Hele-Shaw flow driven by an external electric field

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Abstract

The control of two-dimensional finite blobs of fluid in a Hele-Shaw cell by an external field is considered. The time-dependent free boundary problem is studied both analytically using the Schwarz function of the free boundary and numerically using a boundary integral method. Main problems considered are: (i) the behaviour of an initially circular blob subject to an electric point charge located within the blob, (ii) the delay in cusp formation on the free boundary in the sink driven Hele-Shaw flow due to a strategically placed electric charge.

Introduction

- Aim to solve the free boundary problem [2, 3] given by

$$\nabla^2\phi = \sum_j Q_j \delta(x - x_j, y - y_j), \quad (x, y) \in \ \Omega(t), \tag{1a}$$

$$\phi = \Psi(x, y), \quad (x, y) \in \partial\Omega(t), \tag{1b}$$

$$v_n = \frac{\partial \phi}{\partial n}, \quad (x, y) \in \partial\Omega(t). \tag{1c}$$

- Here Q_j are the strengths of hydrodynamic singularities, ϕ is the velocity potential, Ψ is scalar potential of a background field. $Q_j \equiv 0, \forall j \implies$ boundary driven by background field alone.

Analytical approach:

- Define the Schwarz function as $g(z, t) = \overline{z}$ for $z \in \partial\Omega(t)$, an analytic function in the neighbourhood of $\partial\Omega(t)$.
- It is shown [3] the following equation holds $\forall z \in \Omega(t)$ ($w(z)$ = complex velocity potential):

$$\frac{\partial w}{\partial z} = \frac{1}{2}\frac{\partial g}{\partial t} + \frac{\partial \Psi}{\partial z}. \tag{2}$$

- Given a map $z(\zeta, t)$ from the unit ζ-disk, of the free boundary, determine $g(z, t)$ and hence $\Omega(t)$, $t > 0$.

Numerical method:

- Apply a boundary integral method, where for $z_m \in \partial\Omega(t)$ we have

$$\frac{1}{2\pi i}\oint_{\partial\Omega(t)} \frac{u - iv}{z - z_m}\, dz = \frac{1}{2} w'(z_m) + \sum_j \lim_{z \to z_j} \left[\mathrm{Res}\left(\frac{w'(z)}{z - z_m} \right) \right], \tag{3}$$

- Dynamic boundary condition equivalent to (1c) is given by

$$\frac{\partial \Phi}{\partial s} = (u - \Psi_x)\, x_s + (v - \Psi_y)\, y_s = 0, \tag{4}$$

- Step in time using advection, i.e. $\dot{x} = u$, $\dot{y} = v$ and solve (3)-(4) for new (u, v).

Stability under an electric field

- Boundary motion driven by external electric field (point charge E at $z = 0$) given by

$$\Psi = \frac{E}{4\pi}\log(zg), \quad z \in \partial\Omega(t), \tag{5}$$

- Consider a circular fluid blob (centre $z = 0$) with n small disturbances on the initial boundary, $\partial\Omega(0)$, given by the map ($n \geq 2$ and $|\alpha(t)| \ll 1$ is a real, time varying coefficient)

$$z = \zeta + \alpha\zeta^n. \tag{6}$$

- Using $\overline{\zeta} = \zeta^{-1}$ and inversion of (6) as $z \to 0$, the Schwarz function behaves like

$$g(z, t) \to \frac{\alpha}{z^n} + \frac{n\alpha^2 + 1}{z} + \alpha z^{n-2} + \mathcal{O}(\alpha^2). \tag{7}$$

- Considering the structure of the singularities of $\mathcal{O}(z^{-n})$ on both sides of (2), we have, since $w(z)$ is regular, the following ODE for $\alpha(t)$:

$$\dot{\alpha} = \frac{E}{2\pi}(n - 1)\alpha. \tag{8}$$

- The solution for α is exponentially decaying for negative point charge E only $\implies$ stable.

