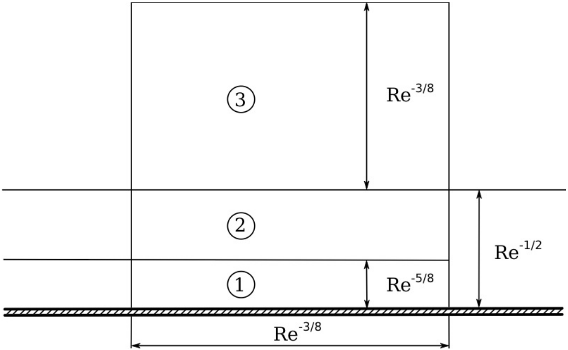
- What is matched asymptotic expansions?



- “smooth blending”

Triple-Deck Structure

- Inspectional analysis: scalings of each region/tier in the vicinity of the interaction region, establish viscous-inviscid interaction.



- Region 1 = Viscous sublayer.
- Region 2 = Main part of boundary-layer.
- Region 3 = Inviscid outer flow .

Formulation of the interaction problem

- In canonical form, the equations of the interaction problem are

$$\bar{U}\frac{\partial \bar{U}}{\partial \bar{X}} + \bar{V}\frac{\partial \bar{U}}{\partial \bar{Y}} = -\frac{d\bar{P}}{d\bar{X}} + \frac{\partial^2 \bar{U}}{\partial \bar{Y}^2},$$
$$\frac{\partial \bar{U}}{\partial \bar{X}} + \frac{\partial \bar{V}}{\partial \bar{Y}} = 0,$$
$$\bar{P} = -\frac{d\bar{A}}{d\bar{X}} + \bar{p}_0,$$

- With the conditions

$$\begin{aligned}
&\bar{U} = V_w, \quad \bar{V} = 0 && \text{at} \quad \bar{Y} = 0,\\
&\bar{U} = \bar{Y} + \bar{A}(\bar{X}) && \text{as} \quad \bar{Y} \to \infty,\\
&\bar{U} = \bar{Y} + \ldots && \text{as} \quad \bar{X} \to -\infty.
\end{aligned}$$

- Here the jump in pressure is given by $\bar{p}_0(\bar{X}) = \alpha_0 H(\bar{X})$, and $\bar{P}$ not known beforehand!

Kaplun's Extension Theorem

Let $f(x;\epsilon)$ be an approximation to $u(x;\epsilon)$ valid to order $\xi(\epsilon)$ in the order domain $[M, N]$. Then there exist order classes $M_e < M$ and $N_e > N$ such that $f(x;\epsilon)$ is an approximation to $u(x;\epsilon)$ valid to order $\xi(\epsilon)$ in the extended order domain $[M_e, N_e]$

$\Rightarrow$ extend solution upstream of interaction region.

Linearising the Equations

- We see previous equations have the basic solution

$$\bar{U} = \bar{Y} + V_w, \quad \bar{V} = 0, \quad \bar{P} = 0, \quad \bar{A} = 0.$$

- To linearise we superimpose the basic solutions with small perturbation as follows (assuming the shockwave is weak)

$$\bar{U} = \bar{Y} + V_w + \epsilon u'(\bar{X}, \bar{Y}), \quad \bar{V} = \epsilon v'(\bar{X}, \bar{Y}), \quad \bar{P} = \epsilon p'(\bar{X}),$$
$$\bar{A} = \epsilon A'(\bar{X}), \quad \bar{p}_0 = \epsilon p_0'(\bar{X}).$$

Linearised Equations and Boundary Conditions

- Neglecting terms of order $O(\epsilon^2)$ we have the following linearised equations

$$(\bar{Y}+V_w)\frac{\partial u'}{\partial \bar{X}}+v'=-\frac{dp'}{d\bar{X}}+\frac{\partial^2 u'}{\partial \bar{Y}^2},$$
$$\frac{\partial u'}{\partial \bar{X}}+\frac{\partial v'}{\partial \bar{Y}}=0,$$
$$p'=-\frac{dA'}{d\bar{X}}+\alpha_0 H(\bar{X}).$$

- With the following corresponding boundary conditions

$$\begin{aligned} u' &= v' = 0 && \text{at} \quad \bar{Y}=0,\\ u' &= A'(\bar{X}) && \text{as} \quad \bar{Y}\to\infty,\\ u' &= 0 && \text{as} \quad \bar{X}\to-\infty. \end{aligned}$$

- To calculate p', need A'.