Circular blob with off centre point charge

- Consider a circular fluid blob centred at $(\epsilon, 0)$ subject to an electric point charge at $z = 0$, then

$$g(z, t) = \epsilon + \frac{R^2}{z - \epsilon}. \tag{9}$$

- For small ϵ, considering the structure of the singularities of $\mathcal{O}(z^{-1})$ and $\mathcal{O}(z^{-2})$ on both sides of (2), we have the following system of ODEs for $R(t)$ and $\epsilon(t)$:

$$R\dot{R} = 0, \tag{10a}$$

$$\frac{R^2\dot{\epsilon}}{2} + \epsilon R\dot{R} - \frac{E}{4\pi}\epsilon = 0, \quad (E < 0). \tag{10b}$$

- Hence, $\dot{R} = 0$, so $R(t) = constant = R_0$, where R_0 is the radius of the initial blob $\implies$ consistent with conservation of area. Hence, (10b) gives solution (ϵ_0 being initial centre)

$$\epsilon(t) = \epsilon_0 \exp\left(\frac{E}{2R_0^2\pi} t \right). \tag{11}$$

- For small ϵ, the fluid blob remains circular throughout motion and it's centre emigrates towards the position of the point charge. Figure 1 shows agreement of ϵ and centre of mass $\overline{\epsilon}$ computed from numerical data.

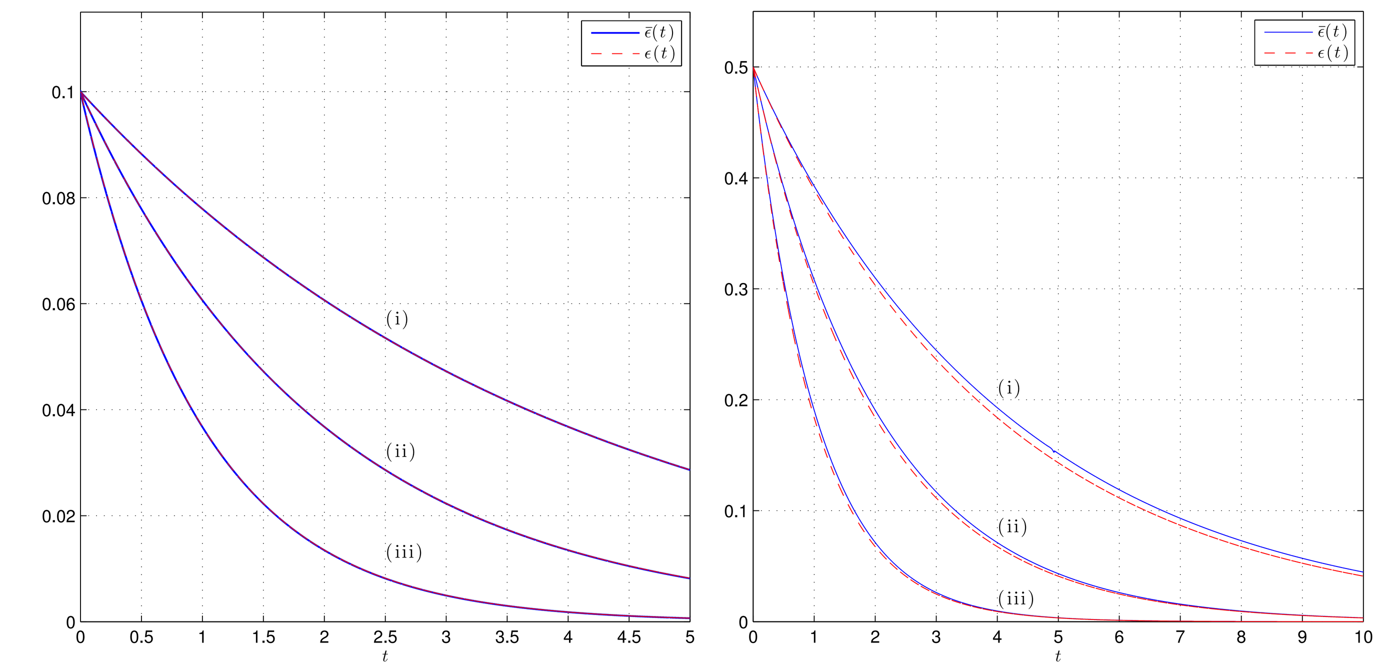

Figure 1: Evolution of the centre of mass, $\overline{\epsilon}(t)$, of a circular fluid blob, superimposed with the analytic solution, $\epsilon(t)$, in the case $\epsilon_0 = 0.1$ (left), $\epsilon_0 = 0.5$ (right). In both cases $R_0 = 1$, point charge strengths (i) $E = -\pi/2$, (ii) $E = -\pi$ and (iii) $E = -2\pi$.

- For large values of ϵ_0, boundary no longer remains circular throughout motion, see Figure 2.