Method of Fourier transform

- Eliminating $p'(\bar{X})$ and taking the Fourier in $\bar{X}$

$$(\bar{Y} + V_w)ik\tilde{u} + \tilde{v} = -k^2\tilde{A} - \alpha_0 + \frac{d^2\tilde{u}}{d\bar{Y}^2},$$

$$ik\tilde{u} + \frac{d\tilde{v}}{d\bar{Y}} = 0.$$

- Differentiating w.r.t $\bar{Y}$ and eliminating $d\tilde{v}/d\bar{Y}$, we have

$$(\bar{Y} + V_w)ik\frac{d\tilde{u}}{d\bar{Y}} = \frac{d^3\tilde{u}}{d\bar{Y}^3}.$$

- Which leads to the *Airy Equation* (for $d\tilde{u}/dz$)

$$\frac{d^3\tilde{u}}{dz^3} - z\frac{d\tilde{u}}{dz} = 0,$$

where the change of variable $z = \theta(\bar{Y} + V_w) = (ik)^{1/3}(\bar{Y} + V_w)$ is made.

Boundary conditions and general solution

- The Airy Equations for $d\tilde{u}/dz$ is to be solved with the conditions

$$\begin{aligned} \tilde{u} =& 0 \qquad\qquad \text{at} \quad \bar{Y} = 0, \\ \tilde{u} =& \tilde{A}(k) \qquad \text{as} \quad \bar{Y} \to \infty, \end{aligned}$$

and from the momentum equation,

$$\frac{d^2\tilde{u}}{d\bar{Y}^2} = k^2\tilde{A} + \alpha_0 \qquad \text{at} \quad \bar{Y} = 0.$$

- The general solution of the Airy Equation is

$$\frac{d\tilde{u}}{d\bar{z}} = C_1 \mathrm{Ai}(z) + C_2 \mathrm{Bi}(z),$$

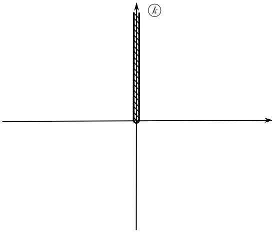
where Ai and Bi are linearly independent.

Limiting the analysis - behaviour of Bi(z)

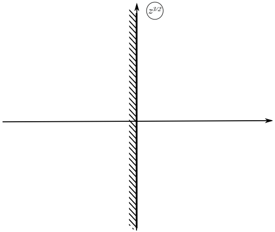
- The asymptotic representation of Bi for large z is

$$\mathrm{Bi}(z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4} e^{\zeta} + \ldots, \qquad \zeta = \frac{2}{3} z^{3/2}.$$

- Recall $\theta = (ik)^{1/3}$, which is three valued function of k.
- Make this single valued for z - take branch cut in the complex k-plane



k-plane



$z^{3/2}$-plane

- exponential growth as $\bar{Y} \to \infty \Rightarrow C_2 = 0$

Fourier transform of displacement function

- So we have $d\tilde{u}/d\bar{z} = C_1\mathrm{Ai}(z)$
- Apply boundary conditions (from earlier)

$$
\begin{aligned}
\tilde{u} =& 0 && \text{at} \quad \bar{Y} = 0, \\
\tilde{u} =& \tilde{A}(k) && \text{as} \quad \bar{Y} \to \infty, \\
\frac{d^2\tilde{u}}{d\bar{Y}^2} =& k^2\tilde{A} + \alpha_0 && \text{at} \quad \bar{Y} = 0,
\end{aligned}
$$

- Eliminate C_1 to get an expression for the Fourier transform of the displacement function

$$
\tilde{A} = \frac{\alpha_0 \int_{\theta V_w}^{\infty} \mathrm{Ai}(s)ds}{\theta^2 \mathrm{Ai}'(\theta V_w) - k^2 \int_{\theta V_w}^{\infty} \mathrm{Ai}(s)ds}.
$$

Pressure - inverse transform

- Recall interaction law $p' = -\frac{dA'}{d\bar{X}} + \alpha_0 H(\bar{X})$
- Fourier transform: $\tilde{p} = -ik\tilde{A} + \mathcal{F}[\alpha_0 H(\bar{X})](k)$
- Substituting the expression for $\tilde{A}$ and taking the inverse Fourier transform, we finally have

$$p' = \underbrace{-i\alpha_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{k \int_{\theta V_w}^{\infty} \mathrm{Ai}(s)ds}{\theta^2 \mathrm{Ai}'(\theta V_w) - k^2 \int_{\theta V_w}^{\infty} \mathrm{Ai}(s)ds} e^{ik\bar{X}} \right] dk}_{I_p} + \underbrace{\alpha_0 H(\bar{X})}_{stepfunction}$$

- So we want to calculate integral I_p!

Applying Jordan's Lemma

- Recall the Fourier transform of a function $\Phi(\bar{X})$ is given by

$$\mathcal{F}[\Phi](k) = \int_{-\infty}^{\infty} \Phi(\bar{X}) e^{ik\bar{X}} d\bar{X} = \int_{-\infty}^{\infty} \Phi(\bar{X}) e^{ik_r\bar{X}} e^{-k_i\bar{X}} d\bar{X}$$

  where $k_r = \Re(k)$ and $k_i = \Im(k)$ and $k_r, k_i \in \mathbb{R}$
- If k_i is positive then we have exponential decay for $\bar{X} > 0$, i.e. downstream
- If k_i is negative then we have exponential decay for $\bar{X} < 0$, i.e. upstream
- Therefore we close the contour in the lower half of the complex plane for upstream influence.

Stationary wall

- When $V_w = 0$, the integral I_p reads

$$I_p = -i\alpha_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{k \int_0^\infty \mathsf{Ai}(s)ds}{\theta^2 \mathsf{Ai}'(0) - k^2 \int_0^\infty \mathsf{Ai}(s)ds} e^{ik\bar{X}} \right] dk.$$

- Singularities found by setting denominator equal to zero, we have

$$k = -i\kappa, \qquad \kappa = \left[3\left|\mathsf{Ai}'(0)\right|\right]^{3/4} > 0, \qquad \textit{simple pole}$$

- Applying Jordan's Lemma and subsequently Cauchy's Residue Theorem, we find pressure upstream given by

$$p' = \alpha_0 \left[\frac{\kappa}{2\kappa^{-1/3}|\mathsf{Ai}'(0)| + 2\kappa} e^{\kappa\bar{X}} \right] + \alpha_0 H(\bar{X}).$$

- So we have exponential decay upstream, i.e. when $\bar{X} < 0$. Note that $H(\bar{X}) = 0$ for $\bar{X} < 0$.

Downstream moving wall

- For $V_w \neq 0$ reads

$$I_p = -i\alpha_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{k \int_{\theta V_w}^{\infty} \mathsf{Ai}(s) ds}{\theta^2 \mathsf{Ai}'(\theta V_w) - k^2 \int_{\theta V_w}^{\infty} \mathsf{Ai}(s) ds} e^{ik\bar{X}} \right] dk$$

- Considered two cases
  - When V_w is large (considering asymptotic behaviour of derivative and integral) we find the singularity is given by

$$k = -iV_w, \qquad \textit{simple pole}$$

$$p' \sim \left[\frac{\alpha_0 3 V_w^{-2}}{2} e^{V_w \bar{X}} \right] + \alpha_0 H(\bar{X})$$

Downstream moving wall

- For $V_w \neq 0$ reads

$$I_p = -i\alpha_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{k \int_{\theta V_w}^{\infty} \mathrm{Ai}(s)ds}{\theta^2 \mathrm{Ai}'(\theta V_w) - k^2 \int_{\theta V_w}^{\infty} \mathrm{Ai}(s)ds} e^{ik\bar{X}} \right] dk$$

- Considered two cases

  - When V_w is small, i.e. as $V_w \to 0$, (considering Taylor expansion) we see the singularity varies by a small amount Δk from the case of the stationary wall, which is given by

$$i\Delta k = \frac{(\kappa)^{5/3} V_w Ai(0)}{\frac{4}{9}(\kappa)^{1/3} - \frac{5}{3}(\kappa)^{2/3} V_w Ai(0)} > 0 \quad \text{for small} \quad V_w.$$

- $\Rightarrow$ as V_w increases, pressure decays faster exponentially.