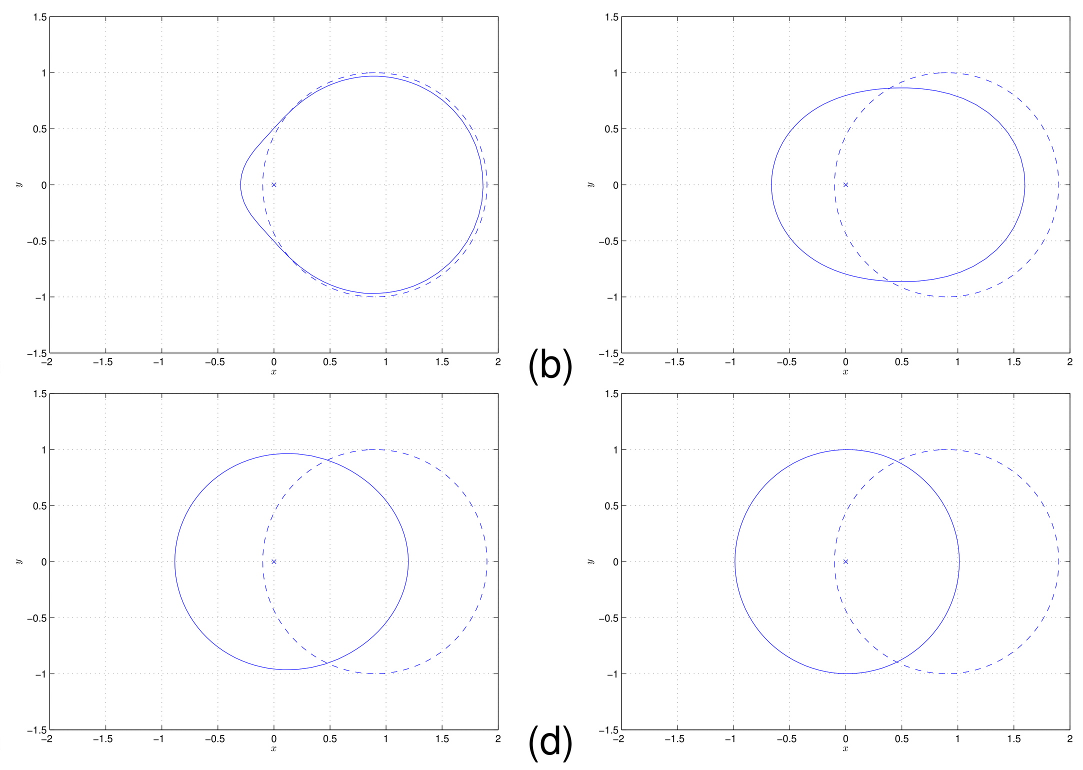

Figure 2: Snapshots (solid line - top left to bottom right) of the time development of an initially circular fluid blob (dashed line) subject to an electric point charge of strength $E = -\pi$ located at the origin (marked by cross), for the case $\epsilon_0 = 0.9$ and $R_0 = 1$.

- Fluid blob moves through pear shaped domains, finally becoming symmetrised about the point charge at $z = 0$, forming a perfect circle as expected.

Extracting all fluid from a blob

- Extracting fluid from a fluid blob via a hydrodynamic sink has been considered in the past, which leads to cusp formation in finite time [1], see Figure 3.
- The boundary is given by the following polynomial map ($a(t)$ and $b(t)$ are real time varying coefficients)

$$z = \zeta(a + b\zeta^n). \tag{12}$$

- Starting with initial parameters $a(0) = 1.0$, $b(0) = 0.1$ (dashed line), the solution breaks down as the boundary map no longer remains univalent beyond cusp time.

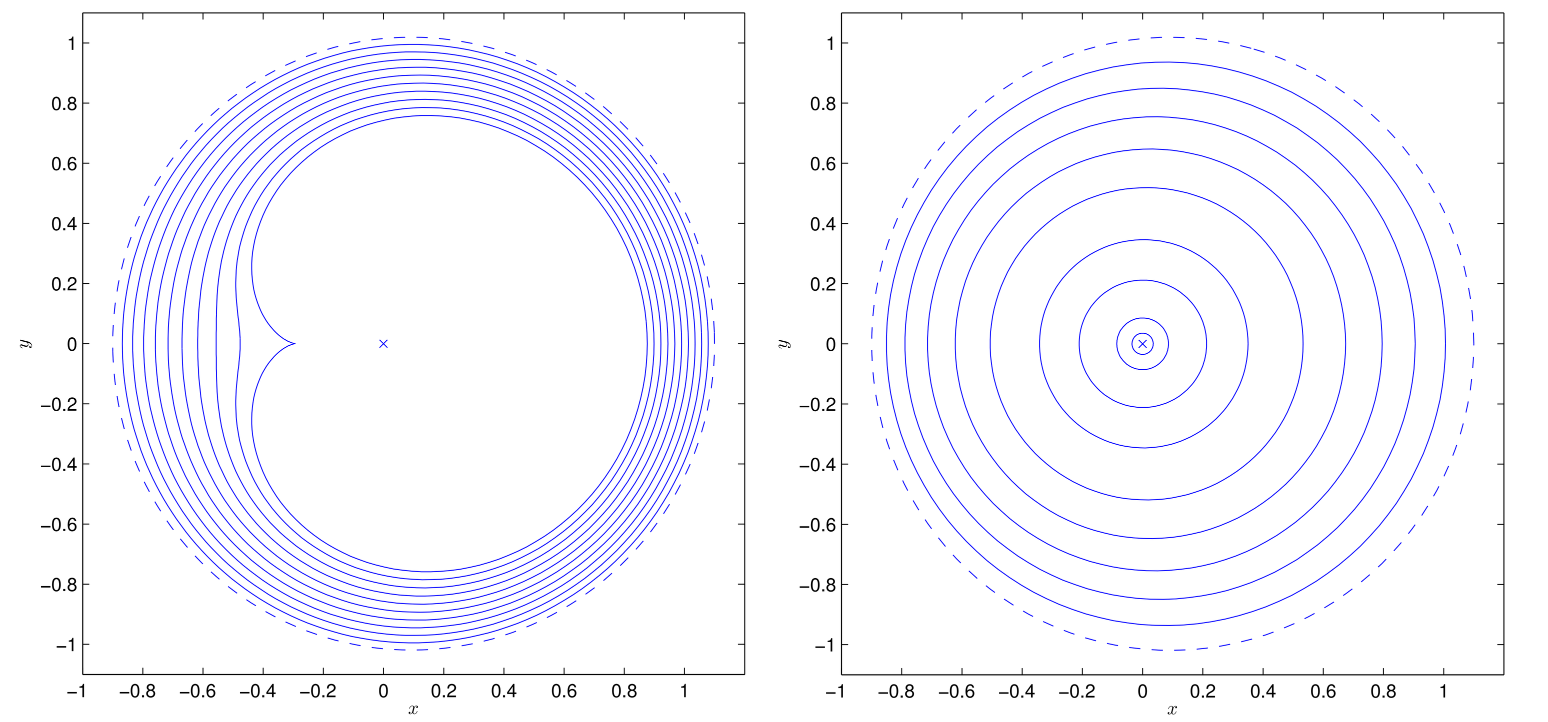

Figure 3: Comparison between cusp formation due to hydrodynamic sink (left, analytic) with $Q = -\pi$ (marked by cross) and superposition of hydrodynamic sink plus electric charge (right, numerical) with $Q = -\pi$, $E = -5\pi$ (marked by cross).

- Superimposing an electric point charge, $E < 0$, with a hydrodynamic sink at the origin provides a competing effect at the boundary, delaying the formation of a cusp.
- Figure 3 (right) shows numerical results of the possibility of extracting all fluid from the fluid blob with such a setup.

Conclusions

- If the boundary motion is driven by an electric field only, it is only stable if the point charge within the fluid domain is negative, in which case the boundary tends to a circle centred at the point charge.
- If the point charge is placed close enough to the centre of an initially circular fluid blob, the blob remains circular throughout its motion.
- The symmeterising effect from the presence of an electric point charge proves desirable in the problem of extracting fluid through a hydrodynamic sink - usually an unstable problem with the formation of cusps or fingers on the boundary.
- Possible applications: control of fluid blobs in microfluidics (recent surge in microfluidic devices with biochemical applications), oil extraction.

References

[1] Cummings, L. J., Howison, S. D. & King, J. R. 1999, Two-dimensional Stokes and Hele-Shaw flows with free surfaces. *J. Appl. Math.*, **10**, 635–680.

[2] Entov, V. M., Etingof, P. & Kleinbock, D. Ya. 2007, On a generalized two-fluid Hele-Shaw flow. *J. Appl. Math.*, **18**, 103–128.

[3] McDonald, N. R. 2011, Generalised Hele-Shaw flow: A Schwarz function approach. *Eur. J. Appl. Math.*, 1–16.

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