Upstream moving wall

- The integral in this case is given by

$$I_p = -i\alpha_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{k \int_{-\theta|V_w|}^{\infty} \mathrm{Ai}(s) ds}{\theta^2 \mathrm{Ai}'(-\theta|V_w|) - k^2 \int_{-\theta|V_w|}^{\infty} \mathrm{Ai}(s) ds} e^{ik\bar{X}} \right] dk = -i\alpha_0 \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{F(k)}{G(k)} \right] dk$$

- The singularities are given by

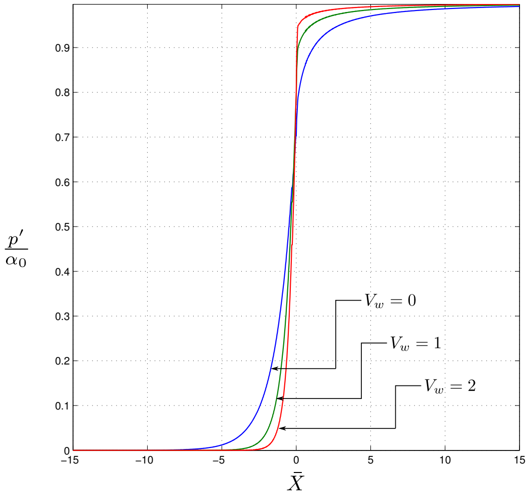
$$(ik)^{2/3} \mathrm{Ai}'(-\theta|V_w|) - k^2 \int_{-\theta|V_w|}^{\infty} \mathrm{Ai}(s) ds = 0$$

- Complicated, there are infinitely many roots as the Airy equation is oscillatory for negative values.
- So the integral would be represented by an infinite sum as

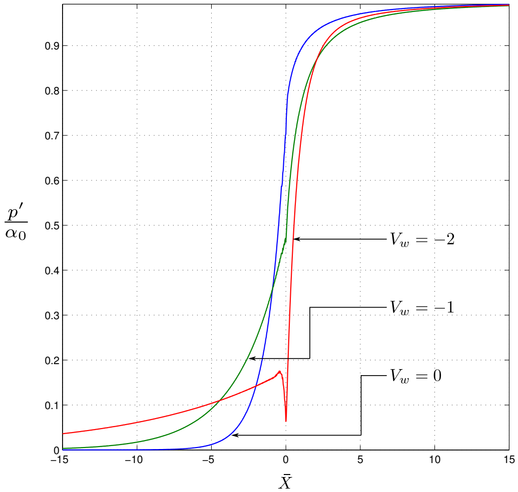
$$\alpha_0 \sum_{n=0}^{\infty} \mathrm{Res}\left(\frac{F}{G}, \tilde{k}_n \right)$$

- Ask the computer!

Downstream Moving Wall



Upstream Moving Wall



Conclusions

- Using the Navier-Stokes equations matched asymptotic expansion, gain model for interaction problem - predict pressure distribution.
- Stationary plate: exponential decay upstream - Remarkably similar to result by Sir James Lighthill (before triple deck theory).
- For downstream moving plate - for increasing speed, region of influence shrinks, $\bar{X} \sim 1/V_w$.
- Upstream moving plate - we see the pressure distribution on either side of shock impingment doesnt converge smoothly in the boundary-layer. solution no longer smooth - lead to discontinuity?
- Also have much shallower algebraic decay upstream - introduce new scaling for upstream region, viscous dissipation effect greater than variations in pressure.

Thank you for listening!
Questions?